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RESUMO

Funções completamente monotônicas e funções de Bernstein são sabidamente boas escolhas de funções que produzem soluções únicas em processos interpolatórios que usam bases de funções radiais. Neste trabalho, mostramos que tais funções, bem como extensões (complexas) contínuas das mesmas, podem ser empregadas em interpolação na esfera de Hilbert complexa.

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INTERPOLATION ON THE COMPLEX HILBERT SPHERE USING POSITIVE DEFINITE AND CONDITIONALLY NEGATIVE DEFINITE KERNELS

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ABSTRACT

We show that completely monotone functions, Bernstein functions, and continuous extensions of these two types of functions, can be used for interpolation to scattered data over the complex Hilbert sphere.

1. INTRODUCTION

Let S^∞ be the unit sphere of the complex Hilbert space ℓ^2. Given any finite subset E of S^∞, we show how to construct large classes of continuous functions g defined on the unit disk $\overline{\Delta(0,1)} := \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$ that solve the problem of interpolating arbitrary data on E by a function in the linear space

$$L_E := \operatorname{span}\{g(\langle \cdot, w\rangle) : \ w \in E\}.$$

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Here, $\langle\cdot,\cdot\rangle$ is the inner product in ℓ^2. The existence of a unique interpolant from L_E is equivalent to the invertibility of the interpolation matrix $A_{E,g}$ with (z,w)–entry given by $g(\langle z,w\rangle)$. A preliminary investigation on this interpolation procedure was already done in [8]. For a wide range of functions g, it was shown there that the interpolation matrices $A_{E,g}$ are positive definite for any given set $E \subset S^\infty$. This interpolation problem is motivated by a form of "radial basis interpolation" on real spheres as surveyed in [3].

Most of the functions g we are going to deal with have the property that the kernel $g\circ\langle\cdot,\cdot\rangle$ is either positive definite or conditionally negative definite on S^∞. Recall that if X is a nonempty set , a function $f: X\times X \to \mathbb{C}$ is called a *positive definite kernel* on X if and only if

$$\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} f(z_\mu, z_\nu) \geq 0$$

for all $n \geq 1$, $\{z_1,\ldots,z_n\} \subset X$, and $\{c_1,\ldots,c_n\} \subset \mathbb{C}$. It is called *conditionally negative definite* on X if and only if it is hermitian, *i.e.*, $f(z,w) = \overline{f(w,z)}$ for all $z,w \in X$, and

$$\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} f(z_\mu, z_\nu) \leq 0$$

for all $n \geq 2$, $\{z_1,\ldots,z_n\} \subset X$, and $\{c_1,\ldots,c_n\} \subset \mathbb{C}$ with $\sum_{\mu=1}^{n} c_\mu = 0$. If the inequalities in the above definitions are strict whenever $\sum_{\mu=1}^{n}|c_\mu| > 0$, then f is called *strictly positive* (respectively *strictly conditionally negative*) *definite* on X. Thus, interpolation matrices $A_{E,g}$ built out of strictly positive definite kernels of the form $g\circ\langle\cdot,\cdot\rangle$ are positive definite and therefore invertible. Those built out of strictly conditionally negative definite kernels may be singular. Meanwhile, if the additional condition $g(1) \geq 0$ holds, then the matrices $A_{E,g}$ have the property that $|\det A_{E,g}| > 0$ and hence they are invertible too.

A characterization of the continuous functions f for which $f\circ\langle\cdot,\cdot\rangle$ is positive definite on S^∞ was given by J. P. R. Christensen and P. Ressel in ([4]). They are expansions of the form

$$f(\zeta) = \sum_{m,n=0}^{\infty} a_{m,n}\zeta^m \overline{\zeta^n},$$

in which $a_{m,n} \geq 0$ and $\sum_{m,n=0}^{\infty} a_{m,n} < \infty$. The continuous functions g for which $g \circ \langle \cdot, \cdot \rangle$ is conditionally negative definite on S^{∞} are of the form

$$g(\zeta) = g(1) + \sum_{m,n=0}^{\infty} a_{m,n} \left(1 - \zeta^m \overline{\zeta^n}\right),$$

in which $a_{m,n} \geq 0$ and $\sum_{m,n=0}^{\infty} a_{m,n} < \infty$. A quick proof of this uses exercise 5-4.10 in [2]. If g has any one of these series representations we write $K(g)$ to denote the set of all pairs (m, n) for which the coefficient $a_{m,n}$ in the series is positive.

We construct strictly positive definite and strictly conditionally negative definite kernels on S^{∞} employing completely monotone functions, Bernstein functions, and continuous extensions of these two types of functions. A $C^{\infty}-$function $F : (0, \infty) \to \mathbb{R}$ is said to be *completely monotone*, if $(-1)^s F^{(s)} \geq 0$ for all integers $s \geq 0$, and is said to be a *Bernstein function*, if $F \geq 0$ and $(-1)^s F^{(s)} \leq 0$ for all integers $s \geq 1$. A theorem of Bernstein ([16]) guarantees that a function F is completely monotone if and only if

$$F(x) = \int_0^{\infty} e^{-xs} d\lambda(s),$$

in which λ is a positive measure on $(0, \infty)$. In particular, $lim_{x \to 0^+} F(x) < \infty$ if and only if λ is a bounded measure. Similarly, a function F is a Bernstein function if and only if

$$F(x) = a + bx + \int_0^{\infty} (1 - e^{-xs}) d\lambda(s),$$

in which $a, b \geq 0$, λ is a positive measure on $(0, \infty)$, and $\int_0^{\infty} s(1+s)^{-1} d\lambda(s) < \infty$. In the above representation we already have $lim_{x \to 0^+} F(x) = a$ and, therefore, a Bernstein function is always continuous at zero. We observe that the representing measure λ for functions in these two classes is uniquely determined.

Completely monotone and Bernstein functions are known to be very useful in problems of multivariate scattered data interpolation. In fact, let g be an appropriate function in either one of these two classes. If $x_1, \ldots, x_n$ are distinct points in $\mathbb{R}^m$ then interpolation of data at the x_μ is always possible by a function of the form

$$\phi(x) = \sum_{\mu=1}^{n} a_\mu g(||x - x_\mu||^2), \quad x \in \mathbb{R}^m,$$

where $||\cdot||$ stands for the usual norm in $\mathbb{R}^m$ ([7], [10]). If the points are on the surface of the unit sphere S^{m-1} of $\mathbb{R}^m$, then interpolation is always possible by a function of the form

$$\phi(x) = \sum_{\mu=1}^{n} c_\mu g(d(x, x_\mu)), \quad x \in S^{m-1},$$

in which d denotes the geodesic distance on S^{m-1} ([9]).

Since $|\exp(-\zeta s)| = \exp(-(\mathrm{Re}\,\zeta)s)$, for $s \geq 0$ and $\zeta \in \mathbb{C}$, and $|1-\exp(-\zeta s)| \leq s|\zeta|$ and $|1-\exp(-\zeta s)| \leq 2$, for $s > 0$ and $\zeta \in \mathbb{C}$, then continuous at zero completely monotone functions and Bernstein functions extend to continuous functions defined in the closed half-plane $\{\zeta \in \mathbb{C} : \mathrm{Re}\zeta \geq 0\}$. Furthermore, these extensions have extended integral representations in this domain. We denote by $\mathcal{CM}$ the set of all nonconstant, continuous at zero completely monotone functions and by $\mathcal{B}$ the set of all Bernstein functions which are neither constant nor linear. In addition, we denote by $\mathcal{CME}$ the set of continuous extensions of functions in $\mathcal{CM}$ and by $\mathcal{BE}$ the set of continuous extensions of functions in $\mathcal{B}$. If F is an element of either one of these four classes and λ is the corresponding (uniquely determined) measure in its integral representation, then we write $F = F_\lambda$.

An outline of the paper is as follows. First, we investigate the invertibility of Schur exponentials of some matrices that are closely related to the interpolation problem introduced above. Using these results and the so-called Oppenheim's inequality, we fix a function of the form $F \circ g$, in which F belong to one of the four classes in the previous paragraph and g is conditionally negative definite on X, and give sufficient conditions to guarantee its strict positive (conditional negative) definiteness on X. Finally, the case in which X is the unit sphere S^∞ is discussed.

2. STRICTLY POSITIVE (CONDITIONALLY NEGATIVE) DEFINITE KERNELS

In this section we first extend to the hermitian case, two results concerning invertibility of Schur exponentials of symmetric matrices. These results are crucial to this work. Using them, we show that completely monotone functions, Bernstein functions, and continuous extensions of both completely monotone and Bernstein functions can be used in multivariate scattered data interpolation problems. The case in which the data is over the complex Hilbert sphere is fully discussed.

Let $A = (A_{\mu\nu})$ be a matrix of order n. The Schur exponential $\exp(A)$ of A is, by definition, the matrix with $(\mu, \nu)-$ entry given by $\exp(A_{\mu\nu})$. The main results in this paper depend on the invertibility of Schur exponentials. It is well known that $\exp(A)$ is nonnegative definite if either A is nonnegative definite or almost nonnegative definite in the sense that it is hermitian and $\sum_{\mu,\nu=1}^{n} c_\mu c_\nu A_{\mu\nu} \geq 0$ when $\sum_{\mu=1}^{n} c_\mu = 0$ ([6], [13]). Thus, to guarantee invertibility of $\exp(A)$ in these two cases, we only need to know under what circumstances the Schur exponentials will be positive definite.

In Lemmas 2.1 and 2.2 below, we present two conditions under which Schur exponentials are positive definite. Before proving these results, we recall two well-known results concerning positive definiteness. The first one, the so-called Oppenheim's inequality ([11]), states that if $A = (A_{\mu\nu})$ and $B = (B_{\mu\nu})$ are nonnegative definite matrices of same order n and $A \circ B$ denotes their Schur product $(A_{\mu\nu}B_{\mu\nu})$, then $\det(A \circ B) \geq A_{11} \cdots A_{nn} \det B$. The second one, a result proved by Schoenberg ([14]) more than half century ago, states that the matrix $(\exp(-||x_\mu - x_\nu||^2))$ is positive definite whenever $x_1, \ldots, x_n$ are distinct points on $\mathbb{R}^m$.

Hereafter, we write $|| \cdot ||$ to denote the usual norm in $\mathbb{C}^n$ and we use the dot notation for the corresponding inner product. If A is a matrix we define $\mathrm{Re}A := (\mathrm{Re}\, A_{\mu\nu})$.

LEMMA 2.1. *Let A be a nonnegative definite matrix. If A has no pair of identical rows then* $\exp(\mathit{Re}\, A)$ *is positive definite.*

PROOF. Let n denote the order of A. Since A is nonnegative definite, it has a nonnegative definite square root B, hence we can write the (μ,ν)-entry $A_{\mu\nu}$ of A in the form $A_{\mu,\nu} = B_\mu.B_\nu$, where B_μ is the μ-th column of B. Using the parallelogram law, we have that

$$||B_\mu - B_\nu||^2 = ||B_\mu||^2 - B_\mu.B_\nu - \overline{B_\mu.B_\nu} + ||B_\nu||^2, \quad 1 \le \mu,\nu \le n,$$

whence

$$e^{\mathrm{Re}(B_\mu.B_\nu)} = e^{\frac{1}{2}||B_\mu||^2} e^{\frac{1}{2}||B_\nu||^2} e^{-\frac{1}{2}||B_\mu - B_\nu||^2}$$

If $c_1,\ldots,c_n \in \mathbb{C}$, the above equality gives us

$$\begin{aligned}\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} e^{\mathrm{Re}(B_\mu.B_\nu)} &= \sum_{\mu,\nu=1}^{n} c_\mu e^{\frac{1}{2}||B_\mu||^2} \overline{c_\nu e^{\frac{1}{2}||B_\nu||^2}} e^{-\frac{1}{2}||B_\mu - B_\nu||^2} \\ &= \sum_{\mu,\nu=1}^{n} c_\mu e^{\frac{1}{2}||B_\mu||^2} \overline{c_\nu e^{\frac{1}{2}||B_\nu||^2}} e^{-\frac{1}{2}||(\mathrm{Re}\,B_\mu,\mathrm{Im}\,B_\mu)-(\mathrm{Re}\,B_\nu,\mathrm{Im}\,B_\nu)||^2}.\end{aligned}$$

If A has no pair of identical rows then $B_\mu \neq B_\nu$ for $\mu \neq \nu$ and, consequently, the n vectors $(\mathrm{Re}\,B_\mu, \mathrm{Im}\,B_\mu)$, $1 \le \mu \le n$, of $\mathbb{R}^{2n}$ are pairwise distinct. Thus, by Schoenberg's result, $\left(\exp(-\frac{1}{2}||(\mathrm{Re}\,B_\mu, \mathrm{Im}\,B_\mu) - (\mathrm{Re}\,B_\nu, \mathrm{Im}\,B_\nu)||^2)\right)$ is positive definite. Thus, the condition $\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} \exp(\mathrm{Re}A_{\mu\nu}) = 0$ implies that $c_\mu \exp(\frac{1}{2}||B_\mu||^2) = 0$, $1 \le \mu \le n$, i.e., $c_1 = \ldots = c_n = 0$. Therefore, $\exp(\mathrm{Re}\,A)$ is positive definite. ■

LEMMA 2.2 *Let A be an almost nonnegative definite matrix. If A has constant diagonal and no pair of identical rows then* $\exp(\mathrm{Re}\,A)$ *is positive definite.*

PROOF. Define $B_{\mu\nu} = A_{\mu\nu} - A_{\mu n} - \overline{A_{\nu n}} + A_{nn}$, $1 \le \mu,\nu \le n$, in which n is the order of A. The matrix B is almost nonnegative definite and the entries in its $n-$th row and $n-$th column equal zero. Hence, B is in fact nonnegative definite. In particular, $\mathrm{Re}\,B$ is nonnegative definite too. Since

$$\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu}\, e^{\mathrm{Re}\,A_{\mu\nu}} = e^{-A_{nn}} \sum_{\mu,\nu=1}^{n} c_\mu e^{\mathrm{Re}\,A_{\mu n}} \overline{c_\nu e^{\mathrm{Re}\,A_{\nu n}}} e^{\mathrm{Re}\,B_{\mu\nu}}$$

and $\exp(\mathrm{Re}\, B)$ is nonnegative definite, it is enough to prove, by the previous lemma, that B has no pair of identical rows under the given hypothesis. But, this is quite evident from the definition of B and the fact that the entries in the main diagonal of A are real. ■

Since the conclusions of the above lemmas refer to real matrices, we first investigate the strict positive (conditional negative) definiteness of real kernels. In what follows, X denotes a nonempty set and Δ_X denotes its diagonal, *i.e.*, $\Delta_X := \{(z,z) : z \in X\}$.

THEOREM 2.3. *Let h be a function of the form $h = F_\lambda \circ g$, in which $F_\lambda \in \mathcal{CM}$ and g is a real conditionally negative definite kernel on X. Let $z_1, \ldots, z_n$ be points on X and form the $n \times n$ matrix A with entries $A_{\mu\nu} = h(z_\mu, z_\nu)$. Then,*
i) A is nonnegative definite;
ii) If the z_μ are distinct and Δ_X is the set of zeros of g then A is positive definite.

PROOF. It suffices to prove the theorem when the measure λ defining F_λ is a Dirac measure at $s > 0$, *i.e.*, when $h = \exp(-sg)$, $s > 0$. In this case, we have that $A_{\mu\nu} = \exp(-sg(z_\mu, z_\nu)$. Since g is conditionally negative definite on X, $(-sg(z_\mu, z_\nu))$ is almost nonnegative definite. Thus, A is nonnegative definite.
If the hypotheses in ii) hold, then $(g(z_\mu, z_\nu))$ has no pair of identical rows. Indeed, if $g(z_\mu, z_\xi) = g(z_\nu, z_\xi)$, for $1 \leq \xi \leq n$ and distinct μ and ν then $g(z_\mu, z_\nu) = 0$. Hence, $(z_\mu, z_\nu) \in \Delta_X$, a contradiction. Thus, $(-sg(z_\mu, z_\nu))$ is almost nonnegative definite, it has no pair of identical rows and its diagonal entries equal 0. Therefore, A is positive definite by Lemma 2.2. ■

THEOREM 2.4. *Let h be a function of the form $h = F_\lambda \circ g$, in which $F_\lambda \in \mathcal{B}$ and g is a real conditionally negative definite kernel on X. Let $z_1, \ldots, z_n, (n \geq 2)$ be points on X and form the $n \times n$ matrix A with entries $A_{\mu\nu} = h(z_\mu, z_\nu)$. Then,*
i) $-A$ is almost nonnegative definite;
ii) If the z_μ are distinct and Δ_X is the set of zeros of g then $-A$ is almost positive definite, i.e., $-\sum_{\mu,\nu=1}^n c_\mu \overline{c_\nu} A_{\mu\nu} > 0$ whenever $\sum_{\mu=1}^n c_\mu = 0 \neq \sum_{\mu=1}^n |c_\mu|$.

PROOF. Once again it suffices to prove the theorem only when the measure λ defining F_λ is a Dirac measure at $s > 0$. In this case, $F_\lambda = a + 1 + bg - \exp(-sg)$, where $a, b \geq 0$ and hence $A_{\mu\nu} = a + 1 + bg(z_\mu, z_\nu) - \exp(-sg(z_\mu, z_\nu))$. As in the proof of the previous theorem, $(\exp(-sg(z_\mu, z_\nu)))$ is nonnegative definite and so, $-A$ is almost nonnegative definite because it is a sum of almost nonnegative definite matrices.
If the hypotheses in ii) hold, then the same argument used in the proof of the previous theorem shows that $(-sg(z_\mu, z_\nu))$ has no pair of identical rows. Since its diagonal entries are all equal to 0, then $(\exp(-sg(z_\mu, z_\nu)))$ is in fact positive definite. Hence, $(\exp(-sg(z_\mu, z_\nu)))$ is almost positive definite and therefore so is A. ■

The matrix $-A$ in Theorem 2.4-ii) is in fact invertible. This follows from the fact that its trace is nonpositive and from an application of the Courant-Fisher Theorem.

COROLLARY 2.5 *Let h be a function as described in Theorem 2.3 (respectively Theorem 2.4). A necessary and sufficient condition in order that h be strictly positive (respectively conditionally negative) definite on X is that Δ_X be the set of zeros of g.*

PROOF. We prove the corollary when h has the form described in Theorem 2.3. The proof in the other case is similar. That the condition is sufficient, follows directly from Theorem 2.3. To prove the necessity of the condition, suppose that Δ_X does not contain all zeros of g. Then, there is a pair $(z_\mu, z_\nu) \in X \times X$, $z_\mu \neq z_\nu$, such that $g(z_\mu, z_\nu) = 0$, whence $h(z_\mu, z_\nu) = F_\lambda(0)$, $1 \leq \mu, \nu \leq 2$. Thus, the 2×2 matrix $(h(z_\mu, z_\nu))$ is not positive definite and, therefore, h is not strictly positive definite on X. ■

Similar results hold for complex kernels. Additional hypotheses are required, but in the case we are mainly interested, *i.e.*, when $X = S^\infty$, these hypotheses are quite natural.

THEOREM 2.6. *Let h be a function of the form $h = F_\lambda \circ g$, in which $F_\lambda \in \mathcal{CME}$ (respectively $F_\lambda \in \mathcal{BE}$) and g is kernel on X. If g has a real summand g_0 such that g_0 and $g - g_0$ are conditionally negative definite on X and such that Δ_X is the set of zeros of g_0, then h is* *strictly positive (respectively conditionally negative) definite on X.*

PROOF. We only prove the theorem when $F_\lambda \in \mathcal{CME}$. The other case when $F_\lambda \in \mathcal{BE}$, is dealt with similarly. Without loss we assume that h is of the form $h = \exp(-sg_0 - s(g - g_0))$. If $z_1, \ldots, z_n$ are distinct points on X then the matrix $A = (h(z_\mu, z_\nu)$ can be written in the form $A = B \circ C$, in which $B_{\mu\nu} = \exp(-sg_0(z_\mu, z_\nu))$ and $C_{\mu\nu} = \exp(-s(g - g_0)(z_\mu, z_\nu))$. By Theorem 2.3, B is positive definite. From our hypotheses, C is nonnegative definite with no zero diagonal entries. An appeal to Oppenheim's inequality now finishes the proof. ■

Having developed the results for kernels on a general set, we now return to the sphere S^∞. Due to the nature of our interpolation problem described at the beginning, we restrict ourselves to conditionally negative definite kernels of the form $g \circ \langle \cdot, \cdot \rangle$, where g has a series representation as reported in the introduction.

There is no difficulty in showing that the problem of whether $g \circ \langle \cdot, \cdot \rangle$ is strictly conditionally negative definite or not depends only on $K(g)$ and not on the actual values of the coefficients $a_{m,n}$. This dependence can be clearly noticed in our next results.

THEOREM 2.7. *Let $g(\zeta) = \sum_{(m,n)\in K} a_{m,n}(1 - \zeta^m \overline{\zeta}^{\,n})$, in which $K \subset (\mathbb{N} \times \mathbb{N}) \setminus \{(0,0)\}$, $a_{m,n} \geq 0$ for all $(m,n) \in K$, and $\sum_{(m,n)\in K} a_{m,n} < \infty$. The following are equivalent:*
i) The equation $g(\zeta) = 0$ has a unique solution in $\overline{\Delta(0,1)}$;
ii) The set $K_-(g) := \{m - n : (m,n) \in K(g)\}$ has a relatively prime subset;
iii) The system of equations $\exp(ikx) = 1$, $k \in K_-(g)$, has a unique solution in $[0, 2\pi)$.

PROOF. If i) is true, order the elements of $|K_-(g)| := \{|m - n| : (m,n) \in K_-(g)\}$, say $k_1, k_2, \ldots$, and denote by d_j, $j \geq 1$, the greatest common divisor of $k_1, \ldots, k_j$. We obviously have $1 \leq d_{j+1} \leq d_j$, $j \geq 1$. Define $d := d_{j_0}$, where $j_0 := \min\{j : d_j = d_{j+k},\ k \geq 0\}$. If $d > 1$, then $\zeta = \exp(2\pi i/d)$ is a solution of $\zeta^m \overline{\zeta}^{\,n} = 1$, for all $(m,n) \in K(g)$, contradicting i). Thus, $d = 1$ and $\{k_1, \ldots, k_{j_0}\}$ is a relatively prime subset of $|K_-(g)|$. Therefore, $K_-(g)$ has a relatively prime subset.

Next, assuming that $\{k_1, \dots, k_l\}$ is a relatively prime subset of $K_-(g)$ we show that 0 is the unique solution of $\exp(ikx) = 1$, $k \in K_-(g)$, in $[0, 2\pi)$. Suppose that $y > 0$ is a solution of the above system. Since $\exp(i|k_j|y) = 1$, $1 \le j \le l$, there are positive integers $\alpha_1, \dots, \alpha_l$ such that $|k_j|y = 2\alpha_j\pi$, $1 \le j \le l$. Since $\{|k_1|, \dots, |k_l|\}$ is relatively prime, we can select integers $\beta_1, \dots, \beta_j$ such that $\sum_{j=1}^{l} |k_j|\beta_j = 1$ so that $y = \sum_{j=1}^{l} |k_j|y\beta_j = 2\pi \sum_{j=1}^{l} \alpha_j\beta_j$. Hence, $\alpha_s = |k_s| \sum_{j=1}^{l} \alpha_j\beta_j$, $1 \le s \le l$, and, in particular, $|k_1| \le \alpha_1$. Thus, $|k_1|y = 2\alpha_1\pi \ge 2\pi|k_1|$. Since $|k_1| \ne 0$, we finally obtain $y \ge 2\pi$.
At last, assume that iii) holds and suppose that $\zeta \ne 1$ is a solution of $g(\zeta) = 0$. Since $|\zeta| = 1$ we can write $\zeta = \exp(i\theta)$, with $\theta \in (0, 2\pi)$. It follows that $\exp(i(m-n)\theta) = 1$, for $(m, n) \in K(g)$, contradicting iii). ■

THEOREM 2.8. *Let h be a function of the form $h = F_\lambda \circ g \circ \langle \cdot, \cdot \rangle$, in which $F_\lambda \in \mathcal{CM}$ (respectively $F_\lambda \in \mathcal{B}$) and $g \circ \langle \cdot, \cdot \rangle$ is a real negative definite kernel on S^∞. In order that h be strictly positive (respectively strictly conditionally negative) definite on S^∞ it is necessary and sufficient that $K_-(g)$ have a relatively prime subset.*

PROOF. We only prove the theorem under the assumption that $F_\lambda \in \mathcal{CM}$. The proof when $F_\lambda \in \mathcal{B}$ is dealt with similarly. We intend to use Theorem 2.3. Observe that

$$h(z, w) = F_\lambda(g(\langle z, w \rangle)) = \int_0^\infty e^{-sg(1)} e^{-s(g - g(1))(\langle z, w \rangle)} d\lambda(s)$$

and therefore, by Oppenheim's inequality, we can assume that $g(1) = 0$. In doing so, we only have to prove that Δ_{S^∞} is the set of zeros of $g \circ \langle \cdot, \cdot \rangle$. If the equation $g(\zeta) = 0$ has a unique solution in $\overline{\Delta(0,1)}$ and $g(\langle z, w \rangle) = 0$ then $\langle z, w \rangle = 0$, so $\langle z - w, z - w \rangle = 0$, *i.e.*, $z = w$. Conversely, if $\zeta \in \overline{\Delta(0,1)} \setminus \{1\}$ is such that $g(\zeta) = 0$ then $|\zeta| = 1$ and we can write

$$\zeta = e^{i\theta_1} \overline{e^{i\theta 2}} = \langle (e^{i\theta_1}, 0, \cdots), (e^{i\theta_2}, 0, \cdots) \rangle, \quad \theta_1 \ne \theta_2 \pmod{2\pi},$$

so $g \circ \langle \cdot, \cdot \rangle$ vanishes outside Δ_{S^∞}. Thus, Δ_{S^∞} is the set of zeros of $g \circ \langle \cdot, \cdot \rangle$ if and only if the equation $g(\zeta) = 0$ has a unique solution in $\overline{\Delta(0,1)}$. By Theorem 2.7, $\zeta = 1$ is the unique solution of $g(\zeta) = 0$ in $\overline{\Delta(0,1)}$ if and only if $K_-(g)$ has a relatively prime subset. The proof is now complete. ∎

COROLLARY 2.9. *Let h be a function of the form $h = F_\lambda \circ g \circ \langle \cdot, \cdot \rangle$, in which $F_\lambda \in \mathcal{CME}$ (respectively $F_\lambda \in \mathcal{BE}$) and $g \circ \langle \cdot, \cdot \rangle$ is conditionally negative definite on S^∞. In order that h be strictly positive (respectively strictly conditionally negative) definite on S^∞ it is sufficient that $\overline{K} := \{m - n : (m,n), (n,m) \in K(g)\}$ have a relatively prime subset.*

PROOF. If $\overline{K}$ has a relatively prime subset, then g has a real conditionally negative definite summand g_0 such that $K(g_0)$ contains a relatively prime subset. Thus, the set of zeros of g_0 is precisely Δ_{S^∞}. Since $(g - g_0) \circ \langle \cdot, \cdot \rangle$ is conditionally negative definite on S^∞, then the result follows from Theorems 2.5 and 2.6 ∎

COROLLARY 2.10. *Let $F_\lambda \in \mathcal{CM}$ (respectively $F_\lambda \in \mathcal{B}$) and let $g \circ \langle \cdot, \cdot \rangle$ be a conditionally negative definite kernel on S^∞ such that $g(1) \geq 0$. In order that $F_\lambda \circ (g + \overline{g}) \circ \langle \cdot, \cdot \rangle$ be strictly positive definite (respectively strictly conditionally negative) definite on S^∞ it is necessary and sufficient that $K_-(g)$ have a relatively prime subset.*

PROOF. It suffices to observe that a conditionally negative definite function is real if and only if the coefficients $a_{m,m}$ in its series representation satisfy $a_{m,n} = a_{n,m}$ for all (m,n). ∎

As an example consider a function g of the form $g(\zeta) = 1 - \mathrm{Re}\,\zeta + g_1(\zeta)$, in which $g_1 \circ \langle \cdot, \cdot \rangle$ is conditionally negative definite on S^∞ ($g_1 \equiv 0$ makes g simpler). By the preceding results we have that $\exp(-g) \circ \langle \cdot, \cdot \rangle$ is strictly positive definite and $log(1 + g) \circ \langle \cdot, \cdot \rangle$ and $g^\alpha \circ \langle \cdot, \cdot \rangle$, $0 < \alpha < 1$, are strictly conditionally negative definite.

As we said before, when employing strictly positive definite and strictly conditionally negative definite kernels one has the advantage that the resulting interpolation matrices are at least hermitian and therefore some well-known methods can be used to do the numerical work. We now show how to use the previous results to construct a variety of non-hermitian kernels that still produces nonsingular interpolation matrices. We look at kernels having interpolation matrices of the form $A+iB$, where A and B are hermitian. It should be noted that matrices of this form are not invertible even if A and B are. Meanwhile, they are nonsingular whenever one of the following matrices A, B, and $A+B$ is either positive definite or almost positive definite with a nonpositive trace. Assuming, for example, that $A+B$ is positive definite, a proof of this fact goes as follows: if c is a vector in the null space of $A+iB$, then $(A+iB)c=0$ and so $c^*Ac+ic^*Bc=0$. Since A and B are hermitian we see that $c^*Ac=c^*Bc=0$ and hence $c^*(A+B)c=0$. The positive definiteness of $A+B$ now implies that $c=0$. Very interesting material concerning matrices of the form $A+iB$ can be found in chapter 2 of [1] and in [12].

From the above comments, we now see that a kernel of the form $h=(f+ig)\circ\langle\cdot,\cdot\rangle$ can generate an interpolant in L_E (for an arbitrary E) when one of the kernels $f\circ\langle\cdot,\cdot\rangle$, $g\circ\langle\cdot,\cdot\rangle$, and $(f+g)\circ\langle\cdot,\cdot\rangle$ is either strictly positive definite or strictly conditionally negative definite assuming only nonnegative values on Δ_{S^∞}.

Finally, we observe that continuous functions defined on S^∞ can be uniformly approximated (on compact subsets) by sets of functions generated by strictly positive definite kernels. Indeed, proceeding as in [15], but using a complex version of the Stone-Weierstrass Theorem, it is not hard to see that the set $\{z\to f(\langle\cdot,w\rangle) : w\in S^\infty\}$ is fundamental in $C(S^\infty,\mathbb{C})$ whenever $f\circ\langle\cdot,\cdot\rangle$ is positive definite and $K(f)=\mathbb{N}\times\mathbb{N}$. Results in [5] are also related to these matters.

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