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ON THE STRUCTURAL STABILITY OF PLANAR QUASIHOMOGENEOUS POLYNOMIAL VECTOR FIELDS

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ABSTRACT. Denote by H_{pqm} the space of all planar (p,q)-quasihomogeneous vector fields of degree m endowed with the coefficient topology. In this paper we characterize the set Ω_{pqm} of the vector fields in H_{pqm} that are structurally stable with respect to perturbations in H_{pqm}, and determine the exact number of the topological equivalence classes in Ω_{pqm}. The characterisation is applied to give an extension of the Hartman-Grobmann Theorem for such family of planar polynomial vector fields. It follows from the main result in this paper that, for a given $X \in H_{pqm}$ we give a explicit method to decide whether it is structurally stable with respect to perturbation in H_{pqm} before finding the vector field induced by X in the Poincaré-Lyapunov sphere. This work is an extension and an improvement of the Llibre-Perez-Rodriguez's paper [17], where the homogeneous case was considered. More precisely, if both p and q are odd, the main results of this paper are similar to those of the Llibre-Perez-Rodriguez's paper; if either p or q is odd while the other is even, we present some results which do not appear in the above mentioned paper. For example, one of the interesting results is that there may be triples (p,q,m) such that $H_{pqm} \neq \emptyset$ but $\Omega_{pqm} = \emptyset$, which does not occur in the homogeneous case.

1. Introduction and statement of the main results

The structural stability of planar vector fields has been a subject of great interest in the global qualitative theory of dynamical systems since the sixties. The first definition of structural stability of planar vector fields goes back to Andronov and Pontrjagin (1937) [1] and Peixoto (1962) [20]. There are many paper about this subject. In 1990 [19] Shafer characterized the planar gradient polynomial vector fields which are structurally stable with respect to perturbations in the set of all C^r planar vector fields and in the set of all planar polynomial vector fields. In 1993, Jarque and Llibre [12] found the similar characterisation to planar Hamiltonian polynomial vector field with respect to perturbations in the same sets of Shafer. In 1996 Llibre, Perez and Rodriguez [17] characterized the structural stable homogeneous vector fields with respect to perturbation in the same restricted set. In 2000 the same authors [18] extended most of the results of the previous paper to systems of the form $X = (P_m, Q_n)$, where P_m and Q_n are polynomial functions of degree m and n, respectively, $m, n > 1$. In 2005 Jarque, Llibre and Shafer [13] provided sufficient conditions for a planar polynomial foliation to be structurally stable under several different types of

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perturbation. In 2008 [14], Jarque, Llibre and Shafer obtained the full characterisation of structural stability of polynomial foliations of degree 1 and 2, both in the Poincaré sphere and in the plane, together with a complete catalogue of phase portrait of stable systems.

However the complete and explicit characterisation of all planar structurally stable polynomial vector fields of degree m with respect to polynomial perturbations, is an open problem. In many papers about this subject we found the study of some families of polynomials vector fields, specially modulo limit cycles, as in [2].

A function $f(x, y)$ is called a *(p, q)-quasihomogeneous function of degree m* if $f(\lambda^p x, \lambda^q y) = \lambda^m f(x, y)$ for all $\lambda \in \mathbb{R}$. If $P(x, y)$ and $Q(x, y)$ are (p, q)- quasihomogeneous polynomials of degree $p - 1 + m$ and $q - 1 + m$, respectively, we say that $X = (P, Q)$ is a *planar (p, q)-quasihomogeneous polynomial vector field of degree m*. The system of differential equations associated to X is

$$\frac{dx}{dt} = P(x, y) = \sum_{pi+qj=p-1+m} a_{ij}x^i y^j, \quad \frac{dy}{dt} = Q(x, y) = \sum_{pi+qj=q-1+m} b_{ij}x^i y^j. \tag{1}$$

Here p, q and m are positive integers and $P(x, y)$ and $Q(x, y)$ are coprime in the ring $\mathbb{R}[x, y]$. To be short we denote this by $(P(x, y), Q(x, y)) = 1$.

Observe that the above definition is the natural one for the following reasons[5]:

(1) When $p = q = 1$ it coincides with the usual definition of homogeneous vector field of degree m.
(2) The differential equation $dy/dx = Q/P$ associated with X, is invariant by the change of variables $\bar{x} = \lambda^p x, \bar{y} = \lambda^q y$.
(3) Homogeneous vector fields can be integrated using polar coordinates whereas (p, q)-quasihomogeneous vector fields can be integrated using the (p, q)-polar coordinates. These generalized polar coordinates were introduced by Lyapunov in his study of the stability of degenerate critical points[9]. In this paper we will describe this change of coordinates and some of their main properties in Section 2.

Let H_{pqm} be the set of all planar (p, q)-quasihomogeneous vector fields of degree m.

It follows from (1) that $H_{pqm} \neq \emptyset$ if and only if $pi + qj = p - 1 + m$ and $pi' + qj' = q - 1 + m$ have non-negative integer solutions (i, j) and (i', j') respectively. Therefore if $(p, q, m) = (3, 7, 2)$, then $H_{372} = \emptyset$; and if $(p, q, m) = (1, 2, 2)$, then $H_{122} \neq \emptyset$.

Although there are integers p, q and m such that $H_{pqm} = \emptyset$, there has been a substantial amount of work devoted to understanding the properties of the quasihomogeneous vector fields. For instance, the quasihomogeneous vector fields have appeared in several works about the Hilbert's Sixteenth Problem[5, 16], and about integrability of planar vector fields[7]. The (p, q)-polar coordinates, used to study the quasihomogeneous vector fields, have also been applied to study properties of planar differential equations[3, 15]. It is well known that (see [7]), given a polynomial $f \in \mathbb{R}[x, y]$ we can write it in the form $f = f_m + f_{m+1} + \ldots + f_{m+n}$, where f_k is a (p, q)-quasihomogeneous polynomial function of degree k.

Before stating the main results of this paper, we give some definition and notations. To study the behaviour of the trajectories of a planar differential system near infinity we use the Poincaré-Lyapunov compactification, see for instance [9]. In the Poincaré-Lyapunov compactification we prefer to work on a hemisphere, calling it as Poincaré-Lyapunov disk.

The induced vector field in the Poincaré-Lyapunov disc is called *the Poincaré-Lyapunov compactification of the vector field* X, denoted by $E(X)$.

Roughly speaking, we shall say that two vector fields $X, Y \in H_{pqm}$ are *topologically equivalent* if there exists a homeomorphism h in the Poincaré-Lyapunov disc, carrying orbits of the flow induced by $E(X)$ onto orbits of the flow induced by $E(Y)$, preserving sense but not necessarily parametrization; h is termed an equivalence homeomorphism between X and Y. Moreover, as in the homogeneous case, the coefficients of X, P and Q, are polynomials so, there exists a positive integer $k < \infty$, such that every $X = (P, Q) \in H_{pqm}$ could be identified with a unique point in $\mathbb{R}^k$, by the identification of the coefficients of X with points from $\mathbb{R}^k$. The number k could be chosen in the following way: let k_1 and k_2 be the numbers of the non-negative integer solutions, in case they exist, for the equations $pi + qj = p - 1 + m$ and $pi' + qj' = q - 1 + m$ respectively, if there exists. Then $k = k_1 + k_2$. We take in H_{pqm} the topology induced by the Euclidean norm in $\mathbb{R}^k$.

Further, a vector field $X \in H_{pqm}$ is *structurally stable* with respect to perturbation in H_{pqm} if there exists a neighborhood U of X in H_{pqm} such that for all $Y \in U$, X and Y are topologically equivalent. We also need to observe that this definition of structural stability does not require that the equivalence homeomorphism is near the identity map on the Poincaré-Lyapunov sphere. It is important to say that in certain cases, for open manifolds, the restriction of h to a neighborhood of identity in H_{pqm} is not superfluous (see [8, 11, 19]), but here it is redundant. This follows from the Peixoto results in [20], as in the homogeneous case. Since the coefficient topology is equivalent to the C^1 topology and the Poincar Lyapunov sphere is a manifold in the Peixoto conditions we don't need to require that our equivalence homeomorphism be near the identity map. Peixoto showed in [20] that on an orientable differentiable compact connected 2-manifold without boundary, if a C^1 vector field X is equivalent to all vector fields in a neighborhood U of X in the C^1 topology, then the equivalence homeomorphism between X and any vector field in U can be chosen sufficiently close to the identity map.

Denoted by Ω_{pqm} the set of all vector fields in H_{pqm} which are structurally stable with respect to perturbations in H_{pqm}. In this paper we characterise the vector fields in Ω_{pqm} and determine the exact number of the topological equivalence classes in Ω_{pqm}. By using this characterisation we give an extension of The Hartman-Grobman Theorem for a kind of planar vector fields.

We would like to point out that the results of this paper are an extension of the results in [17] and an improvement of them. For a given (p, q)-quasihomogeneous vector field of degree m, we give an explicit method to decide whether it is structurally stable with respect to perturbation in H_{pqm}, before finding the vector field induced by X in the Poincaré-Lyapunov sphere, as it has taken place in others papers such as [17], for instance. Roughly speaking, if both p and q are odd, the study of quasihomogeneous system are similar to those of the homogeneous one in [17]; if either p or q is odd while the other is even, then we should use different ideas to study the (p, q)-quasihomogeneous vector fields. In fact, we present some results which do not appear in [17] for the latter case. For example, one of the interesting results is that there are triples (p, q, m) such that $H_{pqm} \neq \emptyset$ but $\Omega_{pqm} = \emptyset$, which does not occur in the homogeneous case. We also note that Proposition 3 is not necessarily in [17], but it is crucial for our analysis. Lemma 4 will tell us what is the difference between a general quasihomogeneous vector field and a homogeneous one.

Now we shall present the main results of this paper. Let

$$\eta(x, y) = pxQ(x, y) - qyP(x, y). \tag{2}$$

In Lemma 9 (see Section 2 below) it is proved that if $\eta(x, y) \equiv 0$, then $(P, Q) = (px, qy)$. This degenerated case will not be considered in this paper. Moreover we understand that the following convention (a) holds up without loss of generality. See Lemma 8 in Section 2.

Convention. *In this paper we understand that*

(a) p *is odd and* $(p, q) = 1$, *unless the opposite is claimed, and*
(b) $\eta(x, y) \not\equiv 0$.

In Section 2 we shall study the phase portraits of the vector fields in H_{pqm} and prove the following theorem.

Theorem 1. *Let* $X \in H_{pqm} \neq \emptyset$. *If* $\eta(1, y)$ *has no zero and* $\eta(0, 1) \neq 0$, *then* $\eta(x, y) \neq 0$, *for* $(x, y) \neq (0, 0)$ *and, the origin of system (1) is*

(a) *a global center if and only if* $I_X = 0$, *where*

$$I_X = \int_{-\infty}^{+\infty} \frac{P(1, u)}{\eta(1, u)} du. \tag{3}$$

(b) *a global stable (respectively, unstable) focus if and only if* $I_X < 0$ *(respectively,* $I_X > 0$*).*

In [16] I_X is given by the integral of (p, q)-trigonometric functions. Here I_X depends on the integral of the rational function $P(x, y)/\eta(x, y)$.

In Section 2 it is proved that, if $\eta(1, y) = 0$ has a real zero or $\eta(0, 1) = 0$ then the system (1) has at least one invariant curve. Therefore, Theorem 1 implies that the origin is a center if and only if $\eta(x, y) \neq 0,\ I_X = 0$, or if and only if $\eta(1, y) \neq 0,\ \eta(0, 1) \neq 0,\ I_X = 0$.

The following theorem is proved in Section 3 and it gives the characterisation of the set Ω_{pqm}.

Theorem 2. *Denoted by* Ω_{pqm} *the set of all vector fields in* H_{pqm} *which are structurally stable with respect to perturbations in* $H_{pqm} \neq \emptyset$. *If* $\Omega_{pqm} \neq \emptyset$, *then the vector field* $X \in \Omega_{pqm}$ *if and only if one of the following conditions is satisfied:*

(a) *If* $\eta(1, y)$ *has no zero and* $\eta(0, 1) \neq 0$ *then* $I_X \neq 0$;
(b) *If the condition in (a) does not hold, then all the zeros of* $\eta(1, y)$ *are simple if they exist, and* $\partial\eta(0, 1)/\partial x \neq 0$, *if* $\eta(0, 1) = 0$.

In order to compute the number of topological equivalence classes in Ω_{pqm} it is necessary to have some additional information about the normal form of the structural stable vector field $X = (P, Q)$. They are given by the following proposition.

Proposition 3. *Let $X = (P, Q) \in H_{pqm} \neq \emptyset$ and $\eta(x, y)$ is given by (2). Then $\eta(x, y)$ is a (p, q)- quasihomogeneous polynomial of degree $p + q + m - 1$ having the form*

$$\eta(x, y) = \sum_{pi+qj=p+q+m-1} c_{ij} x^i y^j. \tag{4}$$

Suppose $X \in \Omega_{pqm} \neq \emptyset$, then there exist a unique integer r such that

(a) *if $\eta(0, 1) \neq 0$, $\eta(1, 0) \neq 0$ then $p + q + m - 1 = (r + 1)pq$ with $r \geq 0$ and*

$$\begin{aligned} \eta(x, y) &= \sum_{l=0}^{r+1} c_{lq,(r+1-l)p} (x^q)^l (y^p)^{r+1-l}, \\ P(x, y) &= \sum_{l=0}^{r} a_{lq,(r+1-l)p-1} x^{lq} y^{(r+1-l)p-1}, \\ Q(x, y) &= \sum_{l=0}^{r} b_{(r+1-l)q-1,lp} x^{(r+1-l)q-1} y^{lp}, \end{aligned} \tag{5}$$

with $a_{0,(r+1)p-1} \neq 0$, $b_{(r+1)q-1,0} \neq 0$, $c_{0,(r+1)p} \neq 0$, $c_{(r+1)q,0} \neq 0$;

(b) *if $\eta(0, 1) \neq 0$, $\eta(1, 0) = 0$, then $p + m - 1 = rpq$ with $r \geq 0$ and*

$$\begin{aligned} \eta(x, y) &= \left(\sum_{l=0}^{r} c_{lq,(r-l)p+1} (x^q)^l (y^p)^{r-l} \right) y, \\ P(x, y) &= \sum_{l=0}^{r} a_{lq,(r-l)p} (x^q)^l (y^p)^{r-l}, \\ Q(x, y) &= \sum_{l=0}^{r} b_{(r-l)q-1,lp+1} x^{(r-l)q-1} y^{lp+1}, \end{aligned} \tag{6}$$

with $a_{0,rp} \neq 0$, $b_{rq-1,1} \neq 0$, $c_{0,rp+1} \neq 0$, $c_{rq,1} \neq 0$;

(c) *if $\eta(0, 1) = 0$, $\eta(1, 0) \neq 0$, then $q + m - 1 = rpq$ with $r \geq 0$ and*

$$\begin{aligned} \eta(x, y) &= x \left(\sum_{l=0}^{r} c_{1+lq,(r-l)p} (x^q)^l (y^p)^{r-l} \right), \\ P(x, y) &= x \left(\sum_{l=0}^{r-1} a_{1+lq,(r-l)p-1} x^{lq} y^{(r-l)p-1} \right), \\ Q(x, y) &= \sum_{l=0}^{r} b_{(r-l)q,lp} (x^q)^{r-l} (y^p)^l, \end{aligned} \tag{7}$$

with $a_{1,rp-1} \neq 0$, $b_{rq,0} \neq 0$, $c_{1,rp} \neq 0$, $c_{1+rq,0} \neq 0$;

(d) *if* $\eta(0,1)=0,\ \eta(1,0)=0$, *then* $m-1=(r-1)pq$ *with* $r\geq 1$ *and*

$$\begin{aligned}
\eta(x,y) &= xy\left(\sum_{l=0}^{r-1} c_{1+lq,(r-1-l)p+1}(x^q)^l(y^p)^{r-1-l}\right),\\
P(x,y) &= x\left(\sum_{l=0}^{r-1} a_{1+lq,(r-1-l)p}(x^q)^l(y^p)^{r-1-l}\right),\\
Q(x,y) &= y\left(\sum_{l=0}^{r-1} b_{(r-1-l)q,1+lq}(x^q)^{r-1-l}(y^p)^l\right),
\end{aligned}\tag{8}$$

with $a_{1,(r-1)p}\neq 0,\ b_{(r-1)q,1}\neq 0\ c_{1,(r-1)p}\neq 0,\ c_{1+(r-1)q,1}\neq 0.$

Proposition 3 says that, if $X\in\Omega_{pqm}$ then there is a non-negative integer r such that r satisfies at least one of the following equations:

$$\begin{array}{ll}
\mathcal{E}_1: p+q+m-1=(r+1)pq, & \mathcal{E}_2: p+m-1=rpq,\\
\mathcal{E}_3: q+m-1=rpq, & \mathcal{E}_4: m-1=(r-1)pq.
\end{array}\tag{9}$$

An interesting consequence of the above proposition is the following: there are p,q and m such that $H_{pqm}\neq\emptyset$ but $\Omega_{pqm}=\emptyset$. This is proved by choosing the triple (p,q,m) such that there is no r satisfying any equation in (9).

Example. Let $(p,q,m)=(1,7,2)$ and consider H_{172}. Since $X=(ax^2,bx^8+cxy)\in H_{172}$ for each $(a,b,c)\in\mathbb{R}^3$, we have $H_{172}\neq\emptyset$. It is easy to check that there is no r satisfying any equation in (9) then $\Omega_{172}=\emptyset$.

Denote by

$$\Theta_i=\{(p,q,m): \text{ there exists } r \text{ satisfying } \mathcal{E}_i\}.\tag{10}$$

We shall classify Ω_{pqm} by $\Theta_i,\ i=1,2,3,4$, and study the number of equivalence classes in Ω_{pqm}. The following lemma shows what happens if $(p,q,m)\in\Theta_i\cap\Theta_j,\ j\neq i$. It also tells us that if $p=q=1$ (the homogeneous case), then $(1,1,m)$ satisfies all equations in (9).

Lemma 4. *Let* r_i *be solution of* $\mathcal{E}_i$. *If* $(p,q,m)\in\Theta_i\cap\Theta_j,\ i\neq j$, *then* $r_i=r_j$. *More precisely,*

(a) $p=q=1$ *if one of the following condition holds:*
  (a.1) $(p,q,m)\in\Theta_1\cap\Theta_4$;
  (a.2) $(p,q,m)\in\Theta_2\cap\Theta_3$;
  (a.3) (p,q,m) *satisfies three equations of* (9);
(b) $p=1$ *if one of the following two condition holds:*
  (b.1) $(p,q,m)\in\Theta_1\cap\Theta_2$;
  (b.2) $(p,q,m)\in\Theta_3\cap\Theta_4$;
(c) $q=1$ *if one of the following two condition holds:*
  (b.1) $(p,q,m)\in\Theta_1\cap\Theta_3$;
  (b.2) $(p,q,m)\in\Theta_2\cap\Theta_4$.

If $p=1,\ (1,q,m)\in\Theta_1\cap\Theta_2$*(resp.* $\Theta_3\cap\Theta_4$*)*, $X=(P(x,y),Q(x,y))\in\Omega_{1qm}\neq\emptyset$, *then* X *with* (6) *(resp.* (8)*) can be changed into system* (1) *with* (5) *(resp.* (7)*) by the* $(1,q)$*-quasihomogeneous polynomial transformation* $y\to y-\lambda x^q$, $0\neq\lambda\in\mathbb{R}$.

It follows from the above lemma that for p, q and m given, if there exists the number r, it is unique, it can not take different values, even if (p,q,m,r) satisfies two or more equations of (9). It also says that there exists a triple (p,q,m) such that $(p,q,m) \in \Theta_i$ but $(p,q,m) \notin \Theta_j$, $j \neq i$.

Based on Lemma 4, without loss of generality, we assume that the following conventions hold, provided $X \in \Omega_{pqm} \neq \emptyset$:

(i) $p = 1$, if $p = 1$ or $q = 1$;
(ii) In order to compute the number of topological equivalence classes in Ω_{pqm}, we consider the case $(1,q,m) \in \Theta_1$ (resp. Θ_3) if $(1,q,m) \in \Theta_1 \cap \Theta_2$ (resp. $(1,q,m) \in \Theta_3 \cap \Theta_4$).

Denote by C_{pqm} the number of topological equivalence classes in Ω_{pqm}. The following theorem gives us the value of C_{pqm} for each p,q and m.

Theorem 5. *Let r be an integer, defined in Proposition 3, and $\Omega_{pqm} \neq \emptyset$. Assume that the above conventions (i) and (ii) hold.*

(a) *Suppose that both p and q are odd.*
(a.1) *If r is odd, then*

$$C_{pqm} = \begin{cases} 1 + \dfrac{1}{2} \displaystyle\sum_{j=1}^{(r+1)/2} \sum_{n|j} (\mathcal{P}_{2n} + I_{2n}), & \text{if } (p,q,m) \in \Theta_1 \backslash \displaystyle\bigcup_{i=2}^{4} \Theta_i, \\ -1 + \dfrac{1}{2} \displaystyle\sum_{j=1}^{(r+1)/2} \sum_{n|j} (\mathcal{P}_{2n} + I_{2n}), & \text{if } (p,q,m) \in \displaystyle\bigcup_{i=2}^{4} \Theta_i, \end{cases}$$

where

$$\mathcal{P}_{2n} = \frac{1}{n}\left(2^{2n} - \sum_{l|n, l\neq n} l\mathcal{P}_{2l}\right), \quad I_{2n} = 2^{n+1} - \sum_{l|n, l\neq n} I_{2l},$$

(a.2) *If r is even, then*

$$C_{pqm} = -1 + \frac{1}{2} \sum_{j=1}^{r/2} \sum_{n|2j+1} (\mathcal{P}_{2n} + I_{2n}),$$

where

$$\mathcal{P}_{2n} = \frac{1}{n}\left(2^{n} - \sum_{l|n, l\neq n} l\mathcal{P}_{2l}\right), \quad I_{2n} = 2^{(n+1)/2} - \sum_{l|n, l\neq n} I_{2l}, \tag{11}$$

(b) *Suppose that p is odd and q are even and $\mathcal{P}_{2n}$ is given by (11).*

(b.1) *If* $(p,q,m) \in \Theta_1 \setminus \bigcup_{i=2}^{4} \Theta_i$, *then*

$$C_{pqm} = \begin{cases} \dfrac{1}{2}\left(r+3+\displaystyle\sum_{j=1}^{(r+1)/2} \sum_{n|2j} \mathcal{P}_{2n} \right), & \text{if } r \text{ is odd,} \\ \dfrac{1}{2}\left(r-2+\displaystyle\sum_{j=1}^{r/2} \sum_{n|2j+1} \mathcal{P}_{2n} \right), & \text{if } r \text{ is even,} \end{cases}.$$

(b.2) *If* $(p,q,m) \in \Theta_2$, *then*

$$C_{pqm} = \begin{cases} \dfrac{1}{2}\left(r-1+\displaystyle\sum_{j=1}^{(r+1)/2} \sum_{n|2j} \mathcal{P}_{2n} \right), & \text{if } r \text{ is odd,} \\ \dfrac{1}{2}\left(r-2+\displaystyle\sum_{j=1}^{r/2} \sum_{n|2j+1} \mathcal{P}_{2n} \right), & \text{if } r \text{ is even,} \end{cases}.$$

(b.3) *If* $(p,q,m) \in \Theta_3 \cup \Theta_4$, *then*

$$C_{pqm} = \begin{cases} r+\dfrac{1}{2}\left(\displaystyle\sum_{j=1}^{(r+1)/2} \sum_{n|j} \mathcal{P}_{2n} \right), & \text{if } r \text{ is odd,} \\ \dfrac{1}{2}\left(r-2+\displaystyle\sum_{j=1}^{r/2} \sum_{n|2j+1} \mathcal{P}_{2n} \right), & \text{if } r \text{ is even,} \end{cases}.$$

Finally we have an extension of the Hartman-Grobman Theorem to vector fields in H_{pqm}. Before stating these results we need some notations. We say that two analytic vector fields X and Y are *locally topologically equivalent at origin (resp. infinity)* if there are two neighborhoods U and V of the origin (resp. the infinity) and a homeomorphism $h: U \to V$ that carries orbits of the flow induced by X onto orbits of the flow induced by Y, preserving sense but not necessarily parametrization (see for instance [17]).

The next two theorems are extensions of the Hartman-Grobman Theorem at the origin and infinity respectively.

Theorem 6. *Let* $X = \sum_{i\geq m} X_i$, *where* $X_i = (P_i(x,y), Q_i(x,y))$ *is a* (p,q)*-quasihomogeneous polynomial vector field of degree* i *if* $H_{pqi} \neq \emptyset$, *and* $X_i = (0,0)$ *if* $H_{pqi} = \emptyset$, *where* $i \in \{m, m+1, \cdots,\}$, $m \geq 1$. *Suppose* $X_m \in \Omega_{pqm}$, *then the phase portrait of* X *and* X_m *are locally topologically equivalent at the origin.*

Now we can get the analogous of Theorem 6 at infinity.

Theorem 7. *Let* $\hat{X} = \sum_{i=0}^{m} X_i$, *where* $X_i = (P_i(x,y), Q_i(x,y))$ *is a* (p,q)*-quasihomogeneous polynomial vector field of degree* i *if* $H_{pqi} \neq \emptyset$, *and* $X_i = (0,0)$ *if* $H_{pqi} = \emptyset$, *where* $i \in \{0, 1, ..., m\}$. *Suppose* $X_m \in \Omega_{pqm}$, *then the phase portraits of* X_m *and* $\hat{X}$ *are locally topologically equivalent at infinity.*

The rest of this paper is organized as follows. In Section 2 we study the phase portraits of (p,q)-quasihomogeneous polynomials of degree m on Poincaré-Lyapunov disk. In Section 3 we prove Theorem 2 which characterises the vector fields $X \in \Omega_{pqm}$. In Section 4 we compute the number C_{pqm} of topological equivalence classes in Ω_{pqm}. Theorem 6 and Theorem 7 are proved in Section 5.

2. Phase portraits of quasihomogeneous vector fields

In this section we shall study the phase portraits of quasihomogeneous vector fields.

As discussed before, one of our conventions is that, if $X \in H_{pqm}$ then $(p,q) = 1$. The next lemma shows that, if $(p,q) \geq 2$, then there exist a triple (p', q', m') with $(p', q') = 1$ such that $X \in H_{p'q'm'}$.

Lemma 8. *Suppose* $(p,q) = k \geq 2$ *in* (1), *then there exists a unique triple* (p', q', m') *with* $(p', q') = 1$ *such that system* (1) *is a* (p', q')*-quasihomogeneous vector field of degree* m'.

Proof. Let $p = kp'$, $q = kq'$. Since $(p,q) = k \geq 2$, we have $(p', q') = 1$. It follows from the definition of $P(x,y)$ in (1) that $kp' - 1 + m = kp'i + kq'j$, which implies that $k | m - 1$. Taking $m' = 1 + (m-1)/k$, the statement follows. □

Lemma 9. *Let* $X = (P, Q) \in H_{pqm} \neq \emptyset$ *and consider the polynomial* $\eta(x,y)$, *defined by* (2).

(a) $\eta(x,y)$ *is a* (p,q)*-quasihomogeneous polynomial of degree* $p + q + m - 1$.

(b) *If* $\eta(x,y) \equiv 0$, *then* $X = (P,Q) = (px, qy)$.

Proof. The item (a) is a consequence of the fact that $P(x,y)$ and $Q(x,y)$ are (p,q)-quasihomogeneous polynomials of degree $m - 1 + p$ and $m - 1 + q$, respectively. Now if $\eta(x,y) \equiv 0$, then $pxQ(x,y) = qyP(x,y)$. As $P(x,y)$ and $Q(x,y)$ are coprime, follows that $P(x,y) | x$ and $Q(x,y) | y$. □

The lemma bellow says that, as in the homogeneous case, there is no limit cycles if $\eta(1,y) = 0$ has zeros, or $\eta(0,1) = 0$.

Proposition 10. *Let* $X \in H_{pqm} \neq \emptyset$.

(a) *If* $\eta(0,1) = 0$, *then* $x = 0$ *is an invariant curve for the flow of* X;

(b) *If there exists* $\lambda \in \mathbb{R}$ *such that* $\eta(1, \lambda) = 0$, *then* $y^p - \lambda^p x^q = 0$ *is an invariant curve of the flow of* X;

(c) X *has no periodic orbit if (a) or (b) holds.*

Proof. As $P(x,y)$ is (p,q)-quasihomogeneous polynomial of degree $p + m - 1$ it follows that $P(0, y^q) = y^{p-1+m} P(0,1) = 0$, if $0 = \eta(0,1) = P(0,1)$. Therefore $x = 0$ is an invariant line of X.

Suppose there exists λ such that $\eta(1,\lambda)=0$. If $\lambda=0$, then $0=\eta(1,0)=pQ(1,0)$. So $Q(x^p,0)=x^{q-1+m}Q(1,0)=0$. Therefore $y=0$ is an invariant curve of X. Otherwise, let $V(x,y)=y^p-\lambda^p x^q$. Then

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{y=\lambda x^{q/p}} &= p\lambda^{p-1}x^{q(p-1)/p}Q(x,\lambda x^{q/p})-q\lambda^p x^{q-1}P(x,\lambda x^{q/p})\\ &= \lambda^{p-1}(x^{1/p})^{pq+m-1}(pQ(1,\lambda)-q\lambda P(1,\lambda))\\ &= \lambda^{p-1}(x^{1/p})^{pq+m-1}\eta(1,\lambda)=0,\end{aligned}$$

where $x^{q/p}:=(x^{1/p})^q$. This ends the proof of (b).

From [16] system (1) has a unique singular point at the origin. If there exists an invariant curve passing through the unique singular point of system (1), then no limit cycle can surround the origin and the statement (c) is proved. □

To study the singular point $(0,0)$ of (1), we introduce the *(p,q)-trigonometric functions $z(\phi)=\mathrm{Cs}\phi$ and $\omega(\phi)=\mathrm{Sn}\phi$* [15, 16] as the solution of the following initial problem

$$\dot z=-\omega^{2p-1},\ \dot\omega=z^{2q-1},\ z(0)=p^{-\frac{1}{2q}},\ \omega(0)=0.$$

It is know that $Cs\phi$ and $Sn\phi$ are $\mathcal{T}$-periodic functions with

$$\mathcal{T}=2p^{-\frac{1}{2q}}q^{-\frac{1}{2p}}\frac{\Gamma(\frac{1}{2p})\Gamma(\frac{1}{2q})}{\Gamma(\frac{1}{2p}+\frac{1}{2q})}.$$

and satisfies

$$p\mathrm{Cs}^{2q}\phi+q\mathrm{Sn}^{2p}\phi=1,\ \frac{d\mathrm{Cs}\phi}{d\phi}=-\mathrm{Sn}^{2p-1}\phi,\ \frac{d\mathrm{Sn}\phi}{d\phi}=\mathrm{Cs}^{2q-1}\phi.$$

For $(p,q)=(1,1)$, we have that $\mathrm{Cs}\phi=\cos\phi$, $\mathrm{Sn}\phi=\sin\phi$, i.e. the $(1,1)$-trigonometric functions are the classical ones.

In the *(p,q)-polar coordinates* (r,ϕ)

$$x=r^p\mathrm{Cs}\phi,\ \ y=r^q\mathrm{Sn}\phi, \tag{12}$$

the planar (p,q)-quasihomogeneous system (1) of degree m is written as

$$\dot r=r^mF(\phi),\ \ \dot\phi=r^{m-1}G(\phi),$$

with

$$F(\phi)=\xi\left(\mathrm{Cs}\phi,\mathrm{Sn}\phi\right),\quad G(\phi)=\eta\left(\mathrm{Cs}\phi,\mathrm{Sn}\phi\right),$$

where

$$\xi(x,y)=x^{2q-1}P(x,y)+y^{2p-1}Q(x,y), \tag{13}$$

and $\eta(x,y)$ is defined in (2).

Taking the change of coordinates

$$\frac{ds}{dt}=r^{m-1},\ \ \theta=\frac{2\pi\phi}{\mathcal{T}}, \tag{14}$$

the above system goes over to

$$r'=rf(\theta),\ \ \theta'=g(\theta). \tag{15}$$

where prime denotes derivative with respect to s,

$$f(\theta) = F\left(\frac{\mathcal{T}\theta}{2\pi}\right), \quad g(\theta) = \frac{2\pi}{\mathcal{T}} G\left(\frac{\mathcal{T}\theta}{2\pi}\right). \tag{16}$$

It is easy to check that $f(\theta)$ and $g(\theta)$ are 2π-periodic functions.

Now, we are able to prove Theorem 1 which discuss about the phase portrait of the vector fields in H_{pqm}, provided that $\eta(1, y)$ has no zero and $\eta(0, 1) \neq 0$.

Proof of Theorem 1. First we shall prove that $\eta(x, y) \neq 0$ for all $(x, y) \neq (0, 0)$.
From Lemma 9(a) if follows that if $x \neq 0$, then

$$\eta(x, y) = (x^{1/p})^{p+q-1+m} \eta\left(1, \frac{y}{x^{q/p}}\right), \tag{17}$$

As $\eta(1, y)$ has no zero, from (17), we get $\eta(x, y) \neq 0$, provided that $x \neq 0$.
In what follows we shall prove that $\eta(0, y) \neq 0$. Suppose it does not happen and there exists $y^* \neq 0$ such that $0 = \eta(0, y^*) = -qy^* P(0, y^*)$. So

$$0 = P(0, y^*) = (|y^*|^{1/q})^{p-1+m} P(0, \operatorname{sgn}(y^*)).$$

where

$$0 = P(0, \operatorname{sgn}(y^*)) = \begin{cases} P(0, 1), & \text{if } y^* > 0. \\ P(0, -1), & \text{if } y^* < 0. \end{cases}$$

If $y^* < 0$, then for any $y < 0$,

$$P(0, y) = (|y|^{1/q})^{p-1+m} P(0, -1) \equiv 0.$$

As $P(0, y)$ is a polynomial in the variable y and it is identically zero for $y < 0$, we have $P(0, y) \equiv 0$ and $P(0, 1) = 0$. Hence we conclude that if there exists $y^* \neq 0$ such that $\eta(0, y^*) = 0$, then $\eta(0, 1) = -qP(0, 1) = 0$. This is a contradition with the hypotheses of this theorem. Therefore, $\eta(0, y) \neq 0$ for $y \neq 0$ and one obtain $\eta(x, y) \neq 0$ for $(x, y) \neq (0, 0)$.
Secondly we shall prove the statements (a) and (b). If $\eta(x, y) \neq 0$, then $g(\theta) \neq 0$ (see (16)). It follows from (15) that

$$\frac{dr}{d\theta} = \frac{rf(\theta)}{g(\theta)},$$

and the first return map is given by

$$r(2\pi, r_0) = \begin{cases} r_0 e^{\tilde{I}_X}, & \text{if } f(\theta) \not\equiv 0, \\ r_0, & \text{if } f(\theta) \equiv 0, \end{cases}$$

where

$$\tilde{I}_X = \int_0^{2\pi} \frac{f(\theta)}{g(\theta)} d\theta,$$

and $(r_0, 0)$ is one point of the positive x-axis. From the first return map we deduce that if $\tilde{I}_X = 0$, then the origin is a center. If $\tilde{I}_X < 0$ (resp. $\tilde{I}_X > 0$), then it is a stable (resp. unstable) focus.
To end this proof we must show that the sign of I_X is equal to the sign of $\tilde{I}_X$.

Let $x = \mathrm{Cs}\phi,\ y = \mathrm{Sn}\phi$. Then

$$\tilde{I}_X = \frac{2\pi}{T}\int_0^T \frac{F(\phi)}{G(\phi)}d\phi = \frac{2\pi}{T}\oint_{px^{2q}+qy^{2p}=1} \frac{\xi(x,y)}{x^{2q-1}\eta(x,y)}dy = \frac{2\pi}{T}(\tilde{I}_+ - \tilde{I}_-),$$

where

$$\tilde{I}_\pm = \int_{-q^{-1/(2p)}}^{q^{-1/(2p)}} \frac{\xi(x,y)}{x^{2q-1}\eta(x,y)}\Big|_{x=x_\pm=\pm p^{-1/(2q)}(1-qy^{2p})^{1/(2q)}} dy.$$

If p is odd and $x \neq 0$, then

$$\xi(x,y) = (x^{1/p})^{2pq-1+m}\xi\left(1, \frac{y}{x^{q/p}}\right), \tag{18}$$

where $\xi(x,y)$ is defined in (13). It follows from (17) and (18) that

$$\tilde{I}_+ = \int_{-q^{-1/(2p)}}^{q^{-1/(2p)}} \frac{\xi(1,\frac{y}{x^{q/p}})}{x^{q/p}\eta(1,y/x^{q/p})}\Big|_{x=x_+} dy.$$

Let $x = x_\pm$ and $u = y/x^{q/p}$. Then $u^{2p} = y^{2p}/x^{2q}$ and $y^{2p} = u^{2p}/(p+qu^{2p})$. Hence

$$\begin{aligned}\tilde{I}_+ &= \int_{-\infty}^{\infty} \frac{\xi(1,u)}{x^{q/p}\eta(1,u)}\Big|_{x=x_+} \frac{1}{2py^{2p-1}}\frac{d}{du}\left(\frac{u^{2p}}{p+qu^{2p}}\right)du \\ &= \int_{-\infty}^{\infty} \frac{p\xi(1,u)}{(p+qu^{2p})\eta(1,u)}du.\end{aligned}$$

When p is odd, (17) and (18) are also true for $x = x_-$. Using the same arguments as above, we get $\tilde{I}_- = -\tilde{I}_+$ and hence $\tilde{I}_X = 2\tilde{I}_+$.

As

$$\int_{-\infty}^{\infty} \frac{p\xi(1,u)}{(p+qu^{2p})\eta(1,u)}du - \int_{-\infty}^{\infty} \frac{P(1,u)}{\eta(1,u)}du = \int_{-\infty}^{\infty} \frac{u^{2p-1}}{p+qu^{2p}}du = 0,$$

the statements (a) and (b) follows, provided that p is odd. □

Our next step is to consider the case where $\eta(1,y)$ has zeros or $\eta(0,1)$ is zero. To this we consider the study of the finite and infinity singularities of X.

To study with more details the singularities at infinity, we use *Poincaré-Lyapunov Compactification*, see for instance [9]. First we blow up the system (1) in the positive x-direction by

$$x = \frac{1}{z^p},\ y = \frac{u}{z^q}, d\tau = \frac{z^{1-m}dt}{p},\ z > 0.$$

This yields the vector field

$$\frac{dz}{d\tau} = -zP(1,u),\ \frac{du}{d\tau} = \eta(1,u). \tag{19}$$

Each points $(z,u) = (0,\lambda)$ satisfying $\eta(1,\lambda) = 0$ is an infinity singularity of system (1) and they have their linear part given by

$$\begin{pmatrix} -P(1,\lambda) & 0 \\ 0 & \frac{\partial\eta}{\partial u}(1,\lambda)\end{pmatrix}. \tag{20}$$

Moreover, if $P(1,\lambda) = 0$, then $0 = \eta(1,\lambda) = pQ(1,\lambda)$, so $P(x,\lambda x^{q/p}) = Q(x,\lambda x^{q/p}) = 0$ for $x > 0$. Hence the algebraic curves $P(x,y) = 0$ and $Q(x,y) = 0$ have infinite intersection points. As $P(x,y)$ and $Q(x,y)$ are coprime polynomials, it follows from Bézout theorem [10] that $P(x,y) = 0$ and $Q(x,y) = 0$, what is a contradiction with our hypothesis. Therefore, $P(1,\lambda) \neq 0$ if $\eta(1,\lambda) = 0$ but $\eta(1,y) \not\equiv 0$. Hence all singular points of system (19) are elementary.

Second we blow up the system (1) in the negative x-direction using the transformation

$$x = -\frac{1}{z^p},\quad y = \frac{u}{z^q}, d\tau = \frac{z^{1-m}dt}{p},\quad z > 0.$$

One gets

$$\frac{dz}{d\tau} = zP(-1,u),\quad \frac{du}{d\tau} = -\eta(-1,u). \tag{21}$$

All points $(0,\lambda)$ satisfying $\eta(-1,\lambda) = 0$ are singular points of system (21). Their linear part are given by

$$\begin{pmatrix} P(-1,\lambda) & 0 \\ 0 & -\frac{\partial\eta}{\partial u}(-1,\lambda) \end{pmatrix}. \tag{22}$$

The singular points of the system (21) are studied in the same way as the ones of system (19). Moreover all of them are elementary.

Finally we blow up in the y-direction by

$$x = \frac{v}{z^p},\quad y = \pm\frac{1}{z^q}, d\tau = \frac{z^{1-m}dt}{q},\quad z > 0.$$

which yields two vector fields of the forms

$$\bar{X}^\infty_\pm : \quad \frac{dz}{d\tau} = \mp zQ(v,\pm 1),\quad \frac{dv}{d\tau} = \mp\eta(v,\pm 1). \tag{23}$$

We only need to determine whether the origin is a singular point of the vector fields $\bar{X}^\infty_\pm$. As $\eta(0,1) = 0$ implies that $x = 0$ is an invariant line of X (see Proposition 10), one gets $P(0,y) \equiv 0$ and $\eta(0,-1) = qP(0,-1) = 0$. Using the same arguments as above, it is proved that $\eta(0,-1) = 0$ implies $\eta(0,1) = 0$. This yields that the origin is a singular point of system $\bar{X}^\infty_+$ if and only if it is a singular point of system $\bar{X}^\infty_-$. We also have $Q(0,\pm 1) \neq 0$ if $\eta(0,\pm 1) = 0$. Hence $(0,0)$ is elementary singular point of system $\bar{X}^\infty_\pm$ if $\eta(0,\pm 1) = 0$.

Lemma 11. *Suppose that p is odd.*

(a) *$(0,\lambda)$ is a singular points of system (19) if and only if $(0,(-1)^q\lambda)$ is a singular point of system (21). Suppose that λ is a zero of $\eta(1,\lambda)$ with multiplicity k, then*
   - (i) *If k is even, then $(0,\lambda)$ and $(0,(-1)^q\lambda)$ are saddle-node of system (19) and (21) respectively;*
   - (ii) *if k is odd and $P(1,\lambda)(\partial^k\eta/\partial u^k)(1,\lambda) > 0$, then $(0,\lambda)$ and $(0,(-1)^q\lambda)$ are saddles of system (19) and (21) respectively;*
   - (iii) *if k is odd and $P(1,\lambda)(\partial^k\eta/\partial u^k)(1,\lambda) < 0$, $P(1,\lambda) > 0$ (resp. $P(1,\lambda) < 0$), then $(0,\lambda)$ is a stable (resp. unstable) node of (19), and $(0,(-1)^q\lambda)$ is a unstable (resp. stable) node if m is even, a stable (resp. unstable) node of system (21) if m is odd.*

(b) *System* $\bar{X}_+^\infty$ *has a singular point at* $(0,0)$, *if and only if system* $\bar{X}_-^\infty$ *has a singular point at the origin, if and only if* $\eta(0,1) = 0$. *Moreover, suppose* $\eta(0,1) = 0$, *then there exists positive integers* k, l *such that* $m = (k-1)p + (l-1)q + 1$, *and*
   (i) *if* k *is even, then system* $\bar{X}_+^\infty$ *(resp.* $\bar{X}_-^\infty$*) has a saddle-node at the origin;*
   (ii) *if* k *is odd and* $Q(0,1)(\partial^k\eta/\partial x^k)(0,1) < 0$, *then system* $\bar{X}_+^\infty$ *(resp.* $\bar{X}_-^\infty$*) has a saddle at the origin;*
   (iii) *if* k *is odd and* $Q(0,1)(\partial^k\eta/\partial x^k)(0,1) > 0$, $Q(0,1) > 0$ *(resp.* $Q(0,1) < 0$*), then system* $\bar{X}_+^\infty$ *has a stable (resp. unstable) node at the origin, and* $\bar{X}_-^\infty$ *has a stable (resp. unstable) node at origin if* l *is odd, a unstable (resp. stable) node at origin if* l *is even, respectively.*

Proof. (a) Since p is odd, it follows from (17) that
$$\eta(-1,(-1)^q\lambda) = (-1)^{p+q-1+m}\eta(1,\lambda), \tag{24}$$
which implies that $\eta(1,\lambda) = 0$ if and only if $\eta(-1,(-1)^q\lambda) = 0$, and hence $(0,\lambda)$ is a singular point of system (19), if and only if $(0,(-1)^q\lambda)$ is a singular point of system (21). The equality (24) also yields
$$\frac{\partial^k\eta(-1,(-1)^q\lambda)}{\partial u^k} = (-1)^{(k-1)q+m}\frac{\partial^k\eta(1,\lambda)}{\partial u^k}. \tag{25}$$
Since $P(-1,(-1)^q\lambda) = (-1)^m P(1,\lambda)$, the statements (i),(ii) and (iii) with $k \geq 2$ in (a) follow from Theorem 2.19 of [9]. If $k = 1$,then all singular points are hyperbolic and the statement in (a) follows from Theorem 2.15 of the same book [9].

(b) Using the same arguments as in the case for system $\bar{X}_\pm$, we get that system $\bar{X}_+^\infty$ has a singular point at $(0,0)$, if and only if system $\bar{X}_-^\infty$ has a singular point at the origin, if and only if $\eta(0,1) = 0$.

Suppose that $\eta(x,y)$ has the form (4). If $\eta(0,1) = 0$, then there are k and l such that $pk + ql = p + q + m - 1$, $c_{kl} \neq 0$ and $c_{ij} = 0$ for $i \leq k-1$, where $k \geq 1$, $l \geq 1$. This gives $(\partial^k\eta/\partial x^k)(0,\pm 1) = k!c_{kl}(\pm 1)^l$.

If $\eta(0,1) = 0$, then $P(0,1) = 0$, which implies that $P(x,y)$ has a divisor x. On the other hand, $Q(0,1) = 0$ also means that $Q(x,y)$ has a divisor x. Since $P(x,y)$ and $Q(x,y)$ are coprime, we have that $Q(0,1) \neq 0$ if $\eta(0,1) = 0$. The other statements in (b) follows by the same arguments as in the proof of (a). □

Before ending this part some comments are necessary. If p is odd, then it follows from (24) that $(0,\lambda)$ is a singular point of system (19), if and only if, $(0,(-1)^q\lambda)$ is a singular point of system (21). In fact, the information found in the positive x-direction also covers the one in the negative x-direction. However because of the proof of Lemma 11 is convenient to study both, the positive and the negative x-direction.

Now, using similar arguments and local charts we have the study of the finite singular points of the system (1). It is important to observe that using (p,q)-polar coordinates changes one gets the vector field (15) which is defined on $\mathbb{S}^1 \times \mathbb{R}$. Although the cylinder $\mathbb{S}^1 \times \mathbb{R}$ is good surface for getting the phase portrait near the origin, it is often less appropriate for making calculations, since we have to deal with expressions of (p,q)-trigonometric functions. Hence we prefer to make the calculations in different charts.

We are going to use the method of quasihomogeneous blow-up (see for instance [9]) in local charts to study the singular point $(0,0)$ of system (1). We first blow up the vector field in the positive x-direction by

$$x = \bar{x}^p, \quad y = \bar{x}^q \bar{y}, \quad d\tau = \frac{\bar{x}^{m-1} dt}{p}, \quad \bar{x} > 0, \tag{26}$$

yielding

$$\dot{\bar{x}} = \bar{x} P(1, \bar{y}), \quad \dot{\bar{y}} = \eta(1, \bar{y}). \tag{27}$$

The points $(0, \lambda)$ satisfying $\eta(1, \lambda) = 0$ are the isolated singular points of (27) on the line $\{\bar{x} = 0\}$. The system in these singular points has their linear part given by

$$\begin{pmatrix} P(1,\lambda) & 0 \\ 0 & \frac{\partial \eta}{\partial \bar{y}}(1,\lambda) \end{pmatrix}.$$

Since we have shown $P(1, \lambda) \neq 0$, all singular points of system (27) are elementary.

Next we blow up the vector field in the negative x-direction, the positive y-direction and the negative y-direction, respectively. Then we obtain three systems whose singular points are elementary. Here we omit the details.

After blown up the origin we obtain elementary singular points so the singular point $(0,0)$ of (1) has been desingularized. After blowing down we get the phase portrait of system (1) near the origin.

From the later considerations we have that the finite and infinite singularities of X are determined by the zeros of $\eta(1, y)$ or $\eta(0, 1)$. The following proposition guarantee that the invariant curves of the system (1) determine the phase portraits of system (1).

Proposition 12. *Suppose that $(1, \lambda_i), (1, \lambda_{i+1})$ are two consecutive infinity singularities of system (1), which correspond to the invariant curves L_{λ_i} and $L_{\lambda_{i+1}}$ of the Proposition 10, then these invariant curves are characteristic orbits at $(0,0)$ and they determine one sector, more precisely:*

(a) *if both are singularities of the node type at infinity then they define a hyperbolic sector at $(0,0)$,*
(b) *if both are singularities of the saddle type at infinity then they define a elliptic sector at $(0,0)$, and*
(c) *if both are hyperbolic and non equivalent singularities at infinity then they define a parabolic sector at $(0,0)$.*

Proof. It follows from the study of the infinity and finite singular points of the system X that they are determined by the zeros of η and each singular point is elementary. Moreover, from the Lemma 11 if $(1, \lambda_i)$ is a zero of η then there is an invariant curve which leaves (or going to) an infinite singular point and goes to (or leaves) a finite singular point. So each pair of zeroes defines a local sector in the disc. These sectors are hyperbolic, parabolic or elliptic sectors because the singular points are elementary. □

It is important to comment that Proposition 10, Lemma 11 and Proposition 12 are fundamental to describe the different phase portrait of the system (1). They will be very important tools in Section 4 to calculate the number of topological equivalence classes in Ω_{pqm}. In fact, the phase portrait of system (1) is determined by the zeros of equations $\eta(1,\lambda)=0$ and if $\eta(0,1)=0$. Moreover, from the later proposition follows that,to describe the phase portrait of the system (1), it is enough to know the sign of the product $P(1,\lambda).\dfrac{\partial\eta}{\partial y}(1,\lambda)$ for each zero of $\eta(1,\lambda)$ and the sign of $P(0,1).\dfrac{\partial\eta}{\partial y}(0,1)$, if $\eta(0,1)=0$, .

3. Structural Stability

In this section we shall prove the Theorem 2 which characterise the set Ω_{pqm} of the (p,q)-quasihomogeneous vector fields of degree m in the plane which are structurally stable with respect to perturbations in H_{pqm}. As observed in the introduction we denote by $E(X)$ the induced (or extended) vector field on Poincaré-Lyapunov sphere and the coefficient topology in H_{pqm}. Then we shall say that two vector fields X and Y in H_{pqm} are equivalent it there exists an equivalence h between the induced vector fields $E(X)$ and $E(Y)$ on the Poincaré-Lyapunov sphere. In our case is redundant to required that the equivalence be near of the identity map because the Poincaré-Lyapunov sphere is a manifold in the Peixoto condictions and the coefficient topology is equivalent to the C^1 topology.

So we can start the prove of the Theorem

Proof of Theorem 2. Assume that $X\in H_{pqm}$ is structurally stable with respect to perturbations in H_{pqm}. Let us prove that (a) and (b) happen. If $\eta(1,y)$ has no zero and $\eta(0,1)\neq 0$, then it follows from Theorem 1 that the origin of the system (1) is a global center if $I_X=0$, and a global (stable or unstable) focus if $I_X\neq 0$. If $I_X=0$ there exists a vector field Y in any neighborhood of X in H_{pqm} such that the phase portrait of Y is a global focus. Therefore X is not a structural stable vector field with respect to perturbations in H_{pqm}. This yields that if $X\in H_{pqm}$, then $I_X\neq 0$, i.e., the condition (a) is satisfied.

Moreover, the origin is the unique singular point of system (1)[16], and the phase portrait of X is completely determined by the zeros of $\eta(1,y)$ and $\eta(0,1)$ (see Lemma 11 and Proposition 12). If $\eta(1,y)$ has a zero or $\eta(0,1)=0$, then it follows from Proposition 10(c) that there is no periodic orbits in the set of all solutions of the system (1).

If $\eta(1,y)$ has a multiple zero then, by Lemma 11, we conclude that there exist vector field Y in a neighborhood of X in H_{pqm} such that X and Y have different number of sectors. So X is not structural stable with respect to H_{pqm}. If $\eta(0,1)=0$ and $\partial\eta(0,1)/\partial x=0$, then by the same arguments we get that X is not structural stable with respect to H_{pqm}. Therefore, the condition (b) is satisfied.

Now, we assume that (a) or (b) are satisfied and we shall prove that X is a structurally stable vector field with respect to perturbations in H_{pqm}. First we suppose that (a) is true. Then there exists a neighborhood V of Y and a neighborhood W of X in H_{pqm} such that if $y^*\in V$ then $\eta(1,y^*)\neq 0$ and for all $Y\in W$ we have $sign(I_X)=sign(I_Y)$. Then it follows from Theorem 1 that X and Y are topologically equivalent and X is structurally stable with respect to perturbations in H_{pqm}.

Assume that there are s points $(1,y_i)$ such that $\eta(1,y_i)=0$ or $\eta(0,1)=0$ and that these zeros are simple. There exists a neighborhood U of X in H_{pqm} such that if $Y\in U$ then there are exactly s points such that $\eta(1,\lambda_i)=0$ or $\eta(0,1)=0$. We can choose this neighborhood such that $sign(P_X \partial\eta_X/\partial u)(1,\lambda_i) = sign(P_Y \partial\eta_Y/\partial u)(1,\lambda_i))$, and if $\eta(0,1) = 0$ then $sign(Q_X \partial\eta_X/\partial x)(0,1) = sign(Q_Y(1,\lambda)\partial\eta_Y/\partial x)(0,1)$. By Lemma 11 and Proposition 12 each vector field $Y \in U$ has the same local sectors as the vector field Y. So they are topologically equivalent and we conclude that X is structurally stable with respect to perturbations in H_{pqm}. □

Here we shall present a concrete example.

Example 13. *Let $p = 1$, $q = m = 2$, then H_{122} is non empty. In fact if $X \in H_{122}$ then X can be written in the form $X(x,y) = (a_1x^2 + a_2y, a_3x^3 + a_4xy)$ with $(a_1,a_2,a_3,a_4) \in \mathbb{R}^4\backslash(0,0,0,0)$. The question is, what are the conditions to X to be structurally stable? By the direct computation we have $\eta(x,y) = a_3x^4 + (a_4 - 2a_1)x^2y - 2a_2y^2$. It follows from Theorem 2 that $X \in \Omega_{122}$ if and only if the following conditions holds: (i) $a_2 \neq 0$, $4a_1^2 + a_4^2 + 4(a_1a_4 + 2a_2a_3) < 0$, $I_X \neq 0$, (ii) $a_2 \neq 0$ and $4a_1^2 + a_4^2 + 4(a_1a_4 + 2a_2a_3) > 0$.*

So the vector field $X_1(x,y) = (x^2 - y/2, x^3 + 2xy)$ and $X_2(x,y) = (x^2 - y, 2x^3 - 3xy)$ are structurally stable vector fields. They are non equivalents and their phase portrait are given in Figure 1. The vector field X_1 has a global unstable focus at the origin. On the other hand, X_2 has four singular points at infinity, two stable and two unstable nodes, therefore, we observe four hyperbolic sectors at origin.

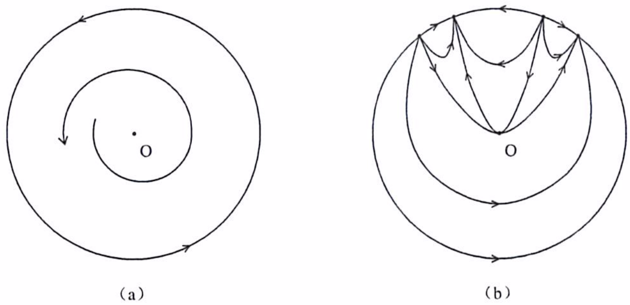

FIGURE 1. Phase portraits of X_1 and X_2 in the Poincaré-Lyapunov disc

4. THE NUMBER OF TOPOLOGICAL EQUIVALENCE CLASSES IN Ω_{pqm}

In this section we compute the number of topological equivalence classes in Ω_{pqm}.

Proof of Proposition 3. By Lemma 9 $\eta(x,y)$ is a (p,q)-quasihomogeneous polynomial of degree $p+q+m-1$ having the form (4). Then (i,j) in (4) is a solution of the equation

$$pi + qj = p + q + m - 1. \tag{28}$$

The pair (i,j) is called an integer (resp. non-negative integer) solution of (28) if i and j are integers (resp. non-negative integers). If (i_0,j_0) is an integer solution of (28), then all the other integer solution are

$$\{(i_0 + lq,\ j_0 - lp) : l \in \mathbb{Z}\}.$$

If $\eta(0,1)\neq 0$, $\eta(1,0)\neq 0$, then the equation (28) has two integer solutions $(0,j')$ and $(i',0)$ satisfying $qj'=p+q+m-1$ and $pi'=p+q+m-1$ respectively. Since $(p,q)=1$, we have $pq|(p+q+m-1)$. Hence there exists r such that $p+q+m-1=(r+1)pq$. Let $(i_0,j_0)=(0,j')=(0,(p+q+m-1)/q)=(0,(r+1)p)$. Then all non-negative integer solution of (28) are $\{(lq,(r+1-l)p): l=0,1,2,\cdots,r+1\}$, which yields $\eta(x,y)$ has the form, defined in (5). On the other hand, if $\eta(0,1)\neq 0$, $\eta(1,0)\neq 0$, then $P(0,1)\neq 0$, $Q(1,0)\neq 0$. This gives that $q|(p+m-1)$ and $p|(q+m-1)$. As $(p+m-1)/q=(p+q+m-1)/q-1=(r+1)p-1$ and $(q+m-1)/p=(p+q+m-1)/p-1=(r+1)q-1$, we get that $P(x,y)$ and $Q(x,y)$ has the form, defined in (5), by the same arguments as above.

Suppose that $\eta(0,1)\neq 0$, $\eta(1,0)=0$. As $X\in\Omega_{pqm}$, it follows from Theorem 2 that $y=0$ is a simple zero of $\eta(1,y)$. By the assumption, the equation (28) has two integer solutions $(0,j')$, $(i',1)$ respectively and $c_{0,j'}\neq 0$, $c_{i',1}\neq 0$, which implies that $q|p+m-1$ and $p|m-1$. Let $m-1=k'p$. Then $p+m-1=(k'+1)p$. Since $(p,q)=1$ we have $q|(k'+1)$. Hence there exists r such that $p+m-1=rpq$. All non-negative integer solution of (28) are $\{lq,(r-l)p+1: l=0,1,2,\cdots,r\}$, which yields that $\eta(x,y)$ has the form in (6). On the other hand, if $\eta(0,1)\neq 0$, $\eta(1,0)=0$, then $P(0,1)\neq 0$, $Q(1,0)=0$. We get $P(x,y)$ and $Q(x,y)$, defined in (6), by the same arguments as above.

If $\eta(0,1)=0$ and $\eta(1,0)\neq 0$, then (28) has two integer solutions $(1,j')$, $(i',0)$ and $x=0$ is a simple zero of $\eta(x,1)$. Using the same arguments as above, we get (c).

If $\eta(0,1)=0$ and $\eta(1,0)=0$, then (28) has two integer solutions $(1,j')$, $(i',1)$ and $x=0$ and $y=0$ are simple zeros of $\eta(x,1)$ and $\eta(1,y)$ respectively. Using the same arguments as above, one gets (d). □

Proof of Lemma 4. For the first part of this lemma, we only prove the case (a.1). Other results are proved by the same arguments.

If $(p,q,m)\in\Theta_1\cap\Theta_4$, then

$$p+q+m-1=(r_1+1)pq,\quad m-1=(r_4-1)pq.$$

Eliminating m from the above equations, we get

$$p+q=(r_1-r_4+2)pq, \tag{29}$$

which implies $p|q$. Since we suppose $(p,q)=1$, one obtains $p=1$. Substituting $p=1$ into (29), we have $1=(r_1-r_4+1)q$. Therefore $q=1$ and $r_1=r_4$.

The second part of this lemma is proved by direct computations. □

Proposition 14. *Let p, q, m, r be the integers defined in Proposition 3.*

(a) *Suppose that p and q are odd, then m is odd (resp. even) if and only if r is odd (resp. even),*

(b) *Suppose that p is odd and q is even.*
 (i) *If $(p,q,m)\in\Theta_1\cup\Theta_2$, then m is even,*
 (ii) *If $(p,q,m)\in\Theta_3\cup\Theta_4$, then m is odd.*

Proof. Suppose $(p,q,m)\in\Theta_1$, then $p+q+m-1=(r+1)pq$. If p and q are odd, then $p+q-1$ is odd. As $m=(r+1)pq-(p+q-1)$, we have that m is odd (resp. even) if and only if r is odd (resp. even). If p is odd and q is even, then $p-1$ is even, which implies that $m=(r+1)pq-q-(p-1)$ is even.

The other statements can be proved by the same arguments. □

Proposition 15. *If $X \in \Omega_{pqm}$, then there exists a number k such that X has $2k$ singular points at infinity with $k \leq r+1$ and $k \equiv r+1$ (mod 2), where r is defined in Proposition 3.*

Proof. Consider the quasihomogeneous polynomial $\eta(x, y)$, defined in (4). Since p is odd, $\eta(1, y)$ has at most $r+1$ real zeros if $\eta(0,1) \neq 0$ and r real zeros if $\eta(0,1) = 0$. On the other hand, $x = 0$ is a simple zero of $\eta(x, 1)$ if $\eta(0,1) = 0$. The statement follows from Lemma 11. □

Let

$$\Omega_{pqm}^{2k} = \{X \in \Omega_{pqm} : E(X) \text{ has } 2k \text{ singular point at infinity}\}.$$

The following corollary follows from Proposition 15:

Corollary 16. *Suppose that r is as defined in Proposition 3, then $\Omega_{pqm} = \bigcup_{k \in J_{m,r}} \Omega_{pqm}^{2k}$, where*

$$J_{m,r} = \left\{k = 2j : 0 \leq j \leq \frac{r+1}{2}\right\}.$$

if r is odd, and

$$J_{m,r} = \left\{k = 2j+1 : 0 \leq j \leq \frac{r}{2}\right\}.$$

if r is even, respectively.

If X and Y are two topological equivalent vector fields in Ω_{pqm}, then they have the same number of singular points at infinity. Let C_{pqm}^k be the number of topological equivalence classes in Ω_{pqm}^{2k}. If $k = 0$, then by Theorem 1 $C_{pqm}^0 = 2$ (a global stable focus and a global unstable focus). It follows from Corollary 16 that

$$C_{pqm} = \sum_{k \in J_{m,r}} C_{pqm}^k. \tag{30}$$

To convenience, we call $\lambda = +\infty$ (resp. $\lambda = -\infty$) a simple zero of $\eta(1, u)$ (resp. $\eta(-1, u)$) if $\eta(0,1) = 0$ (resp. $\eta(0,-1) = 0$) and $\partial\eta(0,1)/\partial x \neq 0$. Define $\text{sgn}(\partial\eta(\pm1, \pm\infty)/\partial u) = -\text{sgn}(\partial\eta(0, \pm1)/\partial x)$ if $\eta(0, \pm1) = 0$. In what follows we suppose that $\lambda_1, \lambda_2, \cdots, \lambda_k$ are zeros of $\eta(1, u)$ and $\lambda_{k+1}, \lambda_{k+2}, \cdots, \lambda_{2k}$ are zeros of $\eta(-1, u)$ with $-\infty < \lambda_1 < \lambda_2 < \cdots < \lambda_k$ and $+\infty > \lambda_{k+1} > \lambda_{k+2} > \cdots > \lambda_{2k}$ respectively, provided $X = (P, Q) \in \Omega_{pqm}^{2k}$.

Proposition 17. *Let $X = (P, Q) \in \Omega_{pqm}^{2k}$, Then*

(a) $(\partial\eta(1, \lambda_i)/\partial u)(\partial\eta(1, \lambda_{i+1})/\partial u) < 0$, $i = 1, 2, \cdots, k-1$, $k \geq 2$;
(b) $(\partial\eta(-1, \lambda_i)/\partial u)(\partial\eta(-1, \lambda_{i+1})/\partial u) < 0$ *for* $i = k+1, \cdots, 2k-1$, $k \geq 2$;
(c) $P(1, \lambda_i) \neq 0$, $i = 1, 2, \cdots, k$, *if* $\lambda_k \neq +\infty$, *and* $P(-1, \lambda_i) \neq 0$, $i = k+1, \cdots, 2k$, *if* $\lambda_{2k} \neq -\infty$;
(d) $Q(0, \pm1) \neq 0$ *if* $\eta(0, \pm1) = 0$; *and*
(e) $(\partial\eta(1, \lambda_k)/\partial u)(\partial\eta(-1, \lambda_{k+1})/\partial u) > 0$, $(\partial\eta(1, \lambda_1)/\partial u)(\partial\eta(-1, \lambda_{2k})/\partial u) > 0$.

Proof. As $X \in \Omega_{pqm}$, all zeros of $\eta(\pm 1, u)$ are simple. The statements (a) and (b) follow if $\lambda_k \neq +\infty$ and $\lambda_{2k} \neq -\infty$.

Suppose $\lambda_k = +\infty$, which yields $\eta(0,1) = 0$ and $\eta(1,u) \neq 0$ for $u > \lambda_{k-1}$. Proposition 3 shows that $\eta(x,y)$ has the form as (7) or (8). If $\partial\eta(1,\lambda_{k-1})/\partial u > 0$, then $\eta(1,u) > 0$, $u \in (\lambda_{k-1}, +\infty)$. This implies either $c_{1,rp} > 0$ in (7) or $c_{1,(r-1)p+1} > 0$ in (8). Since either $\partial\eta(0,1)/\partial x = c_{1,rp}$ or $\partial\eta(0,1)/\partial x = c_{1,(r-1)p+1}$ happens and $\mathrm{sgn}(\partial\eta(\pm 1, \pm\infty)/\partial u) = -\mathrm{sgn}(\partial\eta(0,\pm 1)/\partial x)$, one gets (a) for $i = k$ if $\partial\eta(1,\lambda_{k-1})/\partial u > 0$. By the same arguments we get $\partial\eta(1,\lambda_k)/\partial u > 0$ if $\lambda_k = +\infty$, $\partial\eta(1,\lambda_{k-1})/\partial u < 0$. This proves (a).

If $\lambda_{2k} = -\infty$, then $\eta(-1,u) \neq 0$ for $u \in (-\infty, \lambda_{2k-1})$. Repeating the same arguments as the proof of (a), one gets (b).

The statement (c) and (d) have been proved in Section 2.

Finally we prove (e). Suppose that $\eta(x,y)$ has the form (5). From the definition of λ_k and λ_{k+1} we have $\eta(1,u) \neq 0$ in the interval $(\lambda_k, +\infty)$ and $\eta(-1,u) \neq 0$ in the interval $(\lambda_{k+1}, +\infty)$ respectively. If $\eta(x,y)$ has the form (5), then

$$\eta(\pm 1, u) = c_{0,(r+1)p} u^{(r+1)p} + (\pm 1) c_{1,rp} u^{rp} + \cdots + c_{rq,p} (\pm 1)^{rq} u^p + c_{(r+1)q,0} (\pm 1)^{(r+1)q},$$

which gives $\mathrm{sgn}\eta(1,u) = \mathrm{sgn}\eta(-1,u)$ as $u \to +\infty$. As λ_k and λ_{k+1} are simple zeros of $\eta(1,u)$ and $\eta(-1,u)$ respectively, one gets $(\partial\eta(1,\lambda_k)/\partial u)(\partial\eta(-1,\lambda_{k+1})/\partial u) > 0$. On the other hand, $\mathrm{sgn}\eta(1,u) = \mathrm{sgn}\eta(-1,u)$ as $u \to -\infty$, which implies $(\partial\eta(1,\lambda_1)/\partial u)(\partial\eta(-1,\lambda_{2k})/\partial u) > 0$.

Suppose that $\eta(x,y)$ has the form (7), then $\lambda_k = +\infty$, $\lambda_{2k} = -\infty$, and

$$\eta(x, \pm 1) = x(c_{1,rp}(\pm 1)^{rp} + c_{1+q,(r-1)p}(\pm 1)^{(r-1)p} x^q + \cdots + c_{1+rq,0} x^{rq}),$$

which yields $\mathrm{sgn}(\partial\eta(\pm 1, \pm\infty)/\partial u) = -\mathrm{sgn}(\partial\eta(0,\pm 1)/\partial x) = -\mathrm{sgn}(c_{1,rp}(\pm 1)^{rp})$. As

$$\eta(\pm 1, u) = (\pm 1)(c_{1,rp} u^{rp} + c_{1+q,(r-1)p}(\pm 1)^q u^{(r-1)p} + \cdots + c_{1+rq,0}(\pm 1)^{rq}),$$

$\mathrm{sgn}(\eta(-1,u)) = -\mathrm{sgn}(c_{1,rp})$ for $u \in (\lambda_{k+1}, +\infty)$ and $\mathrm{sgn}(\eta(1,u)) = \mathrm{sgn}(c_{1,rp}(-1)^{rp})$ for $u \in (-\infty, \lambda_1)$ respectively, which shows that $\mathrm{sgn}(\partial\eta(-1,\lambda_{k+1})/\partial u) = -\mathrm{sgn}(c_{1,rp})$ and $\mathrm{sgn}(\partial\eta(1,\lambda_1)/\partial u) = -\mathrm{sgn}(c_{1,rp}(-1)^{rp})$. The statement (e) follows.

If $\eta(x,y)$ is defined as (6) or (8),then one gets (e) by the same arguments. □

Let S be the set of all sequence $(\sigma,\nu) = \{(\sigma_i,\nu_i)\}_{i\in\mathbb{Z}}$ such that $\sigma_i, \nu_i \in \{-1,1\}$ and $\sigma_i\sigma_{i+1} < 0$ for all $i \in \mathbb{Z}$. For each $k \in \mathbb{N}$ we denote

$$S^{2k} = \{(\sigma,\nu) \in S : (\sigma_i,\nu_i) = (\sigma_{i+2k}, \nu_{i+2k}), \text{ for all } i \in \mathbb{Z}\}.$$

A sequence (σ,ν) is *periodic of period l* (or *l-periodic*) if l is the smallest natural number such that $(\sigma_i,\nu_i) = (\sigma_{i+l}, \nu_{i+l})$ for all $i \in \mathbb{Z}$. Obviously each sequence (σ,ν) in S^{2k} is periodic and is completely determined if the elements (σ_i,ν_i) are given for $i = 1, 2, \cdots, l$. As $\sigma_i\sigma_{i+1} = -1$, the period l is an even divisor of $2k$.

The above notations have been used in the study of the homogeneous vector fields in [17].

From Proposition 17 we can associate a sequence of S^{2k} to each vector field $X \in \Omega^{2k}_{pqm}$ by taking

$$(\sigma_i, \nu_i) = \begin{cases} \left(\operatorname{sgn}\left(\frac{\partial\eta(1,\lambda_i)}{\partial u}\right), \operatorname{sgn}(-P(1,\lambda_i))\right), & i = 1, 2, \cdots, k-1; \\ \left(\operatorname{sgn}\left(\frac{\partial\eta(1,\lambda_i)}{\partial u}\right), \operatorname{sgn}(-P(1,\lambda_i))\right), & \text{if } i = k,\ \lambda_k \neq +\infty; \\ \left(\operatorname{sgn}\left(\frac{-\partial\eta(0,1)}{\partial v}\right), \operatorname{sgn}(-Q(0,1))\right), & \text{if } i = k,\ \lambda_k = +\infty; \\ \left(\operatorname{sgn}\left(\frac{-\partial\eta(-1,\lambda_i)}{\partial u}\right), \operatorname{sgn}(P(-1,\lambda_i))\right), & i = k+1, \cdots, 2k-1; \\ \left(\operatorname{sgn}\left(\frac{-\partial\eta(-1,\lambda_i)}{\partial u}\right), \operatorname{sgn}(P(-1,\lambda_i))\right), & \text{if } i = 2k,\ \lambda_{2k} \neq -\infty; \\ \left(\operatorname{sgn}\left(\frac{\partial\eta(0,-1)}{\partial v}\right), \operatorname{sgn}(Q(0,-1))\right), & \text{if } i = 2k,\ \lambda_{2k} = -\infty, \end{cases} \tag{31}$$

where $\eta(v, u)$ is defined by (2).

Proposition 18. *Let $X = (P, Q) \in \Omega^{2k}_{pqm}$ with $0 \neq k \in J_{m,r}$ and let (σ, ν) be the sequence associated to X according to (31).*

(a) *Suppose that both p and q are odd, then $(\sigma_{i+k}, \nu_{i+k}) = (-1)^{m-1}(\sigma_i, \nu_i)$ for $i = 1, 2, \cdots, k$;*

(b) *Suppose that p is odd and q is even.*
   (i) *If $\eta(0,1) \neq 0$, then m is even and $(\sigma_{2k-i+1}, \nu_{2k-i+1}) = -(\sigma_i, \nu_i)$ for $i = 1, 2, \cdots, k$.*
   (ii) *If $\eta(0,1) = 0$, then m is odd, $(\sigma_{2k-i}, \nu_{2k-i}) = (\sigma_i, \nu_i)$ for $i = 1, 2, \cdots, k-1$, and $(\sigma_{2k}, \nu_{2k}) = (-1)^{r-1}(\sigma_k, \nu_k)$, where r is defined in (c) or (d) of Proposition 3.*

Proof. It follows from Lemma 11 (or (24)) that if $\eta(1, \lambda_i) = 0,\ i = 1, 2, \cdots, k$ and $\lambda_k \neq +\infty$, then $\eta(-1, (-1)^q\lambda_i) = 0$.

(a) Suppose that both p and q are odd. As $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, we have $(-1)^q\lambda_1 > (-1)^q\lambda_2 > \cdots > (-1)^q\lambda_k$, which implies $\lambda_{i+k} = (-1)^q\lambda_i,\ i = 1, 2, \cdots, k$. As $P(-1, \lambda_{i+k}) = P((-1)^p, (-1)^q\lambda_i) = (-1)^m P(1, \lambda_i)$, we get $(\sigma_{i+k}, \nu_{i+k}) = (-1)^{m-1}(\sigma_i, \nu_i)$ from (25) for $i = 1, 2, \cdots, k-1$, and for $i = k$ if $\lambda_k \neq +\infty$.

If $\lambda_k = +\infty$, then it follows from Lemma 11 $\eta(0, 1) = \eta(0, -1) = 0$, which implies that $\eta(x, y)$ has the form as (7) or (8). If $\eta(x, y)$ is defined in (7), then $\partial\eta(0, \pm1)/\partial x = c_{1,rp}(\pm1)^{rp} = c_{1,rp}(\pm1)^r$ and $q + m - 1 = rpq$. By Proposition 14 $\partial\eta(0, \pm1)/\partial x = c_{1,rp}(\pm1)^m$. As $Q(0, -1) = (-1)^{q-1+m}Q(0, 1) = (-1)^m Q(0, 1)$, one gets $(\sigma_{2k}, \nu_{2k}) = (-1)^{m-1}(\sigma_k, \nu_k)$.

If $\eta(x, y)$ is defined in (8), then we get $(\sigma_{2k}, \nu_{2k}) = (-1)^{m-1}(\sigma_k, \nu_k)$ by the same arguments.

(b) Suppose that p is odd and q is even. If $\eta(0, 1) \neq 0$, then $\lambda_{2k-i+1} = \lambda_i$. By the same arguments we get $(\sigma_{2k-i+1}, \nu_{2k-i+1}) = (-1)^{m-1}(\sigma_i, \nu_i)$ for $i = 1, 2, \cdots, k$. On the other hand, if $\eta(0, 1) \neq 0$ it follows, from Proposition 3 and Proposition 14(b)(i), that m is even. This proves (i).

If $\eta(0, 1) = 0$ and q is even, then it follows from Proposition 3 and Proposition 14(b)(ii) that m is odd and $\lambda_k = -\lambda_{2k} = +\infty$. By the same arguments as (a), we have $\lambda_{2k-i} = \lambda_i$ and hence $(\sigma_{2k-i}, \nu_{2k-i}) = (\sigma_i, \nu_i)$ for $i = 1, 2, \cdots, k-1$.

If $\eta(x,y)$ is defined in (7), then it follows from (7) that $\partial\eta(0,\pm1)/\partial v = c_{1,rp}(-1)^r$, $Q(0,\pm1) = b_{0,rp}(\pm1)^r$, which gives $(\sigma_{2k},\nu_{2k}) = (-1)^{r-1}(\sigma_k,\nu_k)$. If $\eta(x,y)$ has the form (8), then we get (ii) by the same arguments. □

Following the idea in [17], we say that (σ,ν) is *m-admissible* if there exists $X \in \Omega^{2k}_{pqm}$ satisfying (31) for each pair (p,q). Denote by S^{2k}_m the set of all sequence in S^{2k} that are m-admissible.

The next two proposition characterises the sequences in S^{2k} that are m-admissible. It follows from Proposition 3 that, if $X \in \Omega_{pqm}$, then it is only necessary to consider the numbers p, q, m and r that satisfy one of the equations in (9).

Proposition 19. *Suppose that r is defined as in Proposition 3. Denote by s the number of changes of sign in the sequence $\{\nu_1,\nu_2,\cdots,\nu_k\}$. Let $0 \neq k \in J_{m,r}$ and $(\sigma,\nu) = \{(\sigma_i,\nu_i)\}_{i\in\mathbb{Z}}$ verifying*

(i) $(\sigma_{i+k},\nu_{i+k}) = (-1)^{m-1}(\sigma_i,\nu_i)$ *if both p and q are odd,*
(ii) $(\sigma_{2k-i+1},\nu_{2k-i+1}) = -(\sigma_i,\nu_i)$ *if p is odd and q, m are even,*
(iii) $(\sigma_{2k-i},\nu_{2k-i}) = (\sigma_i,\nu_i)$ *for $i = 1,2,\cdots,k-1$ and $(\sigma_{2k},\nu_{2k}) = (-1)^{r-1}(\sigma_k,\upsilon_k)$ if p, m are odd and q is even.*

Then

(a) *If either $k < r+1$ or $k = r+1$, $s < r$, then (σ,ν) is m-admissible.*
(b) *If $k = r+1$, $s = r$, then (σ,ν) is m-admissible if and only if there exists $j \in \mathbb{Z}$ such that $\sigma_j = \nu_j$.*

Proof. (a) To prove the statement (a) it is sufficient to find $X = (P(x,y),Q(x,y))$ such that $X \in \Omega^{2k}_{pqm}$ and (31) is fulfilled for $i = 1,2,\cdots,k$.

We note that $k < r+1$ implies $s < r$.

Case 1. $(p,q,m) \in \Theta_1$.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ be the finite real numbers with $\lambda_j \neq 0$, $j = 1,2,\cdots,k$ and

$$\eta(x,y) = a(x^{2q}+y^{2p})^{(r+1-k)/2}\prod_{j=1}^{k}(y^p - \lambda_j^p x^q)$$

with $a \in \mathbb{R}$, $a \neq 0$. Since $k \in J_{m,r}$, it follows from Proposition 15 that $(r+1-k)/2 \in \mathbb{N}$, and hence $\eta(x,y)$ is a (p,q) quasihomogeneous polynomial of weight degree $p+q+m-1 = (r+1)pq$, which implies that $\eta(1,u)$ has simple real zeros λ_1, λ_2, $\cdots$, λ_k. Furthermore,

$$\frac{\partial\eta(1,\lambda_i)}{\partial u} = ap\lambda_i^{p-1}(1+\lambda_i^{2p})^{(r+1-k)/2}\prod_{j=1,j\neq i}^{k}(\lambda_i^p - \lambda_j^p) \neq 0, \quad i = 0,1,\cdots,k.$$

If we choose $a = \pm1$ in such way that $\mathrm{sgn}\partial\eta(1,\lambda_1)/\partial u = \sigma_1$, then $\eta(x,y)$ satisfies $\mathrm{sgn}\partial\eta(1,\lambda_i)/\partial u = \sigma_i$, $i = 1,2,\cdots,k$.

To determine $P(x,y)$ we choose $\mu_i \in (\lambda_j,\lambda_{j+1})$ if $\nu_j \neq \nu_{j+1}$. So we obtain s real numbers $\mu_1 < \mu_2 < \cdots < \mu_s$ where s is the number of changes of sign in the sequence $\{\nu_1,\nu_2,\cdots,\nu_k\}$. Let

$$h(x,y) = \prod_{i=1}^{s}(y^p - \mu_i^p x^q).$$

If $\mathrm{sgn}(ah(1,\lambda_1)) = -\nu_1$, then define

$$\alpha(x,y) = \begin{cases} (x^{2q}+y^{2p})^{(r-s-1)/2}(y^p + (1-\lambda_1^p)x^q), & \text{if } r-s \text{ is odd,} \\ (x^{2q}+y^{2p})^{(r-s)/2}, & \text{if } r-s \text{ is even,} \end{cases}$$

If $\mathrm{sgn}(ah(1,\lambda_1)) = \nu_1$, then

$$\alpha(x,y) = \begin{cases} (x^{2q}+y^{2p})^{(r-s-1)/2}(y^p - (1+\lambda_k^p)x^q), & \text{if } r-s \text{ is odd,} \\ (x^{2q}+y^{2p})^{(r-s-2)/2}(y^p + (1-\lambda_1^p)x^q)(y^p - (1+\lambda_k^p)x^q), & \text{if } r-s \text{ is even.} \end{cases}$$

Define $P(x,y)$ and $Q(x,y)$ as follows

$$P(x,y) = -\frac{a}{q}\alpha(x,y)h(x,y)y^{p-1}, \quad Q(x,y) = \frac{1}{px}(\eta(x,y) + qyP(x,y)).$$

As $\eta(0,y) + qyP(0,y) \equiv 0$, $Q(x,y)$ is also a polynomial. It is easy to prove that $X = (P,Q)$ is a (p,q)-quasihomogeneous vector filed of degree m. From Theorem 2 and Proposition 18 it follows that $X \in \Omega_{pqm}^{2k}$ and (σ,ν) associated to X verifies either (i) or (ii). Therefore (σ,ν) is m-admissible.

Case 2. $(p,q,m) \in \Theta_2$.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ be the finite real numbers with $0 \in \{\lambda_1, \lambda_2, \cdots, \lambda_k\}$. Without loss of generality suppose $\lambda_l = 0$. Let

$$\begin{aligned} \eta(x,y) &= a(x^{2q}+y^{2p})^{(r+1-k)/2} y \prod_{i=1, i\neq l}^{k} (y^p - \lambda_i^p x^q), \\ P(x,y) &= -\frac{a}{q}\alpha(x,y)h(x,y), \end{aligned}$$

where $a = \pm 1$, $\alpha(x,y)$, $h(x,y)$ are defined in the same way as in Case 1. Define $Q(x,y)$ by the equation (2). Using the same arguments as in Case 1, $X = (P,Q) \in \Omega_{pqm}^{2k}$ and (σ,ν) associated to X verifies either (i) or (ii). Therefore (σ,ν) is m-admissible.

Case 3. $(p,q,m) \in \Theta_3$.

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_{k-1}$ be the finite real numbers with $\lambda_i \neq 0$, $i = 1, 2, \cdots, k-1$, $\lambda_k = +\infty$, and

$$\eta(x,y) = ax(x^{2q}+y^{2p})^{(r+1-k)/2} \prod_{i=1}^{k-1} (y^p - \lambda_i^p x^q),$$

where a is chose in the same way as in Case 1. If $\mathrm{sgn}(ah(1,\lambda_1)) = -\nu_1$, then define

$$\alpha(x,y) = \begin{cases} (x^{2q}+y^{2p})^{(r-s-1)/2}, & \text{if } r-s \text{ is odd,} \\ (x^{2q}+y^{2p})^{(r-s-2)/2}(y^p + (1-\lambda_1^p)x^q), & \text{if } r-s \text{ is even.} \end{cases}$$

If $\mathrm{sgn}(ah(1,\lambda_1)) = \nu_1$, then

$$\alpha(x,y) = \begin{cases} -x^q(x^{2q}+y^{2p})^{(r-s-3)/2}(y^p + (1-\lambda_1^p)x^q), & \text{if } r-s \text{ is odd,} \\ -x^q(x^{2q}+y^{2p})^{(r-s-2)/2}, & \text{if } r-s \text{ is even.} \end{cases}$$

Let

$$P(x,y) = -\frac{a}{q}xy^{p-1}\alpha(x,y)h(x,y), \quad Q(x,y) = \frac{1}{px}(\eta(x,y) + qyP(x,y)),$$

where $h(x,y)$ is defined as in Case 1. Using the same arguments as in Case 1, $X=(P,Q)\in\Omega_{pqm}^{2k}$ and (σ,ν) associated to X verifies either (i) or (iii). Therefore (σ,ν) is m-admissible.

Case 4. $(p,q,m)\in\Theta_4$.

Let $\lambda_1<\lambda_2<\cdots<\lambda_{k-1}$ be the finite real numbers with $0\in\{\lambda_1,\lambda_2,\cdots,\lambda_{k-1}\}$, and $\lambda_k=+\infty$. Without loss of generality suppose $\lambda_l=0$. Let

$$\eta(x,y)=axy(x^{2q}+y^{2p})^{(r+1-k)/2}\prod_{i=1,i\neq l}^{k-1}(y^p-\lambda_i^p x^q),\ P(x,y)=-\frac{a}{q}x\alpha(x,y)h(x,y),$$

where $h(x,y)$ and $\alpha(x,y)$ are defined as in Case 1 and Case 3 respectively, $a=\pm$ is chose such that $\mathrm{sgn}\partial\eta(1,\lambda_1)/\partial u=\sigma_1$. Define $Q(x,y)$ by the equation (2). We get a a (p,q)-quasihomogeneous vector filed $X=(P,Q)$ of degree m which is structurally stable. The sequence (σ,ν) associated to X verifies either (i) or (iii). Therefore (σ,ν) is m-admissible.

(b) Let $k=r+1$, $s=r$. Suppose that (σ,ν) is m-admissible, then there exists $X=(P,Q)\in\Omega_{pqm}^{2k}$ such that (σ,ν) associated to X verifies one of (i), (ii) and (iii) and $\eta(x,y)$, $P(x,y)$, $Q(x,y)$ have one of the forms listed in Proposition 3.

If $\eta(0,1)\neq 0$, $\eta(1,0)\neq 0$, then $\eta(x,y)$ has the form (5), which implies that $\eta(1,u)=a\prod_{i=1}^{r+1}(u^p-\lambda_i^p)$, where $\lambda_1<\lambda_2<\cdots<\lambda_{r+1}$ with $\lambda_i\neq 0$, $i=1,2,\cdots,r+1$ and $a\in\mathbb{R}$, $a\neq 0$. This gives

$$\eta(x,y)=(x^{1/p})^{p+q+m-1}\eta\left(1,\frac{y}{x^{q/p}}\right)=a\prod_{i=1}^{r+1}(y^p-\lambda_i^p x^q).$$

Since $s=r$, $P(1,y)$ has r real zeros which belongs to the interval $(\lambda_1,\lambda_{k+1})$. It follows from (5) that $P(x,y)$ is necessarily of the form

$$P(x,y)=b\left(\prod_{i=1}^{r}(y^p-\mu_i^p x^q)\right)y^{p-1},\quad \lambda_1<\mu_1<\lambda_2<\cdots<\lambda_r<\mu_r<\lambda_{r+1},$$

with $b\in\mathbb{R}$, $b\neq 0$. $Q(x,y)$ is defined by the equation (2). Hence we have $\eta(0,1)=a=-qb=-qP(0,1)$. Therefore $\sigma_1=\mathrm{sgn}(\partial\eta(1,\lambda_1)/\partial u)=\mathrm{sgn}((-1)^r a)=\mathrm{sgn}((-1)^{r+1}b)=-\mathrm{sgn}(P(1,\lambda_1))=\nu_1$.

If $\eta(0,1)=0$, $\eta(1,0)\neq 0$, then $\eta(1,u)$ has $r+1$ zeros $\lambda_1<\lambda_2<\cdots<\lambda_r<\lambda_{r+1}=+\infty$ with $\lambda_i\neq 0$, $i=1,2,\cdots,r$. By the same arguments as above, we have

$$\eta(x,y)=ax\prod_{i=1}^{r}(y^p-\lambda_i^p x^q),\ P(x,y)=bx\left(\prod_{i=1}^{r-1}(y^p-\mu_i^p x^q)\right)y^{p-1}$$

with $ab\neq 0$, $a,b\in\mathbb{R}$, and $\lambda_1<\mu_1<\lambda_2<\cdots<\mu_{r-1}<\lambda_r<613+\infty$. This gives

$$Q(x,y)=\frac{1}{p}\left(a\prod_{i=1}^{r}(y^p-\lambda_i^p x^q)+qb\left(\prod_{i=1}^{r-1}(y^p-\mu_i^p x^q)\right)y^p\right).$$

Hence $\nu_r=\mathrm{sgn}(-P(1,\lambda_r))=\mathrm{sgn}(-b)$, $\nu_{r+1}=\mathrm{sgn}(-Q(0,1))=\mathrm{sgn}(-a-qb)$. Since we suppose $s=r$, we have $\nu_r\nu_{r+1}=-1$, which implies $ab<0$. This yields $\sigma_1=\mathrm{sgn}(\partial\eta(1,\lambda_1)/\partial u)=\mathrm{sgn}(a(-1)^{r-1})$, $\nu_1=\mathrm{sgn}(-P(1,\lambda_1))=\mathrm{sgn}(-b(-1)^{r-1})$. Finally, one gets $\sigma_1=\nu_1$.

If either $\eta(0,1)\neq 0$, $\eta(1,0)=0$ or $\eta(0,1)=\eta(1,0)=0$, one gets $\sigma_1=\nu_1$ by the same arguments.

Conversely, if $\sigma_1 = \nu_1$ and $k = r+1$, $s = r$, then there exist the vector field $X = (P, Q)$ such that (σ, ν) verifies one of (i), (ii) and (iii), where $P(x, y)$, $Q(x, y)$ are defined as above. This finishes proof. $\square$

Proposition 20. *Let $0 \neq k \in J_{m,r}$ and X, $X' \in \Omega^{2k}_{pqm}$. Denote by (σ, ν), $(\sigma', \nu') \in S^{2k}_m$ the sequences associated to X and X' respectively. Then X and X' are topologically equivalent if and only if there exists $\tau \in \mathbb{N}$ $(0 \leq \tau \leq 2k-1)$ such that one of the following conditions is satisfied for all $i \in \mathbb{Z}$:*

$$(\sigma_i, \nu_i) = (\sigma'_{i+\tau}, \nu'_{i+\tau}),$$
$$(\sigma_i, \nu_i) = (\sigma'_{2k-i+1+\tau}, \nu'_{2k-i+1+\tau}).$$

Proof. The statement is proved by following the arguments in the proof of Proposition 11 of [17]. $\square$

Let $w = (\sigma, \nu) = \{w_i = (\sigma_i, \nu_i),\ i \in \mathbb{Z}\} \in S^{2k}_m$. It follows from Proposition 19 that the sequence $\bar{w} = \{\bar{w}_i = (\sigma_{i+1}, \nu_{i+1}),\ i \in \mathbb{Z}\}$ also belongs S^{2k}_m. Define the application

$$\mathfrak{R}:\ S^{2k}_m \longrightarrow S^{2k}_m$$

such that if $w = (\sigma, \nu) \in S^{2k}_m$, then $\mathfrak{R}(w)$ is the sequence of S^{2k}_m verifying

$$(\mathfrak{R}(w))_i = w_{i+1} \quad \text{for all } i \in \mathbb{Z}. \tag{32}$$

Whenever $f:\ A \longrightarrow A$ is an arbitrary application then $a \in A$ is called a *l-periodic point* of f if $f^l(a) = a$ and $f^i(a) \neq a$ for $i = 1, 2, \cdots, l-1$. The integer l is called the *period* of a. The set $C = \{a, f(a), \cdots, f^{l-1}(a)\} \subset A$ is a *cycle of order l* (*l-cycle*) of f if $a \in A$ is a l-periodic point of f. These notation can be found in many papers, see for instance [17].

From (32) $w \in S^{2k}_m$ is a l-periodic point of $\mathfrak{R}$ if and only if w is a l-periodic sequence. Therefore $C \subset S^{2k}_m$ is a l-cycle of $\mathfrak{R}$ if there exists a l-periodic sequence $w \in S^{2k}_m$ such that $C = \{w, \mathfrak{R}(w), \cdots, \mathfrak{R}^{l-1}(w)\}$. Each sequence $w \in S^{2k}_m$ belongs to some cycle of order l of $\mathfrak{R}$ where l is an even division of $2k$[17].

Define the application

$$\Psi:\ S^{2k}_m \longrightarrow S^{2k}_m,$$

where if $w \in S^{2k}_m$ then $\Psi(w)$ is given by

$$(\Psi(w))_i = w_{2k-i+1}.$$

By the definition we know that Ψ is the application that reverses the order of the elements (σ_1, ν_1), (σ_2, ν_2), $\cdots$, (σ_{2k}, ν_{2k}).

The following two propositions have been obtained in [17] for homogeneous polynomial vector fields. They can be proved in the same way for (p, q)-quasi-homogenous polynomial vector fields. Here we omit the details.

Proposition 21. *Let $0 \neq k \in J_{m,r}$. The following statements are true:*

(a) $\Psi \circ \Psi = Id$;
(b) $\Psi \circ \mathfrak{R} = \mathfrak{R}^{-1} \circ \Psi$.

Proposition 22. *Let $0 \neq k \in J_{m,r}$ and X, $X' \in \Omega^{2k}_{pqm}$. Denote by w and w' the sequences of S^{2k}_m associated X and X' respectively. Then X and X' are topologically equivalent if and only if either w and w', or w and $\Psi(w')$ belong to the same cycle of $\mathfrak{R}$.*

Following the idea in [17], we define in S_m^{2k} the equivalent relation: $w \sim \bar{w}$ if and only if one of the sequences w and $\Psi(w)$ belongs to the cycle $\{\bar{w}, \mathfrak{R}(\bar{w}), \cdots, \mathfrak{R}^{l-1}(\bar{w})\}$. According to Proposition 22, the number of equivalence classes of $\sim$ in S_m^{2k} coincide with the number C_{pqm}^k of topological equivalence classes in Ω_{pqm}^{2k}.

Consider the cycle C of $\mathfrak{R}$ such that $w \in C$. From Proposition 21(b), $\Psi(\mathfrak{R}^i(w)) = \mathfrak{R}^{-i}(\Psi(w))$, and hence the cycle of $\Psi(w)$ is $C' = \{\Psi(w) : w \in C\}$. Therefore the equivalence class of w is $C \cup C'$.

A cycle C of $\mathfrak{R}$ is called *symmetric* if $C = C'$. Denote by D_{pqm}^k the number of cycles of $\mathfrak{R}$ in S_m^{2k} and E_{pqm}^k the number of symmetric cycles respectively. Then

$$C_{pqm}^k = E_{pqm}^k + \frac{D_{pqm}^k - E_{pqm}^k}{2} = \frac{E_{pqm}^k + D_{pqm}^k}{2}. \tag{33}$$

The above expression has been obtained in [17] for homogeneous polynomial systems.

Proposition 23. *Suppose that p and q are odd and $0 \neq k \in J_{m,r}$.*

(a) *If r is odd and $k < r+1$, then*

$$D_{pqm}^k = \sum_{2n|k} \mathcal{P}_{2n}, \ \mathcal{P}_{2n} = \frac{1}{n}\left(2^{2n} - \sum_{l|n, l\neq n} l\mathcal{P}_{2l}\right).$$

(b) *If r is even and $k < r+1$, then*

$$D_{pqm}^k = \sum_{n|k} \mathcal{P}_{2n}, \ \mathcal{P}_{2n} = \frac{1}{n}\left(2^{n} - \sum_{l|n, l\neq n} l\mathcal{P}_{2l}\right).$$

(c) *If r is odd, then $D_{pqm}^{r+1} = (\sum_{2n|r+1} \mathcal{P}_{2n}) - 1$ with $\mathcal{P}_{2n}$ defined as in (a).*

(d) *If r is even, then $D_{pqm}^{r+1} = (\sum_{n|r+1} \mathcal{P}_{2n}) - 1$ with $\mathcal{P}_{2n}$ defined as in (b).*

Proof. It follows from Proposition 14 that m is odd (resp. even) if and only if r is odd (resp. even). The proposition follows by the same arguments as in [17]. □

The sequences verifying Proposition 18(b) are not occurred in [17]. We study D_{pqm}^k for these sequences in the following proposition.

Proposition 24. *Suppose that p is odd and q is even, $0 \neq k \in J_{m,r}$.*

(a) *If $(p, q, m) \in \Theta_1 \cup \Theta_2$, then*

$$D_{pqm}^k = \sum_{n|k} \mathcal{P}_{2n}, \ \mathcal{P}_{2n} = \frac{1}{n}\left(2^{n} - \sum_{l|n, l\neq n} l\mathcal{P}_{2l}\right),$$

for $k < r+1$, and $D_{pqm}^{r+1} = (\sum_{n|r+1} \mathcal{P}_{2n}) - 1$ for $k = r+1$, respectively.

(b) *If either $(p, q, m) \in \Theta_3 \cup \Theta_4$, then $D_{pqm}^k = \sum_{n|k} \mathcal{P}_{2n}$ for $0 \neq k < r+1$ and $D_{pqm}^k = (\sum_{n|r+1} \mathcal{P}_{2n}) - 1$ for $k = r+1$ respectively, where $\mathcal{P}_{2n}$ is given in (11).*

Proof. We note that each sequence $w = (\sigma, \nu) \in S_m^{2k}$ is periodic and its period is an even divisor of $2k$. Let w be the $2n$-periodic sequence in S_m^{2k} with $2n|2k$. Then w is completely determined if the elements (σ_i, ν_i) is given for $i = l+1, l+2, \cdots, l+2n$ for $l \in \mathbb{Z}$. It is obvious that w belongs a $2n$-cycle of $\mathfrak{R}$. If $\mathcal{P}_{2n}$ denotes the number of $2n$-cycles of $\mathfrak{R}$ in S_m^{2k}, then $\mathcal{P}_{2n} = \mathcal{N}_{2n}/(2n)$, where $\mathcal{N}_{2n}$ is the number of $2n$-periodic sequences in S_m^{2k}.

(a) Suppose $(p, q, m) \in \Theta_1 \cup \Theta_2$. If $k < r+1$, then it follows from Proposition 18(b)(i) that there are 2^{n+1} ways of choosing the elements $(\sigma_{k-n+1}, \nu_{k-n+1})$, $(\sigma_{k-n+2}, \nu_{k-n+2}), \cdots, (\sigma_k, \nu_k), \cdots, (\sigma_{k+n}, \nu_{k+n})$. Therefore

$$\mathcal{N}_{2n} = 2^{n+1} - \sum_{l|n, l\neq n} \mathcal{N}_{2l}, \text{ and } \mathcal{P}_{2n} = \frac{1}{n}\left(2^n - \sum_{l|n,l\neq n} l\mathcal{P}_{2l}\right).$$

The statement (a) for $k < r+1$ follows by adding $\mathcal{P}_{2n}$ for all divisors $2n$ of $2k$.

We note that the above computation are valid for the case $k = r+1$, but, by Proposition 19(b), we have to rule out the two sequences satisfying $\sigma_j = v_j$ for all $j \in \mathbb{Z}$. Since these sequence belong to the same 2-cycle of $\mathfrak{R}$, the statement for $k = r+1$ follows.

(b). Suppose $(p, q, m) \in \Theta_3 \cup \Theta_4$. If $k < r+1$, then it follows from Proposition 18(b)(ii) that there are 2^{n+1} ways of choosing the elements $(\sigma_{k-n+1}, \nu_{k-n+1})$, $(\sigma_{k-n+2}, \nu_{k-n+2}), \cdots, (\sigma_k, \nu_k), \cdots, (\sigma_{k+n}, \nu_{k+n})$. Therefore

$$\mathcal{N}_{2n} = 2^{n+1} - \sum_{l|n, l\neq n} \mathcal{N}_{2l}, \text{ and } \mathcal{P}_{2n} = \frac{1}{n}\left(2^n - \sum_{l|n,l\neq n} l\mathcal{P}_{2l}\right),$$

which proves (b) for $k < r+1$.

If $k = r+1$, then one gets D_{pqm}^k by the same arguments as in (a). □

Next we are going to calculate E_{pqm}^k. The following two propositions have been obtained in [17] for homogeneous polynomial vector fields. They can be proved by the same arguments as in [17].

Proposition 25. *Let $k \in J_{m,r}$. Then C is symmetrical if and only if there exists $w \in C$ such that $\Psi(w) = \mathfrak{R}(w)$.*

Proposition 26. *Under the assumptions of Proposition 25, if C is a symmetrical cycle, then there are exactly two elements of C satisfying $\Psi(w) = \mathfrak{R}(w)$.*

Proposition 27. *Let $0 \neq k \in J_{m,r}$. Suppose that both p and q are odd.*

(a) *If r is odd and $k < r+1$, then*

$$E_{pqm}^k = \sum_{2n|k} I_{2n}, \quad \textit{where } I_{2n} = 2^{n+1} - \sum_{l|n,l\neq n} I_{2l}.$$

(b) *If r is even and $k < r+1$, then*

$$E_{pqm}^k = \sum_{n|k} I_{2n}, \quad \textit{where } I_{2n} = 2^{(n+1)/2} - \sum_{l|n,l\neq n} I_{2l}.$$

(c) *If r is odd, then $E_{pqm}^{r+1} = (\sum_{2n|r+1} I_{2n}) - 1$, where I_{2n} is defined in (a).*

(d) *If r is even, then $E_{pqm}^{r+1} = (\sum_{n|r+1} I_{2n}) - 1$, where I_{2n} defined in (b).*

Proof. It follows from Proposition 14 that r is odd (resp. even) if and only if m is odd (resp. even). The proposition follows by the same arguments as in [17]. □

Proposition 28. *Let $0 \neq k \in J_{m,r}$. Suppose that p is odd and q is even.*

(a) *If $(p, q, m) \in \Theta_1 \cup \Theta_2$, then $E_{pqm}^k = 2$ for $k < r + 1$, and $E_{pqm}^{r+1} = 1$ respectively.*

(b) *If $(p, q, m) \in \Theta_3 \cup \Theta_4$, then*

$$E_{pqm}^k = \begin{cases} 4 & \text{if } r \text{ is odd, } k < r+1, \\ 2 & \text{if } r \text{ is even, } k < r+1. \end{cases} \quad \text{and } E_{pqm}^{r+1} = \begin{cases} 3 & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even,} \end{cases}$$

respectively.

Proof. We have known that the period of a symmetrical cycle of $\mathfrak{R}$ is an even divisor of $2k$. From Proposition 25 and Proposition 26, the number of symmetrical $2n$-cycles of $\mathfrak{R}$ will be obtained if we divide by 2 the number of $2n$-periodic sequence w in S_m^{2k} such that $\Psi(w) = \mathfrak{R}(w)$. Since the sequences w in a symmetric cycle verify $\Psi(w) = \mathfrak{R}(w)$, we have

$$w_{2k-i+1} = w_{i+1}. \tag{34}$$

From now we suppose that w belongs a symmetric cycle.

(a) If $(p, q, m) \in \Theta_1 \cup \Theta_2$ and $k < r + 1$, then w verifies Proposition 19(ii), which gives $w_{2k-i+1} = -w_i$. It follows from (34) that $w_{i+1} = -w_i$. This implies that $w_i = (-1)^{i-1} w_1$. Hence w is a 2-periodic sequence. We can take 4 ways of choosing the element w_1. Therefore, $E_{pqm}^k = 2$ for $k < r + 1$.

Since the sequences such that $\sigma_j = \nu_j$ for all $j \in \mathbb{Z}$ verify $\Psi(w) = \mathfrak{R}(w)$ and belong to 2-cycle of $\mathfrak{R}$, we get $E_{pqm}^{r+1} = 1$.

(b) If $(p, q, m) \in \Theta_3 \cup \Theta_4$, then w verifies Proposition 19(iii), which gives $w_{2k-i+1} = w_{i-1}$ for $i = 2, 3, \cdots, k$ and $w_{2k} = (-1)^{r-1} w_k$. It follows from (34) that $w_{i+1} = w_{i-1}$ for $i = 2, 3, \cdots, k$.

If r is odd, then $w_k = w_{2k}$. It follows from Proposition 15 that k is even. The equation $w_{i+1} = w_{i-1}$ for $i = 2, 3, \cdots, k$ and (34) imply that $w_{2k} = w_{2k-2} = \cdots = w_4 = w_2$ and $w_{2k-1} = w_{2k-3} = \cdots = w_3 = w_1$. Therefore, w is a 2-periodic sequence. We can take 8 ways of choosing the element w_1, w_2 and hence $E_{pqm}^k = 4$.

If r is even, then it follows from Proposition 15 that k is odd. The equation $w_{i+1} = w_{i-1}$ and (34) imply that $w_{2k} = w_{2k-2} = \cdots = w_4 = w_2$ and $w_{2k-1} = w_{2k-3} = \cdots = w_{k+2} = w_k = w_{k-2} = \cdots = w_3 = w_1$. Since r is even, we obtain $w_k = -w_{2k} = -w_2$ from Proposition 19(iii) and (34), which gives $w_2 = -w_1$. Therefore, w is a 2-periodic sequence. We can take 4 ways of choosing the element w_1 and hence $E_{pqm}^k = 2$.

We get E_{pqm}^{r+1} by the same arguments as in (a).

□

In the end of this section we prove Theorem 5.

Proof of Theorem 5. We use (30) to get C_{pqm}. One obtains C_{pqm}^k for $k \neq 0$ by (33), Proposition 23, Proposition 24, Proposition 27 and Proposition 28. If $(p, q, m) \in \Theta_2 \cup \Theta_3 \cup \Theta_4$, then system (1) has at least one singular point at infinity, which implies that $C_{pqm}^0 = 0$. If $(p, q, m) \in \Theta_1 \backslash (\Theta_2 \cup \Theta_3 \cup \Theta_4)$, then $C_{pqm}^0 = 2$ if r is odd, and $C_{pqm}^0 = 0$ if r is even, respectively. The statements of Theorem 5 follows from (30). □

5. Local phase portrait at the origin and at infinity

We shall prove Theorem 6 and Theorem 7 in this section.

5.1. At the origin. Consider the vector field $\bar{X} = (\bar{P}, \bar{Q})$ with

$$\bar{P} = \sum_{i \geq m} P_i(x, y), \quad \bar{Q} = \sum_{i \geq m} Q_i(x, y), \tag{35}$$

where $P_i(x, y)$ and $Q_i(x, y)$ are (p, q)- quasihomogeneous polynomials of degree $p - 1 + i$ and $q - 1 + i$ in the variables x and y respectively.

We would like to compare the local behaviour at origin of the vector field $\bar{X}$ given by (35) and the vector field $X_m = (P_m, Q_m)$. For this we apply the quasihomogeneous blow-up method as doing in Section 2. In the (p, q)-polar coordinates (r, ϕ) (see (12)) the system associated to $\bar{X}$ is

$$\dot{r} = \sum_{i \geq m} r^i F_i(\phi), \quad \dot{\phi} = \sum_{i \geq m} r^{i-1} G_i(\phi),$$

where

$$F_i(\phi) = \xi_i\left(\mathrm{Cs}\phi, \mathrm{Sn}\phi\right), \quad G_i(\phi) = \eta_i\left(\mathrm{Cs}\phi, \mathrm{Sn}\phi\right), \tag{36}$$

with

$$\begin{aligned} \xi_i(x, y) &= x^{2q-1} P_i(x, y) + y^{2p-1} Q_i(x, y) \\ \eta_i(x, y) &= px Q_i(x, y) - qy P_i(x, y) \end{aligned}$$

Taking the changes (14), the above system goes over to

$$r' = \sum_{i \geq m} r^{i-m+1} f_i(\theta), \quad \theta' = \sum_{i \geq m} r^{i-m} g_i(\theta). \tag{37}$$

where prime denotes derivative with respect to s and

$$f_i(\theta) = F_i\left(\frac{T\theta}{2\pi}\right), \quad g_i(\theta) = \frac{2\pi}{T} G_i\left(\frac{T\theta}{2\pi}\right). \tag{38}$$

Doing the same transformations in the vector field X_m we obtain that it is equivalent to the following differential system

$$\dot{r} = r f_m(\theta), \quad \dot{\phi} = g_m(\theta), \tag{39}$$

where $f_m(\theta)$ and $g_m(\theta)$ are defined in (38).

Proof of Theorem 6. We split the proof into two cases, as in Theorem 2.

Case 1. $X_m \in \Omega_{pqm}$ and $\eta_m(1, y)$ has no zeros, $\eta_m(0, 1) \neq 0$. Using Theorem 1 and Theorem 2 we conclude that the origin is a global focus (stable or unstable, depending on the sign of I_{X_m}) of X_m. It follows from (36) and (38) that $g_m(\theta) \neq 0$ in this case.

As the critical point of $\bar{X}$ on $r = 0$ are determined by the zeros of $g_m(\theta)$, $r = 0$ is a periodic orbit for (37). Furthermore, since the dominant terms of (37) in a neighborhood of $r = 0$ are given by $r f_m$ and g_m, the orbit $r = 0$ is a limit cycle with the same type of stability for (37) and (39). Therefore the origin is a focus with the same type of stability for $\bar{X}$ and X_m when $\eta_m(1, y)$ has no zeros and $\eta_m(0, 1) \neq 0$.

Case 2. $X_m \in \Omega_{pqm}$, and at least one of the following condition is satisfied: (i) $\eta_m(1, y)$ has zeros, (ii) $\eta_m(0, 1) = 0$. From Theorem 2 all the zeros of $\eta(1, y)$ are simple if there exists, and $\partial\eta_m(0,1)/\partial x \neq 0$ if $\eta_m(0,1)=0$, which implies that all zeros of $g_m(\theta)$ are simple zeros for $\theta \in [0,2\pi]$ by (36) and (38).

Comparing (37) and (39) we observe that the critical point on $r=0$ are the same for both systems, and they are determined by the zeros of $g_m(\theta)$. Furthermore the linear part of these systems in a critical point $(0,\theta^*)$ are

$$\begin{pmatrix} f_m(\theta^*) & 0 \\ g_{m+1}(\theta^*) & g'_m(\theta^*) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_m(\theta^*) & 0 \\ 0 & g'_m(\theta^*) \end{pmatrix}.$$

By the definition of $f_m(\theta)$ and $g_m(\theta)$ we have that if $g_m(\theta^*)=0$, then $f_m(\theta^*)\neq 0$. So all the critical points of (37) and (39) are hyperbolic and the local phase portrait of (37) and (39) in $(0,\theta^*)$ are locally topologically equivalent, provided that θ^* is a simple zero of $g_m(\theta)$. As the same happens in each singular point and they determine the phase portrait of both systems in a neighborhood of $r=0$, we conclude that $\bar{X}$ and X_m are locally topologically equivalent at the origin. □

5.2. **At infinity.** Now we can get the analogous of the Theorem 6 at infinity.

Consider the vector field $\hat{X}=(\hat{P},\hat{Q})$, with

$$\hat{P}=\sum_{i=0}^{m} P_i(x,y), \quad \hat{Q}=\sum_{i=0}^{m} Q_i(x,y), \tag{40}$$

where $P_i(x,y)$ and $Q_i(x,y)$ are (p,q)-quasihomogeneous polynomials of degree $p-1+i$ and $q-1+i$ respectively.

Applying the (p,q)-polar coordinates (r,ϕ) to the vector field $\hat{X}$ given by (40) the associated system is

$$\dot{r}=\sum_{i=0}^{m} r^i F_i(\phi), \quad \dot{\phi}=\sum_{i=0}^{m} r^{i-1} G_i(\phi),$$

where F_i and G_i are given in (36).

Doing $\rho=\dfrac{1}{r}$ the system (40) is written as

$$\dot{\rho}=-\sum_{i=0}^{m} \frac{1}{\rho^{i-2}} F_i(\phi), \quad \dot{\phi}=\sum_{i=0}^{m} \frac{1}{\rho^{i-1}} G_i(\phi). \tag{41}$$

Reparametrizing the time by $dt/ds=\rho^{m-1}$ the system (41) becomes

$$\begin{aligned} \rho' &= -\rho(F_m(\phi)+\rho F_{m-1}(\phi)+\cdots+\rho^m F_0(\phi)), \\ \phi' &= G_m(\phi)+\rho G_{m-1}(\phi)+\cdots+\rho^m G_0(\phi). \end{aligned} \tag{42}$$

where the derivative is with respect to s.

After these changes of coordinates the proof of Theorem 7 is analogous to the proof of Theorem 6.

Proof of Theorem 7. Applying the same changes of coordinates described before to the vector field $X_m=(P_m,Q_m)\in\Omega_{pqm}$ we obtain

$$\rho'=-\rho F_m(\phi), \quad \phi'=G_m(\phi). \tag{43}$$

The infinity of X_m and $\hat{X}$ corresponds to the invariant circle $\rho = 0$ of the systems (43) and (42). As $X_m \in \Omega_{pqm}$, $\rho = 0$ is a limit cycle or contains a finite number of hyperbolic critical points. Comparing the system (43) and (42) we observe that they have the same dominant terms. Therefore, it follows from the same argument used in the proof of Theorem 6 that both systems are topologically equivalent in a neighborhood of $\rho = 0$. So X_m and $\hat{X}$ are topologically equivalent in a neighborhood of the infinity. □

The converse of Theorems 6 and Theorem 7 are not true. In [17], Proposition 19 is proved that the converse of an analogous theorems to homogeneous vector field are not true. As a homogeneous vector field is a $(1,1)$-quasi homogeneous vector field of degree m the result follows.

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