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ABSTRACT. Let (X, A) be a par of topological spaces, $T : X \to X$ a free involution and A a $T-$invariant subset of X. In this context, a question that naturally arises is whether or not all continuous maps $f : X \to \mathbb{R}^k$ have a T -coincidence point, that is, a point $x \in X$ with $f(x) = f(T(x))$. In this paper, we obtain results of this nature under cohomological conditions on the spaces A and X. Also, we prove that this results still being true when we replace the map $f : X \to \mathbb{R}^k$ by an u.s.c. multi-valued map $\varphi : X \multimap \mathbb{R}^k$ with convex values.

1. Introduction

One formulation of the Borsuk-Ulam Theorem [1] is that there is no map from S^m to S^n equivariant with respect to the antipodal map, when $m > n$. In [6], it was proved that if X and Y are Hausdorff, pathwise connected and paracompact spaces equipped with free involutions $T : X \to X$ and $S : Y \to Y$ such that for some natural $n \geq 1$, $\check{H}^r(X; \mathbb{Z}_2) = 0$ for $1 \leq r \leq n$ and $\check{H}^{n+1}(Y/S; \mathbb{Z}_2) = 0$, where Y/S is the orbit space of Y by S, then there is no equivariant map $f : (X, T) \to (Y, S)$. The first aim of this paper is to generalize this result for the following relative case:

Theorem 1.1. *Let X, Y be Hausdorff, pathwise connected and paracompact spaces equipped with free involutions $T : X \to X$ and $S : Y \to Y$. Let A be a pathwise connected and $T-$invariant subset of X. Suppose that for some $n \geq 1$, $\check{H}^r(A, \mathbb{Z}_2) = 0$ for $1 \leq r \leq n-1$, $i^* : \check{H}^n(X, \mathbb{Z}_2) \to \check{H}^n(A, \mathbb{Z}_2)$ is the null homomorphism, where $i : A \hookrightarrow X$ is the inclusion map and $\check{H}^{n+1}(Y/T; \mathbb{Z}_2) = 0$. Then there is no equivariant map $f : (X, T) \to (Y, S)$.*

The following theorem is an important consequence of Theorem 1.1.

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Theorem 1.2. *Let X be a Hausdorff, pathwise connected and paracompact space with a free involution $T : X \to X$. Let A be a pathwise connected and $T-$invariante subset of X. Suppose that for some $n \geq 1$, $\check{H}^r(A, \mathbb{Z}_2) = 0$ for $1 \leq r \leq n-2$ and $i^* : \check{H}^{n-1}(X, \mathbb{Z}_2) \to \check{H}^{n-1}(A, \mathbb{Z}_2)$ is the null homomorphism, where $i : A \hookrightarrow X$ is the inclusion map. Then, if $f : X \to \mathbb{R}^k$ is a map with $n \geq k$, there exists $x \in X$ such that $f(x) = f(T(x))$.*

The second aim of this paper is to prove that Theorem 1.2 still being true when we replace the map $f : X \to \mathbb{R}^k$ by an u.s.c. multi-valued map $\varphi : X \multimap \mathbb{R}^k$ with convex values. Precisely:

Theorem 1.3. *Let X be a Hausdorff, pathwise connected and paracompact space with a free involution $T : X \to X$. Let A be a paracompact, pathwise connected and $T-$invariante subset of X. Suppose that for some $n \geq 1$, $\check{\mathrm{H}}^r(A, \mathbb{Z}_2) = 0$ for $1 \leq r \leq n-2$ and $i^* : \check{\mathrm{H}}^{n-1}(X, \mathbb{Z}_2) \to \check{\mathrm{H}}^{n-1}(A, \mathbb{Z}_2)$ is the null homomorphism, where $i : A \hookrightarrow X$ is the inclusion map. Then, if $\varphi : X \multimap \mathbb{R}^k$ is an u.s.c. multi-valued map such that $\varphi(x)$ is convex for any $x \in X$ and $n \geq k$, there exists $x \in X$ such that $\varphi(x) \cap \varphi(T(x)) \neq \emptyset$.*

The paper is organized as follows. In Section 2, we prove Theorems 1.1 and 1.2 and we show an interesting example for which Theorem 1.2 can be applied. In Section 3, we recall definitions, fix notations, state results needed on multi-valued map and prove Theorem 1.3.

Throughout the paper, we assume that all spaces under consideration are Hausdorff spaces. $\check{\mathrm{H}}^r(\ \ , \mathbb{Z}_2)$ denotes the $r-$dimensional vector space of Čech cohomology with compact carries and coefficients in $\mathbb{Z}_2$.

2. Proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2 and we show an interesting example for which Theorem 1.2 can be applied. We need first to prove the following lemma.

Lemma 2.1. *Let X, Y be Hausdorff and paracompact spaces, equipped with free involutions $T : X \to X$ and $S : Y \to Y$. Let $e \in \check{H}^1(X/T, \mathbb{Z}_2)$ and $u \in \check{H}^1(Y/T, \mathbb{Z}_2)$ be the Euler classes of the $\mathbb{Z}_2$-principal bundles $X \to X/T$ and $Y \to Y/T$, respectively. If $e^{n+1} \neq 0$ and $u^{n+1} = 0$, then there is no equivariant map $f : (X, T) \to (Y, S)$.*

Proof. First consider $B\mathbb{Z}_2$ the classifying space for $\mathbb{Z}_2$, and denote by $\alpha \in \check{H}^1(B\mathbb{Z}_2, \mathbb{Z}_2)$ the Euler class of the universal principal $\mathbb{Z}_2$-bundle over $B\mathbb{Z}_2$.

Since X is a Hausdorff paracompact space, one can take a classifying map $h : X/T \to B\mathbb{Z}_2$ for the principal $\mathbb{Z}_2$-bundle $X \to X/T$, and from $h^* : \check{H}^1(B\mathbb{Z}_2, \mathbb{Z}_2) \to \check{H}^1(X/T, \mathbb{Z}_2)$ one gets the Euler class $e = h^*(\alpha) \in \check{H}^1(X/T, \mathbb{Z}_2)$ of $X \to X/T$. Now suppose $f : (X,T) \to (Y,S)$ an equivariant map, and let $g : Y/S \to B\mathbb{Z}_2$ be a classifying map for $Y \to Y/S$. Then $g \circ \bar{f}$ also is a classifying map for $X \to X/T$ and, therefore, it is homotopic to the previous classifying map $h : X/T \to B\mathbb{Z}_2$, where $\bar{f} : X/T \to Y/S$ is the map induced by f. Since $u = g^*(\alpha)$, we have that $e = h^*(\alpha) = (g\circ\bar{f})^*(\alpha) = \bar{f}^* \circ g^*(\alpha) = \bar{f}^*(u)$, and thus $\bar{f}^*(u^{n+1}) = e^{n+1} \neq 0$, which contradicts the fact that $u^{n+1} = 0$. ☐

Proof of Theorem 1.1. Let $\hat{e} \in \check{H}^1(A/T, \mathbb{Z}_2)$ be the Euler class of the principal $\mathbb{Z}_2$-bundle $A \to A/T$ and let $\bar{i} : A/T \hookrightarrow X/T$ be the map induced by inclusion $i : A \hookrightarrow X$. We have that

$$\bar{i}^*(e) = \hat{e} \in \check{H}^1(A/T, \mathbb{Z}_2),$$

where $e \in \check{H}^1(X/T, \mathbb{Z}_2)$ is the Euler class for $X \to X/T$.

Now, let us consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(X/T) \arrow[r, "p^*"] \arrow[d, "\bar{i}^*"] & \check{H}^0(X) \arrow[r, "\tau"] \arrow[d, "i^*"] & \check{H}^0(X/T,) \arrow[r, "\smile e"] \arrow[d, "\bar{i}^*"] & \check{H}^1(X/T) \arrow[r] \arrow[d, "\bar{i}^*"] & \cdots \\
0 \arrow[r] & \check{H}^0(A/T) \arrow[r, "p^*"] & \check{H}^0(A) \arrow[r, "\tau"] & \check{H}^0(A/T) \arrow[r, "\smile \hat{e}"] & \check{H}^1(A/T) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^r(X/T) \arrow[r, "p^*"] \arrow[d, "\bar{i}^*"] & \check{H}^r(X) \arrow[r, "\tau"] \arrow[d, "i^*"] & \check{H}^r(X/T) \arrow[r, "\smile e"] \arrow[d, "\bar{i}^*"] & \check{H}^{r+1}(X/T) \arrow[r] \arrow[d, "\bar{i}^*"] & \cdots \\
\cdots \arrow[r] & \check{H}^r(A/T) \arrow[r, "p^*"] & \check{H}^r(A) \arrow[r, "\tau"] & \check{H}^r(A/T) \arrow[r, "\smile \hat{e}"] & \check{H}^{r+1}(A/T) \arrow[r] & \cdots
\end{tikzcd}$$

where the rows are Smith-Gysin exact sequences (see [2, 3; Section 3.10 (Sequence 10.5 of p. 161)]) and each square commutes by the naturality, p is the quotient map and τ is the transfer homomorphism. Since A is pathwise connected, $p^* : \check{H}^0(A/T, \mathbb{Z}_2) \to \check{H}^0(A, \mathbb{Z}_2)$ is an isomorphism, hence $\cup\hat{e} : \check{H}^0(A/T, \mathbb{Z}_2) \to \check{H}^1(A/T, \mathbb{Z}_2)$ is injective and thus $\hat{e} = 1 \cup \hat{e} \in \check{H}^1(A/T, \mathbb{Z}_2)$ is a nonzero class. The fact that $\check{H}^r(A, \mathbb{Z}_2) = 0$, for $1 \leq r \leq n-1$ implies that $\cup\hat{e} : \check{H}^r(A/T, \mathbb{Z}_2) \to \check{H}^{r+1}(A/T, \mathbb{Z}_2)$ is an isomorphism for $1 \leq r \leq n-2$ and injective for $r = n-1$, hence $\hat{e}^n \in \check{H}^n(A/T, \mathbb{Z}_2)$ is nonzero. Since $\bar{i}^*(e^n) = \hat{e}^n \neq 0$, we see that $e^n \in \check{H}^n(X/T, \mathbb{Z}_2)$ is nonzero.

Now, we will show that $e^{n+1} \in \check{H}^{n+1}(X/T, \mathbb{Z}_2)$ is nonzero. Suppose $e^{n+1} = e^n \cup e = 0$, then $e^n \in \ker(\cup e) = \mathrm{im}(\tau)$ and there exists a nonzero class $a \in \check{H}^n(X, \mathbb{Z}_2)$ such that $\tau(a) = e^n$. Therefore, $\bar{i}^* \circ \tau(a) = \bar{i}^*(e^n) = \hat{e}^n \neq 0$.

On the other hand, since $i^* : \check{H}^n(X, \mathbb{Z}_2) \to \check{H}^n(A, \mathbb{Z}_2)$ is the null homomorphism and each square commutes in the diagram, we have that $\bar{i}^* \circ \tau(a) = \tau \circ i^*(a) = 0$. Thus $e^{n+1} \neq 0$.

Finally, since $\check{H}^{n+1}(Y/S, \mathbb{Z}_2) = 0$, we have that $u^{n+1} \in \check{H}^{n+1}(Y/S, \mathbb{Z}_2)$ is zero. By Lemma 2.1 there is no equivariant map $f : (X, T) \to (Y, S)$. $\square$

Proof of Theorem 1.2. In fact, suppose $f(x) \neq f(T(x))$ for any $x \in X$. Then one has the equivariant map $F : (X, T) \to (S^{k-1}, a)$ given by

$$F(x) = \frac{f(x) - f(T(x))}{|f(x) - f(T(x))|},$$

where $a : S^{k-1} \to S^{k-1}$ is the antipodal map. Since $n \geq k$, $\check{H}^n(S^{k-1}/a, \mathbb{Z}_2) = 0$, which contradicts Theorem 1.1. $\square$

Example 2.2. Let $T_n = T_1 \sharp \cdots \sharp T_1$ be the n-fold a connected sum of tori $T_1 = S^1 \times S^1$, which is embedded in $\mathbb{R}^3$ symmetrically with respect to the origin. Let $T : T_n \to T_n$ be the antipodal map given by $T(x, y, x) = (-x, -y, -z)$. If n is even, there exists a loop $A = \{(x, 0, z); x^2 + z^2 = 1\} \subset T_n$ homeomorphic to S^1, which separates T_n into two components symmetrical with respect to the origin such that $T(A) = A$. We have that $i^* : \check{H}^1(T_n, \mathbb{Z}_2) \to \check{H}^1(A, \mathbb{Z}_2)$ is the null homomorphism, so by Theorem 1.2, for any continuous map $f : T_n \to \mathbb{R}^2$, there exists $x \in X$ such that $f(x) = f(T(x))$.

Remark 2.3. In the Example 2.2, let us note that $\check{H}^1(T_n, \mathbb{Z}_2) \neq 0$, and therefore [6, Theorem 1 or Theorem A'] cannot be applied to show this result.

3. Preliminaries on multi-valued map and Proof of Theorem 1.3

Let X and Y be two spaces and assume that for each point $x \in X$ a nonempty compact subset $\varphi(x)$ of Y is given; in this case, we say that φ is a multi-valued map from X into Y and we write $\varphi : X \multimap Y$. More precisely, a multi-valued map $\varphi : X \multimap Y$ can be defined as a subset φ of $X \times Y$ for which the following condition is satisfied: for every $x \in X$ the set $\varphi_x = \{y \in Y \mid (x, y) \in \varphi\}$ is a nonempty compact subset of Y.

A multi-valued map $\varphi : X \multimap Y$ is called upper semicontinuous (u.s.c.) if for every open subset U of Y the set $\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \subset U\}$ is an open subset of X.

A compact space X is acyclic (with respect to the functor $\check{\mathrm{H}}^*$) if $\check{\mathrm{H}}^0(X, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\check{\mathrm{H}}^q(X, \mathbb{Z}_2) = 0$ for all $q > 0$. In words, X has the cohomology of a point.

An u.s.c. multi-valued map $\varphi : X \multimap Y$ is called acyclic if for every $x \in X$ the set $\varphi(x)$ is an acyclic set.

Let $\varphi : X \multimap Y$ be an u.s.c. multi-valued map and consider

$$\Gamma_\varphi = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$$

the graph of φ. We have two projections associated to φ: the maps $p : \Gamma_\varphi \to X$ and $q : \Gamma_\varphi \to Y$ given by $p(x, y) = x$ and $q(x, y) = y$.

Below, we listed some properties of the u.s.c. multi-valued mappings.

Lemma 3.1. *Let X be a connected space and $\varphi : X \multimap Y$ an u.s.c. multi-valued map. Suppose $\varphi(x)$ connected for every $x \in X$. Then $\varphi(X) = \bigcup_{x \in X} \varphi(x)$ is a connected set.*

Lemma 3.2. *Let X be a compact space and $\varphi : X \multimap Y$ an u.s.c. multi-valued map. Then $\varphi(X)$ is compact.*

Proposition 3.3. *Let X be a Hausdorff pathwise connected space and $\varphi : X \multimap \mathbb{R}^k$ an u.s.c. multi-valued map with convex values. Then $\varphi(X) = \bigcup_{x \in X} \varphi(x)$ is pathwise connected.*

Proof. Let $x_0, x_1 \in X$ be arbitrary points. It suffices to show that there exists a path $g : I \to \varphi(X)$ such that $g(0) \in \varphi(x_0)$ and $g(1) \in \varphi(x_1)$. In order to prove, let $\gamma : I \to X$ be a path such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.
Step 1. We will prove that for each $\varepsilon > 0$ there exists a path $g : I \to \mathbb{R}^k$ such that $\operatorname{dist}(g(t), \varphi(\gamma(t))) < \varepsilon$, for every $t \in I$, where

$$\operatorname{dist}(g(t), \varphi(\gamma(t))) = \inf_{u \in \varphi(\gamma(t))} \|g(t) - u\|.$$

Given $y \in \mathbb{R}^k$, we denote $B(y, \varepsilon) = \{x \in \mathbb{R}^k \mid \|y - x\| < \varepsilon\}$ the open ball centered at y with radius ε. Let $K = \varphi(\gamma([0, 1]))$ be the compact subset of $\mathbb{R}^k$ and consider the open covering

$$K \subset \bigcup_{y \in K} B(y, \varepsilon).$$

Since K is compact, we can find $y_1, \ldots, y_n \in K$ such that

$$K = B(y_1, \varepsilon) \cup \cdots \cup B(y_n, \varepsilon).$$

Let $\chi_i : K \to [0,1]$, $1 \leq i \leq n$, be a partition of unity dominated by $\{B(y_1,\varepsilon),\dots,B(y_n,\varepsilon)\}$. Finally, define $g : I \to \mathbb{R}^k$ by

$$g(t) = \sum_{i=1}^{n} \left(\frac{\max[\chi_i(\varphi(\gamma(t)))]}{\sum_{j=1}^{n} \max[\chi_j(\varphi(\gamma(t)))]} \right) y_i.$$

Fixe $t \in I$ and suppose $\max[\chi_i(\varphi(\gamma(t)))] > 0$. Then, $\varphi(\gamma(t)) \cap B(y_i,\varepsilon) \neq \emptyset$. It follows that $\operatorname{dist}(y_i, \varphi(\gamma(t))) < \varepsilon$. Since $\varphi(\gamma(t))$ is convex, if $y_{i_1},\dots,y_{i_l}$ satisfies $\operatorname{dist}(y_{i_j}, \varphi(\gamma(t))) < \varepsilon$, $j = 1,\dots,l$, then any convex combination $y = \sum_{j=1}^{l} a_j y_{i_j}$, $a_j \geq 0$, $1 \leq j \leq l$, $\sum_{j=1}^{l} a_j = 1$, satisfies $\operatorname{dist}(y, \varphi(\gamma(t))) < \varepsilon$. This prove that $\operatorname{dist}(g(t), \varphi(\gamma(t))) < \varepsilon$ for any $t \in I$.

Step 2. We claim that we can define a sequence $\{g_n\}$ of continuous functions from I to $\mathbb{R}^k$ with the following properties:

$$\operatorname{dist}(g_n(t), \varphi(\gamma(t))) < \frac{1}{2^n}, \quad n = 1,2,\dots, \quad t \in I. \tag{3.1}$$

$$\|g_n(t) - g_{n-1}(t)\| < \frac{1}{2^{n-2}}, \quad n = 2,3,\dots, \quad t \in I. \tag{3.2}$$

In fact, for $n = 1$, it is enough to take $\varepsilon = 1/2$ in *Step 1*. Now, suppose we have defined paths g_n satisfying (3.1) and (3.2) up to $n = l$. We shall define g_{l+1} satisfying (3.1) and (3.2) as follows:

Let $\Phi : I \to \mathbb{R}^k$ be defined by

$$\Phi(t) = \overline{B(g_l(t), 1/2^l)} \cap \varphi(\gamma(t)).$$

By (3.1), $\Phi(t) \neq \emptyset$ for every $t \in I$. Thus, Φ is a well-defined u.s.c. multivalued with convex values. Analogous to *Step 1*, there exists a path $g : I \to \mathbb{R}^k$ such that $\operatorname{dist}(g(t), \Phi(t)) < 1/2^{k+1}$ for every $t \in I$. Define $g_{k+1}(t) = g(t)$ for every $t \in I$. Since $\operatorname{dist}(g(t), \Phi(t)) < 1/2^{k+1}$ and $\Phi(t) = \overline{B(g_k(t), 1/2^k)} \cap \varphi(\gamma(t))$, it follows that $\operatorname{dist}(g_{k+1}(t), \varphi(\gamma(t))) < 1/2^{k+1}$, for every $t \in I$, proving (3.1). Moreover, fixed $t \in I$, there exists $y \in \overline{B(g_k(t), 1/2^k)} \cap \varphi(\gamma(t))$ such that $\|g_{k+1}(t) - y\| < 1/2^{k+1}$. It follows that

$$\|g_{k+1}(t) - g_k(t)\| \leq \|g_{k+1}(t) - y\| + \|y - g_k(t)\| < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}},$$

proving (3.2).

Step 3. Since the series $\sum 1/2^n$ converges, $\{g_n\}$ is a Cauchy sequence, uniformly converging to a continuous function g. Since $\varphi(\gamma(t))$ is closed, by (3.1) it follows that $g(t) \in \varphi(\gamma(t))$, for every $t \in I$. Thus, the proof is completed. $\square$

Remark 3.4. In proof of Proposition 3.3, more than to prove that $\varphi(X)$ is pathwise connected, we prove that for any path $\gamma : I \to X$ in X there exists a path $g : I \to \varphi(X)$ such that $g(t) \in \varphi(\gamma(t))$, for every $t \in I$. Thus, the graph Γ_φ is pathwise connected too.

Lemma 3.5. *Let X, Y be Hausdorff spaces and $\varphi : X \multimap Y$ a u.s.c. multi-valued map. Then, for every convergent nets $(x_\alpha)_{\alpha\in J} \to x$ in X and $(y_\alpha)_{\alpha\in J} \to y$ in Y such that $y_\alpha \in \varphi(x_\alpha)$ for any $\alpha \in J$, we have $y \in \varphi(x)$.*

A continuous function $p : X \to Y$ is said perfect if it is closed, surjective and $p^{-1}(y)$ is compact, for each $y \in Y$.

Lemma 3.6 ([5], pp. 260). *Let $p : X \to Y$ be a perfect function. If Y is paracompact, so is X.*

Lemma 3.7. *Let $\varphi : X \multimap Y$ an u.s.c. multi-valued map. Then, the projection $p : \Gamma_\varphi \to X$ is a perfect function. In particular, if X is paracompact, so is Γ_φ.*

Proof. It is clear that p is continuous and surjective. Moreover, $p^{-1}(x) = \{x\} \times \varphi(x)$ is compact because $\varphi(x)$ is compact. We need to show that p is closed. Let $F \subset \Gamma_\varphi$ be a closed subset of Γ_φ. Let $\{x_\alpha\}_{\alpha\in A}$ be a net in $p(F)$ and suppose $x_\alpha \to x$. For each $\alpha \in A$, there exists $y_\alpha \in \varphi(x_\alpha)$ such that $(x_\alpha, y_\alpha) \in F$. Since $K = \{x_\alpha\}_{\alpha\in A} \cup \{x\}$ is compact and φ is u.s.c., we have $\varphi(K)$ compact. Thus, the net $\{y_\alpha\}_{\alpha\in A}$ has a convergent subnet in $\varphi(K)$. Without loss of generality, suppose $y_\alpha \to y \in \varphi(K)$. Since φ is u.s.c., it follows that $y \in \varphi(x)$. Now, $(x_\alpha, y_\alpha) \to (x, y) \in \Gamma_\varphi$ and, since F is closed in Γ_φ, it follows that $(x, y) \in F$. Hence, $x \in p(F)$. Therefore, p is closed. □

In [7], we have:

Theorem 3.8. *Let X, Y be Hausdorff paracompact spaces and $p : X \to Y$ a continuous, closed onto map such that $p^{-1}(y)$ is acyclic for every $y \in Y$. Then, the induced homomorphism*

$$p^* : \check{\mathrm{H}}^*(Y, \mathbb{Z}_2) \xrightarrow{\simeq} \check{\mathrm{H}}^*(X, \mathbb{Z}_2)$$

is an isomorphism.

Proof of Theorem 1.3. Let $\varphi : X \multimap \mathbb{R}^k$ be an u.s.c. multi-valued map such that $\varphi(x)$ is convex for every $x \in X$. Let $\tilde{X}$ and $\tilde{A}$ be the subsets defined by

$$\tilde{X} = \{(x, T(x), u, v) \in X^2 \times \mathbb{R}^{2k} \mid u \in \varphi(x),\ v \in \varphi(T(x))\}$$

$$\tilde{A} = \{(x, T(x), u, v) \in A^2 \times \mathbb{R}^{2k} \mid u \in \varphi(x),\ v \in \varphi(T(x))\}.$$

The space $\tilde{X}$ is the graph of the u.s.c. multi-valued map $\Phi : \{(x, T(x)) \mid x \in X\} \to \mathbb{R}^{2k}$ given by

$$\Phi(x, T(x)) = \varphi(x) \times \varphi(T(x))$$

and $\tilde{A}$ is the graph of Φ restrict to the set $\{(a, T(a)) \mid a \in A\}$. By Lemma 3.7, $\tilde{X}$ and $\tilde{A}$ are paracompact spaces. Moreover, since X is pathwise connected and $\Phi(x, T(x))$ is convex for each $x \in X$, by Proposition 3.3 and Remark 3.4, $\tilde{X}$ and $\tilde{A}$ are pathwise connected spaces. The map $\tilde{T} : \tilde{X} \to \tilde{X}$ defined by

$$\tilde{T}(x, T(x), u, v) = (T(x), x, v, u)$$

is a free involution on $\tilde{X}$. Moreover, $\tilde{A}$ is $\tilde{T}$-invariant. Now, using Theorem 3.8, we can prove that $\check{\mathrm{H}}^r(\tilde{A}, \mathbb{Z}_2) = 0$ for $1 \leq r \leq n-2$ and $j^* : \check{\mathrm{H}}^{n-1}(\tilde{X}, \mathbb{Z}_2) \to \check{\mathrm{H}}^{n-1}(\tilde{A}, \mathbb{Z}_2)$ is the null homomorphism, where $j : \tilde{A} \hookrightarrow \tilde{X}$ is the inclusion map. In fact, let $s : \tilde{X} \to X$ be defined by

$$s(x, T(x), u, v) = x$$

and $s|_{\tilde{A}} : \tilde{A} \to A$ be the restriction of s to $\tilde{A}$. Then, s and $s|_{\tilde{A}}$ are continuous, closed and onto maps. Moreover, $s^{-1}(x) = \{(x, T(x))\} \times \varphi(x) \times \varphi(T(x))$, which is acyclic for each $x \in X$. Hence, by Theorem 3.8, $s^* : \check{\mathrm{H}}^*(X, \mathbb{Z}_2) \to \check{\mathrm{H}}^*(\tilde{X}, \mathbb{Z}_2)$ and $(s|_{\tilde{A}})^* : \check{\mathrm{H}}^*(A, \mathbb{Z}_2) \to \check{\mathrm{H}}^*(\tilde{A}, \mathbb{Z}_2)$ are isomorphisms. From the isomorphism $(s|_{\tilde{A}})^*$, it follows that $\check{\mathrm{H}}^r(\tilde{A}, \mathbb{Z}_2) = 0$ for $1 \leq r \leq n-2$. Note that $i \circ (s|_{\tilde{A}}) = s \circ j$, following the commutative diagram

$$\begin{tikzcd}
\check{\mathrm{H}}^{n-1}(\tilde{X}, \mathbb{Z}_2) \arrow[r, "j^*"] & \check{\mathrm{H}}^{n-1}(\tilde{A}, \mathbb{Z}_2) \\
\check{\mathrm{H}}^{n-1}(X, \mathbb{Z}_2) \arrow[r, "i^*"] \arrow[u, "s^*", "\simeq"'] & \check{\mathrm{H}}^{n-1}(A, \mathbb{Z}_2) \arrow[u, "\simeq", "(s|_{\tilde{A}})^*"']
\end{tikzcd}$$

Since $i^* : \check{\mathrm{H}}^{n-1}(X, \mathbb{Z}_2) \to \check{\mathrm{H}}^{n-1}(A, \mathbb{Z}_2)$ is the null homomorphism and s^* and $(s|_{\tilde{A}})^*$ are isomorphisms, it follows that $j^* : \check{\mathrm{H}}^{n-1}(\tilde{X}, \mathbb{Z}_2) \to \check{\mathrm{H}}^{n-1}(\tilde{A}, \mathbb{Z}_2)$ is the null homomorphism.

Finally, define $f : \tilde{X} \to \mathbb{R}^k$ putting

$$f(x, T(x), u, v) = u.$$

If $n \geq k$, by Theorem 1.2, there exists $(x, T(x), u, v) \in \tilde{X}$ such that $f(x, T(x), u, v) = f(\tilde{T}(x, T(x), u, v))$. Thus, $u = v$.

Therefore, $u = v \in \varphi(x) \cap \varphi(T(x))$. $\square$

Example 3.9. Let $T_{2n} = T_1 \sharp \cdots \sharp T_1$ be the $2n$-fold a connected sum of tori $T_1 = S^1 \times S^1$, which is embedded in $\mathbb{R}^3$ symmetrically with respect to the origin. As in example 2.2, we have that for every $\varphi : T_{2n} \multimap \mathbb{R}^2$ u.s.c. with convex values there exists $x \in T_{2n}$ such that $\varphi(x) \cap \varphi(-x) \neq \emptyset$.

3.1. Maximizing simultaneously two related functions.

Let $f : X \times Y \to \mathbb{R}$ be a real map defined in a cartesian product, $X \times Y$, where Y is compact. For each $x \in X$, let $f_x : Y \to \mathbb{R}$ be the map defined by $f_x(y) = f(x, y)$, for every $y \in Y$. Thus, f define a family of maps, $\{f_x : Y \to \mathbb{R}\}_{x \in X}$. If X admits a free involution $T : X \to X$, we ask if there exists a point $x_0 \in X$ such that the maps f_{x_0} and $f_{T(x_0)}$ can be simultaneously maximized, that is, there exists a point $y_0 \in Y$ such that

$$f_{x_0}(y_0) \geq f_{x_0}(y) \quad \text{and} \quad f_{T(x_0)}(y_0) \geq f_{T(x_0)}(y), \quad \text{for every } y \in Y.$$

This problem is related to a Borsuk-Ulam problem for a multivalued map. Namely, let $\alpha : X \to \mathbb{R}$ be the map defined by

$$\alpha(x) = \max f_x(Y), \quad \text{for each } x \in X,$$

and $\varphi : X \multimap Y$ be the multi-valued defined by

$$\varphi(x) = \{y \in Y \mid f_x(y) = \alpha(x)\}.$$

Then, the maps f_{x_0} and $f_{T(x_0)}$ can be simultaneously maximized if and only if $\varphi(x_0) \cap \varphi(T(x_0)) \neq \emptyset$.

Let $Y \subset \mathbb{R}^n$ be a convex subset. A function $f : Y \to \mathbb{R}$ is said quasiconcave if, for each $\lambda \in (0, 1)$, we have $f(\lambda y_1 + (1 - \lambda) y_2) \geq \min\{f(y_1), f(y_2)\}$, for all $y_1, y_2 \in Y$. In particular, if $f : Y \to \mathbb{R}$ is quasiconcave and Y is compact, then the set $\{y \in Y \mid f(y) = \max f(Y)\}$ is a nonempty, compact and convex subset of Y.

In view of Theorem 1.3 and Example 3.9, we can assert that:

Corollary 3.10. *If $Y \subset \mathbb{R}^2$ is a compact and convex subset of $\mathbb{R}^2$ and $f : T_{2n} \times Y \to \mathbb{R}$ is a continuous function such that $f_x : Y \to \mathbb{R}$ is quasiconcave for each $x \in T_{2n}$, then there exists $x_0 \in T_{2n}$ such that f_{x_0} and f_{-x_0} can be simultaneously maximized.*

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