Instituto de Ciências Matemáticas de São Carlos

ISSN - 0103-2577

Eigenvalue Estimates for Distance Matrices Associated with Conditionally Negative Definite Functions on Spheres

VALDIR A. MENEGATTO

Nº 37

NOTAS DO ICMSC
Série Matemática

São Carlos
Mai./1996

RESUMO

Encontramos estimativas para os autovalores de matrizes de interpolação associadas com certas funções condicionalmente negativas definidas em esferas. Como consequência, estimativas sobre o número de condição espectral de tais matrizes são obtidas.

Version of February 7, 1996

EIGENVALUE ESTIMATES FOR DISTANCE MATRICES ASSOCIATED WITH CONDITIONALLY NEGATIVE DEFINITE FUNCTIONS ON SPHERES

VALDIR A. MENEGATTO [1]
ICMSC-Universidade de São Paulo, Caixa Postal 668
13560-970, São Carlos, SP, Brasil
Email:menegatt@icmsc.sc.usp.br

ABSTRACT. For interpolation matrices generated by a certain class of conditionally negative definite functions on spheres, we give a method for obtaining bounds on the eigenvalues of the matrix. As a consequence, estimates on the spectral condition number of the matrix are provided.

1. INTRODUCTION

In the last decade, there has been a significant amount of work on scattered data interpolation on spheres. Among several approaches to interpolation on spheres, the radial basis method proved to be very effective. It can be described as follows: Given data $\alpha_1, \alpha_2 \ldots, \alpha_n \in \mathbb{R}$ at distinct centres $x_1, x_2, \ldots, x_n$ over the unit sphere S^m in $\mathbb{R}^{m+1}$ and a continuous function $g : [0, \pi] \to \mathbb{R}$, one wants to find an interpolant of the form

$$s(x) = \sum_{j=1}^{n} c_j g(d_m(x, x_j))$$

satisfying the interpolation conditions $s(x_i) = \alpha_i$, $1 \leq i \leq n$. In the above expression, d_m denotes the great-circle or geodesic distance on S^m, *i.e.*,

$$d_m(x, y) = \arccos\langle x, y\rangle, \quad x, y \in S^m,$$

in which $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^{m+1}$. The interpolation problem has a unique solution if and only if the interpolation matrix with ij-entry given by $g(d_m(x_i, x_j))$ is invertible.

[1] Partially supported by CNPq-Brazil

Investigation of this method was started by Cheney and Light ([8]) and Xu and Cheney ([19]) and then further developed by several authors. The survey paper [5] resumes the theoretical results in this area.

We will be specially concerned with this type of interpolation, using conditionally negative definite functions on S^m. A continuous function g is conditionally negative definite on S^m if and only if

$$\sum_{i,j=1}^{n} c_i c_j g(d_m(x_i, x_j)) \leq 0$$

for all $n \geq 2$, $\{x_1, x_2, \ldots, x_n\} \subset S^m$, and $\{c_1, c_2, \ldots, c_n\} \subset \mathbb{R}$ with $\sum_{i=1}^{n} c_i = 0$. These functions are representable in the form ([11])

$$g(t) = g(0) + \sum_{k=1}^{\infty} a_k(1 - p_k^{\lambda}(\cos t)), \quad \lambda = (m-1)/2, \quad a_k \geq 0, \quad \sum_{k=1}^{\infty} a_k < \infty, \tag{1.1}$$

where $p_k^{\lambda}(\cdot)$ are Gegenbauer polynomials normalized by $p_k^{\lambda}(1) = 1$. Thus, $p_k^{\lambda}(\cdot) = P_k^{\lambda}(\cdot)/P_k^{\lambda}(1)$, where P_k^{λ} denotes a standard Gegenbauer polynomial as defined in [18]. The interpolation matrices produced by this type of function are, up to constants, Euclidean distance matrices ([3, Proposition 3.3.2]). Recall that a matrix $D = (D_{ij})$ of order n is a *Euclidean distance matrix* provided there exist points $y_1, y_2, \ldots y_r$ in $\mathbb{R}^r$, $(r \leq n-1)$ such that $D_{ij} = ||y_i - y_j||_2^2$, $1 \leq i, j \leq n$ ([6]).

If g is strictly conditionally negative definite, *i.e.*, if the above inequalities are strict whenever at least one of $c_1, c_2, \ldots, c_n$ does not vanish, and if, in addition, it is nonnegative then the corresponding interpolation problem has a unique solution for all $n \geq 2$. The corresponding interpolation matrices have one positive eigenvalue and $n-1$ negative ones.

A sufficient condition in order that a function g be strictly conditionally negative definite on S^m is that its series representation as above be such that $a_k > 0$ for arbitrarily long sequences of consecutive evens and of consecutive odds. This follows from [15, Theorem 6.9]. By a theorem of Menegatto ([12]), a function of the form $F \circ g$, in which $F : [0, \infty) \to [0, \infty)$ is continuous and F' is nonconstant and completely monotonic and g is conditionally negative definite with $g(t) > 0$ for $t > 0$, is strictly conditionally negative definite on S^m. In particular, any function F as above is itself strictly conditionally negative definite on S^m.

In order to quantify the interpolation method, estimations on the condition number of the interpolation matrices are needed. Our purpose in writing this paper is to obtain eigenvalue estimates for interpolation matrices associated with conditionally negative definite functions f for which $f \circ d_m$ is, up to a nonnegative constant, a metric transform embeddable in Hilbert space. By a result of

Bochner ([4]), such functions have a representation in the form

$$f(t) = f(0) + \sqrt{\sum_{k=1}^{\infty} a_k(1 - p_k^\lambda(\cos t))}, \quad f(0) \geq 0, \quad a_k \geq 0, \quad \sum_{k=1}^{\infty} a_k < \infty. \tag{1.2}$$

If $a_k > 0$ for at least one odd k, then $f(t) > f(0)$ for $t > 0$, and therefore f is strictly conditionally negative definite on S^m.

Due to the nature of our method, the estimates we obtain only depend on the maximum and minimum geodesic separation of the centres. For a given set of centres $\{x_1, x_2, \ldots x_n\} \subset S^m$, the minimum geodesic separation is defined as

$$\min\{d_m(x_i, x_j),\ 1 \leq i < j \leq n\}$$

and the maximum geodesic separation as

$$\max\{d_m(x_i, x_j),\ 1 \leq i < j \leq n\}.$$

The boundedness of S^m allows us to obtain estimates that are independent of the number of centres.

The paper is organized as follows. In Section 2, we derive upper bounds on the absolute value of the eigenvalues. Initially, based upon a result in ([10]) we obtain an upper bound for matrices associated with a general conditionally negative definite function, depending only on the minimum geodesic separation. Bounds associated with functions as in (1.2) are then obtained, but they depend on both the maximum and minimum geodesic separation. In Section 3,we obtain a lower bound for the eigenvalues associated with strictly conditionally negative definite functions representable as in (1.2). Once again, a dependence on both geodesic separations exist. In Section 4, we apply our results to some concrete examples.

2. Upper estimates on the eigenvalues

We begin by obtaining upper bounds on the eigenvalues for matrices associated with a general conditionally negative definite function on S^m. Precisely, we estimate the spectral radius of the interpolation matrix. The idea of the proof of the theorem is from [10]. Recall that the *spectral radius* of an $n \times n$ matrix A is the number $\rho(A)$ given by

$$\rho(A) = \max\{|\alpha| \ : \alpha \text{ is an eigenvalue of } A\}.$$

For normal matrices, the spectral radius coincides with the numerical radius and with the spectral norm of the matrix, *i.e.*,

$$\rho(A) = \max\{|c^*Ac| \ : \ ||c||_2 = 1\} = \max\{\sqrt{\alpha} \ : \ \alpha \text{ is an eigenvalue of } A^*A\}. \tag{2.1}$$

In addition, the following inequality relates $\rho(A)$ to the ℓ^2-norm $||A||_2$ of A:

$$\frac{1}{\sqrt{n}}||A||_2 \leq \rho(A) \leq ||A||_2. \tag{2.2}$$

We observe that the upper bound above is sharp but the lower bound is not. The reader is invited to consult [7, Chapter 5] for a complete discussion on matrix norms.

Theorem 2.1. *Let m be a positive integer and set $\lambda = (m-1)/2$. Let g be a function as in (1.1). If $x_1, \ldots, x_n$ are distinct points on S^m and A is the matrix with entries $A_{ij} = g(d_m(x_i, x_j))$, then the following estimate holds:*

$$\rho(A) \leq n \left(|g(0)| + 2\sum_{k=1}^{\infty} a_k \right)$$

Proof: For $c \in \mathbb{R}^n$ we have that

$$\begin{aligned} |c^t A c| &= \left| \sum_{i,j=1}^{n} c_i c_j \left(g(0) + \sum_{k=1}^{\infty} a_k \left(1 - p_k^\lambda(\cos d_m(x_i, x_j)) \right) \right) \right| \\ &\leq \left| \sum_{i,j=1}^{n} c_i c_j \left(g(0) + \sum_{k=0}^{\infty} a_k \right) \right| + \left| \sum_{i,j=1}^{n} c_i c_j \sum_{k=1}^{\infty} a_k p_k^\lambda(\cos(d_m(x_i, x_j))) \right|. \end{aligned}$$

Recalling Schoenberg's characterization of positive definite functions on spheres ([16]), we obtain

$$|c^t A c| \leq \sum_{i,j=1}^{n} c_i c_j \left(|g(0)| + \sum_{k=0}^{\infty} a_k + \sum_{k=1}^{\infty} a_k p_k^\lambda(\langle x_i, x_j \rangle) \right).$$

Now, we require the addition theorem for spherical harmonics ([13]):

$$p_k^\lambda(\langle x, y \rangle) = \frac{\omega_{m+1}}{P_k^\lambda(1) N_k^{m+1}} \sum_{l=1}^{N_k^{m+1}} S_k^l(x) S_k^l(y).$$

In this formula, $\{S_k^j, 1 \leq j \leq N_k^{m+1}\}$ is an orthonormal set of spherical harmonics of order k and dimension $m+1$, and ω_{m+1} is the surface area of S^m. Defining $a_0 := |g(0)| + \sum_{k=0}^{\infty} a_k$ and using the addition theorem and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |c^t A c| &= \sum_{k=0}^{\infty} a_k \sum_{i,j=1}^{n} c_i c_j p_k^\lambda(\langle x_i, x_j \rangle) \\ &= \sum_{k=0}^{\infty} a_k \frac{\omega_{m+1}}{P_k^\lambda(1) N_k^{m+1}} \sum_{i,j=1}^{n} c_i c_j \sum_{l=1}^{N_k^{m+1}} S_k^l(x_i) S_k^l(x_j) \\ &= \sum_{k=0}^{\infty} a_k \frac{\omega_{m+1}}{P_k^\lambda(1) N_k^{m+1}} \sum_{l=1}^{N_k^{m+1}} \left(\sum_{i=1}^{n} c_i S_k^l(x_i) \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq \|c\|_2^2 \sum_{k=0}^{\infty} a_k \frac{\omega_{m+1}}{P_k^{\lambda}(1) N_k^{m+1}} \sum_{l=1}^{N_k^{m+1}} \sum_{i=1}^{n} (S_k^l(x_i))^2 \\
&= n\, \|c\|_2^2 \sum_{k=0}^{\infty} a_k \\
&= n\, \|c\|_2^2 \left(|g(0)| + 2 \sum_{k=1}^{\infty} a_k \right).
\end{aligned}$$

The result now follows from (2.1). □

The elimination of n in the estimate above can be done using the boundedness of S^m. The precise statement is as follows.

Corollary 2.2. *Let m, g, and A be as in the previous theorem. If the points $x_1, \ldots, x_n$ have minimum geodesic separation ϵ then*

$$\rho(A) \leq \csc^{m+1} \frac{\epsilon}{2} \left(|g(0)| + 2 \sum_{k=0}^{\infty} a_k \right).$$

Proof: If two points have geodesic distance at least ϵ then their Euclidean distance is at least $2 \sin \epsilon/2$. Since any two points on S^m have euclidean distance equal to 2, a subset of S^m with minimum geodesic separation ϵ contains at most

$$\left(\frac{2}{2 \sin \epsilon/2} \right)^{m+1} = \frac{1}{\sin^{m+1} \epsilon/2}$$

points. Thus, the result follows from the previous theorem. □

Although the previous theorems hold for general conditionally negative definite functions, their application to functions representable as in (1.2) may be complicated. Next, we try to minimize this possible complication. Several technical lemmas are needed. The first one concerns the behavior of certain constants that appear in our estimates. For $k \in \mathbb{N} \setminus \{0\}$ and $\lambda > 0$ we define

$$c_{k,\lambda} := \frac{\Gamma(k/2 + \lambda)}{\Gamma(\lambda)\Gamma(k/2 + 1)},$$

where $\Gamma(\cdot)$ is the usual gamma function. An interesting observation to be used later is that if $\lambda = (m-1)/2 > 0$ then $P_k^{\lambda}(1) = c_{2k,2\lambda}$, $k \geq 1$.

Lemma 2.3. *Let $m \in \mathbb{N} \setminus \{0, 1\}$ and set $\lambda = (m-1)/2$. Then, $c_{2k,2\lambda} > c_{k,\lambda}$, $\quad k \in \mathbb{N} \setminus \{0\}$.*

Proof: We show that $c_{k,\lambda}/c_{2k,2\lambda} < 1$ under the given hypotheses. Using the Gauss multiplication formula ([1])

$$\sqrt{\pi}\,\Gamma(2t) = 2^{2t-1}\,\Gamma(t)\,\Gamma(t+\frac{1}{2}), \quad 2t \neq 0,-1,-2,\ldots$$

we have that

$$\begin{aligned}
\frac{c_{k,\lambda}}{c_{2k,2\lambda}} &= \frac{\Gamma(k/2+\lambda)\,\Gamma(2\lambda)\,\Gamma(k+1)}{\Gamma(\lambda)\,\Gamma(k/2+1)\,\Gamma(2\lambda+k)} \\
&= \frac{\Gamma(1/2+\lambda)\,\Gamma(k+1)}{2^k\,\Gamma(k/2+1)\,\Gamma((k+1)/2+\lambda)} \\
&= \frac{\Gamma(1/2+\lambda)\,\Gamma(k+2)}{2^k(k+1)\,\Gamma(k/2+1)\,\Gamma((k+1)/2+\lambda)} \\
&= \frac{2\,\Gamma(1/2+\lambda)\,\Gamma((k+1)/2+1)}{\sqrt{\pi}\,(k+1)\,\Gamma((k+1)/2+\lambda)} \\
&= \frac{\Gamma(1/2+\lambda)\,\Gamma((k+1)/2)}{\sqrt{\pi}\,\Gamma((k+1)/2+\lambda)}.
\end{aligned}$$

Now, we use induction on m. If $m = 2, 3$, we consider two cases. If, on the one hand, $k = 1$, then we have that

$$\frac{c_{1,1/2}}{c_{2,1}} = \frac{\Gamma(1)^2}{\sqrt{\pi}\,\Gamma(3/2)} = \frac{2}{\pi} < 1,$$

and

$$\frac{c_{1,1}}{c_{2,2}} = \frac{\Gamma(3/2)\,\Gamma(1)}{\sqrt{\pi}\,\Gamma(2)} = 1/4 < 1.$$

On the other hand, if $k \geq 2$, then

$$\frac{c_{k,1/2}}{c_{2k,1}} = \frac{\Gamma((k+1)/2)}{\sqrt{\pi}\,\Gamma(k/2+1)} < \frac{1}{\sqrt{\pi}} < 1,$$

and

$$\frac{c_{k,1}}{c_{2k,2}} = \frac{\Gamma((k+1)/2)}{2\,\Gamma((k+3)/2)} < 1/2 < 1,$$

because Γ is increasing on $[3/2,\infty)$. Next, assume that $c_{k,\lambda}/c_{2k,2\lambda} < 1$ for $m = 2, 3, \ldots, l$, and consider $\lambda = (l+1-1)/2 = l/2$. Then,

$$\begin{aligned}
\frac{c_{k,\lambda}}{c_{2k,2\lambda}} &= \frac{\Gamma(+3/2+\lambda-1)\,\Gamma((k+1)/2)}{\sqrt{\pi}\,\Gamma((k+3)/2+\lambda-1)} \\
&= \frac{(1/2+\lambda-1)\,\Gamma(1/2+\lambda-1)\,\Gamma((k+1)/2)}{\sqrt{\pi}\,((k+1)/2+\lambda-1)\,\Gamma((k+1)/2+\lambda-1)} \\
&< \frac{\Gamma(1/2+\lambda-1)\,\Gamma((k+1)/2)}{\sqrt{\pi}\,\Gamma((k+1)/2+\lambda-1)} \\
&= \frac{c_{k,\lambda-1}}{c_{2k,2\lambda-2}},
\end{aligned}$$

and the result follows. □

Lemma 2.4. *Let $m \in \mathbb{N} \setminus \{0,1\}$ and set $\lambda = (m-1)/2$. Then the Gegenbauer polynomials satisfy the following inequality on the interval $[-1,1]$:*

$$|P_k^\lambda(t)| \leq c_{2k,2\lambda}\, t^2 + c_{k,\lambda}\,(1-t^2), \quad k \geq 2.$$

Proof: The inequality is due to Lohöfer ([9]) who proved an even sharper inequality holding for general Gegenbauer functions. □

Lemma 2.5. *Let $m \in \mathbb{N} \setminus \{0,1\}$ and set $\lambda = (m-1)/2$. If $e, \delta \in (0,\pi]$ and $x, y \in S^m$ are such that $\epsilon \leq d_m(x,y) \leq \delta$ then*

$$1 - p_k^\lambda(\cos d_m(x,y)) \leq \left(1 + \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) + \left(1 - \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) \max\{|\cos\epsilon|, |\cos\delta|\}.$$

Proof: First observe that for $t \in [-1,1]$, the following inequality holds

$$|P_k^\lambda(t)| \leq c_{2k,2\lambda}\,|t| + c_{k,\lambda}\,(1-|t|), \quad k \geq 1.$$

It is obvious when $k = 1$. If $k > 1$, it follows from the inequality

$$c_{2k,2\lambda}\, t^2 + c_{k,\lambda}\,(1-t^2) \leq c_{2k,2\lambda}\,|t| + c_{k,\lambda}\,(1-|t|), \quad |t| \leq 1.$$

and Lemma 2.4. The previous inequality follows from Lemma 2.3 and direct computation. The desired inequality can now be obtained

$$\begin{aligned} 1 - p_k^\lambda(\cos d_m(x,y)) &\leq 1 + \frac{c_{2k,2\lambda}\,|\cos d_m(x,y)| + c_{k,\lambda}\,(1 - |\cos d_m(x,y)|)}{c_{2k,2\lambda}} \\ &= \left(1 + \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) + \left(1 - \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) |\cos d_m(x,y)| \\ &\leq \left(1 + \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) + \left(1 - \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) \max\{|\cos\epsilon|, |\cos\delta|\}. \end{aligned}$$

□

Lemma 2.6. *Let A and B be Hermitian matrices of order n and let their eigenvalues be arranged in increasing order. If $\alpha_j(A)$, $\alpha_j(B)$, and $\alpha_j(A+B)$ denote the j-th eigenvalues of A, B, and $A+B$ respectively, according to this arrangement, then*

$$\alpha_j(A) + \alpha_1(B) \leq \alpha_j(A+B) \leq \alpha_j(A) + \alpha_n(B), \quad 1 \leq j \leq n.$$

Proof: This is a direct application of the Courant-Fisher theorem. See [7]. □

Now, we are in position to state and prove the main result of this section.

Theorem 2.7. *Let $m \in \mathbb{N} \setminus \{0,1\}$ and set $\lambda = (m-1)/2$. Let f be a function as in (1.2). Let $x_1, x_2, \ldots, x_n$ be distinct points on S^m with minimum geodesic separation ϵ and maximum geodesic separation δ. If A is the matrix with entries $A_{ij} = f(d_m(x_i, x_j))$ then*

$$\rho(A) \leq f(0) + \csc^{m+1}\frac{\epsilon}{2}\sqrt{\sum_{k=1}^{\infty} a_k\left(1+\frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) + \max\{|\cos\epsilon|, |\cos\delta|\}\sum_{k=1}^{\infty} a_k\left(1-\frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right)}.$$

Proof: We prove the theorem by estimating $||A||_2$. We first handle the case when $f(0) = 0$. In this case, the i-th element in the main diagonal of A^2 is given by

$$\sum_{j=1}^{n} A_{ij}^2 = \sum_{j=1}^{n}\sum_{k=1}^{\infty} a_k(1 - p_k^\lambda(\cos d_m(x_i, x_j))),$$

so

$$\text{trace } A^2 = \sum_{k=1}^{\infty} a_k \sum_{i,j=1}^{n}(1 - p_k^\lambda(\cos d_m(x_i, x_j))).$$

By the previous lemma, we deduce that

$$\begin{aligned} \text{trace } A^2 &\leq \sum_{k=1}^{\infty} a_k \sum_{i,j=1}^{n}\left(\left(1+\frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) + \left(1-\frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right)\max\{|\cos\epsilon|, |\cos\delta|\}\right) \\ &= n^2\left(\sum_{k=1}^{\infty} a_k\left(1+\frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) + \max\{|\cos\epsilon|, |\cos\delta|\}\sum_{k=1}^{\infty} a_k\left(1-\frac{c_{k,l}}{c_{2k,2\lambda}}\right)\right). \end{aligned}$$

Therefore,

$$||A||_2 \leq n\sqrt{\sum_{k=1}^{\infty} a_k\left(1+\frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) + \max\{|\cos\epsilon|, |\cos\delta|\}\sum_{k=1}^{\infty} a_k\left(1-\frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right)}.$$

If $f(0) \neq 0$, we write $A_{ij} = f(0) + B_{ij}$, in which $B_{ij} = f(d_m(x_i, x_j)) - f(0)$. Combining Lemma 2.6 and the previous part we then see that an upper bound for the biggest eigenvalue of A is

$$f(0) + n\sqrt{\sum_{k=1}^{\infty} a_k\left(1+\frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right) + \max\{|\cos\epsilon|, |\cos\delta|\}\sum_{k=1}^{\infty} a_k\left(1-\frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right)}.$$

The theorem now follows from the proof of Corollary 2.2. □

Remark. In practice, the bound given by Theorem 2.7 may be not so easy to handle. If we use the fact that $\max\{|\cos\epsilon|, |\cos\delta|\} \leq 1$ we obtain the bound

$$f(0) + \csc^{m+1}\frac{\epsilon}{2}\sqrt{2\sum_{k=1}^{\infty} a_k}$$

which is of same nature of that presented in Corollary 1.2.

3. Lower estimates on the eigenvalues

In this section, we obtain lower bounds on the eigenvalues for matrices associated with strictly conditionally negative definite functions having a representation as in (2.1). The procedure here is similar to the one used in the previous section. We begin obtaining lower estimates for the Gegenbauer polynomials similar to those obtained in Lemma 2.4.

Lemma 3.1. *Let $m \in \mathbb{N} \setminus \{0,1\}$ and set $\lambda = (m-1)/2$. If $\epsilon, \delta \in (0,\pi]$ and $x, y \in S^m$ are such that $\epsilon \leq d_m(x,y) \leq \delta$ then*

$$1 - p_k^\lambda(\cos d_m(x,y)) \geq \left(1 - \frac{c_{k,\lambda}}{c_{2k,2l}}\right)(1 - \max\{|\cos\epsilon|, |\cos\delta|\}, \quad k \geq 1.$$

Proof: We use the same estimates we have obtained in the proof of Lemma 2.4. They lead to

$$\begin{aligned} 1 - p_k^\lambda(\cos d_m(x,y)) &\geq \frac{c_{2k,2\lambda} - c_{2k,2\lambda}|\cos d_m(x,y)| - c_{k,\lambda} + c_{k,\lambda}|\cos d_m(x,y)|}{c_{2k,2\lambda}} \\ &= \left(1 - \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right)(1 - |\cos d_m(x,y)|) \\ &\geq \left(1 - \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right)(1 - \max\{|\cos\epsilon|, |\cos\delta|\}) \end{aligned}$$

□

The main theorem of this section is as follows.

Theorem 3.2. *Let $m \in \mathbb{N} \setminus \{0,1\}$ and set $\lambda = (m-1)/2$. Let f be a function as in (1.2) satisfying the additional requirement $\sum_{k=0}^{\infty} a_{2k+1} > 0$. Let $x_1, \ldots, x_n$ be distinct points on S^m with minimum geodesic separation ϵ and maximum geodesic separation δ. If A is the matrix with entries $f(d_m(x_i, x_j))$ then*

$$\rho(A) \geq f(0) + \sqrt{n(1 - \max\{|\cos\epsilon|, |\cos\delta|\})\sum_{k=1}^{\infty} a_k \left(1 - \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right)}.$$

Proof: First, we assume $f(0) = 0$. Proceding as in the proof of Theorem 2.7 but using Lemma 3.1 instead, we obtain

$$||A||_2 \geq n \sqrt{(1 - \max\{|\cos \epsilon|, |\cos \delta|\}) \sum_{k=1}^{\infty} a_k \left(1 - \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right)}.$$

Hence , by (2.2),

$$\rho(A) \geq \sqrt{n(1 - \max\{|\cos \epsilon|, |\cos \delta|\}) \sum_{k=1}^{\infty} a_k \left(1 - \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right)}.$$

If $f(0) \neq 0$, we combine the previous part with Lemma 2.6 to obtain the result. □

Remarks. i) The previous theorem does not apply when $\delta = \pi$ due to the symmetric nature of the inequality in Lemma 2.4. This drawback may be fixed with the help of a different inequality or perhaps with a different line of proof.

ii) Another procedure to obtain lower bounds is as follows: write the radicand in the expression defining A as a regular Euclidean distance matrix and then use the eigenvalue estimates given in either [2] or [17]. For instance, using the estimates presented in [2], one gets the following lower bound for the eigenvalues of A

$$f(0) + \frac{0.35...}{\sqrt{n'}} \sqrt{(1 - \max\{|\cos \epsilon|, |\cos \delta|\}) \sum_{k=1}^{\infty} a_k \left(1 - \frac{c_{k,\lambda}}{c_{2k,2\lambda}}\right)}.$$

Here, n' is the smallest odd bigger than or equal to n. Unfortunately, this bound is not as sharp as that one obtained in the previous theorem.

4. Estimates on the spectral condition number: two examples

In this section, we fix $m = 2$ and use the eigenvalue estimates obtained in earlier sections to estimate the spectral condition number of interpolation matrices A associated with the functions $f_1(t) = 2\sin t/2$ and $f_2(t) = \sqrt{t}$. Before doing that, recall that for invertible normal matrices, the spectral condition number for inversion is given by

$$\text{cond } A = \rho(A)\rho(A^{-1}).$$

Thus, assuming that $f(0) = 0$ and employing the notation in Theorem 3.2, we have that

$$\text{cond } A \leq \csc^{(m+1)/2} \epsilon/2 \; \frac{(1 + c_{k,\lambda}/c_{2k,2\lambda}) + \max\{|\cos \epsilon|, |\cos \delta|\}(1 - c_{k,\lambda}/c_{2k,2\lambda})}{(1 - c_{k,\lambda}/c_{2k,2\lambda}) - \max\{|\cos \epsilon|, |\cos \delta|\}(1 - c_{k,\lambda}/c_{2k,2\lambda})}.$$

We start with the function f_1. It is interesting to observe that the metric transform $f_1 \circ d_m$ is the Euclidean chord distance. Hence, eigenvalue and condition number estimates can be obtained using the results contained in [2] and in [17]. To obtain estimates using our results, we proceed as follows. From elementary trigonometry, we have that

$$2\sin\frac{t}{2} = \sqrt{2\,(1-\cos t)}, \quad 0 \le t \le \pi,$$

so that f_1 meets the requirements of Theorems 2.7 and 3.2. From theorem 2.7, we have that

$$\begin{aligned}\rho(A) &\le n\sqrt{2}\sqrt{\left(1+\frac{c_{1,1}}{c_{2,2}}\right)+\max\{|\cos\epsilon|,|\cos\delta|\}\left(1-\frac{c_{1,1}}{c_{2,2}}\right)}\\ &= n\sqrt{2}\sqrt{\frac{\pi+2}{\pi}+\max\{|\cos\epsilon|,|\cos\delta|\}\frac{\pi-2}{\pi}}.\end{aligned}$$

On the other hand, Theorem 3.2 gives us

$$\rho(A) \ge \sqrt{2n}\sqrt{\frac{\pi-2}{\pi}-\max\{|\cos\epsilon|,|\cos\delta|\}\frac{\pi-2}{\pi}}.$$

Thus,

$$\text{cond } A \le \csc^{(m+1)/2}\epsilon/2\sqrt{\frac{\pi+2+(\pi-2)\max\{|\cos\epsilon|,|\cos\delta|\}}{\pi-2-(\pi-2)\max\{|\cos\epsilon|,|\cos\delta|\}}}.$$

For the function f_2, one has to use the following series ([14, p 702])

$$\arcsin x = \sum_{k=0}^{\infty} b_{2k+1}P_{2k+1}^{1/2}(x), \quad |x| \le 1,$$

in which

$$b_{2k+1} = \frac{\pi(4k+3)}{8}\left(\frac{(1/2)_k}{(k+1)!}\right)^2.$$

In the above expression $(\cdot)_k$ denotes the Pochhammer symbol. Changing x into $\cos t$ and arranging we obtain

$$f_2(t) = \sqrt{\sum_{k=0}^{\infty} b_{2k+1}(1-p_{2k+1}^{1/2}(\cos t))},$$

so that f_2 meets the requirements of our theorems. Proceding as above we obtain the following bound

$$\text{cond } A \le \csc^{(m+1)/2}\epsilon/2\left[\pi(1-\max\{|\cos\epsilon|,|\cos\delta|\})\sum_{k=0}^{\infty} b_{2k+1}\left(1-\frac{c_{2k+1,1/2}}{c_{4k+2,1}}\right)\right]^{-1/2}.$$

References

1. Artin, E., The Gamma Function, Holt, Rinehart and Winston, New York, 1964.
2. Ball, K., *Eigenvalues of Euclidean distance matrices*, J. Approx. Theory, **68** (1992), 74-82.

3. Berg, C, J. P. R. Christensen, and P. Ressel, Harmonic Analysis on Semigroups - Theory of Positive Definite and Related Functions, Springer-Verlag, New York, 1984.
4. Bochner, S. *Hilbert distances and positive definite functions*, Ann. Math., **42** (1941), 647-656.
5. Cheney, E. W., *Approximation using positive definite functions*, in Approximation Theory VIII, C. K. Chui and L. L. Schumaker (eds), (1995), 1-24.
6. Hayden, T. L., J. Wells, W. Liu, and P. Tarazaga, *The cone of distance matrices*, Linear Algebra and its Appl., **144** (1991), 153-169.
7. Horn, R. A. and C. A. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
8. Light, W. A. and E. W. Cheney, *Interpolation by periodic radial basis functions*, J. Math. Analysis and Appl., **168** (1992), 35-48.
9. Lohöfer, G., *Inequalities for Legendre functions and Gegenbauer functions*, J. Approx. Theory, **64** (1991), 226-234.
10. Levesley, J., Z. Luo, and X. Sun, *Norm estimates of interpolation matrices and their inverses associated with strictly positive definite functions on spheres*, preprint.
11. Menegatto, V. A., Interpolation on Spherical Domains, PhD Dissertation, The University of Texas at Austin, Austin, 1992.
12. Menegatto, V. A., *Strictly positive definite kernels on the Hilbert sphere*, Appl. Anal., **55** (1994), 91-101.
13. Müller, C., Spherical Harmonics, Lecture Notes in Mathematics, Vol. 17, Springer-Verlag, Berlin, 1966.
14. Prudnikov, A. P. at al., Integrals and Series, Vol. II, Gordon and Breach Science Publishers, New York, 1986.
15. Ron, A. and X. Sun, *Strictly positive definite functions on spheres*, to appear, Math. Comp..
16. Schoenberg, I. J., *Positive definite functions on sphere*, Duke Math. J., **44** (1938), 522-536.
17. Sun, Xingping, *Norm estimates for inverses of Euclidean distance matrices*, J. Approx. Theory, **70** (1992), 339-347.
18. Szegö, G., Orthogonal Polynomials, Amer. Math. Soc. Publ., Vol. XXIII, New York, Revised Edition, 1959.
19. Xu, Y. and E. W. Cheney, *Strictly positive definite functions on spheres*, Proc. Amer. Math. Soc., **116** (1992), 977-981.

NOTAS DO ICMSC

SÉRIE MATEMÁTICA

036/96 CARTER, J.S.; RIELGER, J.H.; SAITI, M. - A combinatorial description of knotted surfaces and their isotopies.

035/96 CARVALHO, A.N.; DLOTKO, T. - Parabolic problems in H'with fast grawing nonlinearities

034/96 KUSHNER, A.L. - Finite determination on algebraic sets.

033/95 CARVALHO, A.N.; OLIVA, S.M.; PEREIRA, A.L.; BERNAL, A.R. - Attractors for parabolic problem with nonlinear baundary conditions.

032/95 GIBSON, C.G.; HOBBS, C.A.; MARAR, W.L. - On the bifurcation of general two-dimensional spatial motions[1]

031/95 LUCAS, L.A., MANZOLI-NETO, O. and SAEKI, O. - An unknotting theorem for $S^p \times S^q$ EMBEDDED IN S^{P+q+2}

030/95 BIASI, C. and GODOY, S.M.S. - On the aleatory variable $Y_n = 10^n X - [10^n X]$ for large n

029/95 CARVALHO, A.N., CHOLEWA, J.W. and DLOTKO, T. - Ezamples of global attractors in parabolica problems

028/95 LUCAS, L.A., MANZOLI-NETO, O. and SAEKI, O. - A generalization of Alexander's torus theorem to higher dimensions

027/95 BIASI, C. and GONÇALVES, D.L. - Equivalent conditions for m-equivalences module a class C of groups