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Generalized Hermite Interpolation on Spheres via Positive Definite and Related Functions

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RESUMO

Assim como no caso de interpolação usual, mostramos que funções positivas definidas e condicionalmente definidas em esferas podem ser utilizadas com sucesso em interpolação Hermitiana em esferas.

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GENERALIZED HERMITE INTERPOLATION ON SPHERES VIA POSITIVE DEFINITE AND RELATED FUNCTIONS

V. A. MENEGATTO [1] and A. P. PERON

ABSTRACT

We describe how positive definite and conditionally negative definite functions on spheres can be used to carry out generalized Hermite interpolation on spheres.

1. INTRODUCTION

Let S^m be the unit sphere in the Euclidean space $\mathbb{R}^{m+1}$, and let d_m be the great-circle distance on S^m, i.e.,

$$d_m(x,y) = \arccos\langle x,y\rangle, \quad x,y \in S^m,$$

where $\langle x,y\rangle$ stands for the usual inner product in $\mathbb{R}^{m+1}$. Let $\Gamma = \{L_{\mu i}, 1 \leq \mu \leq r, 1 \leq i \leq n\}$ be a set of linearly independent continuous linear functionals defined on $C(S^m)$, the space of all real continuous functions defined on S^m. In this paper, we study the nonsingularity of matrices of the form $A = (A_{\mu\nu})_{\mu,\nu=1}^{r}$, in which $A_{\mu\nu}$ denotes the $n \times n$ matrix with ij-entry

$$L_{\mu i}^{x} L_{\nu j}^{y} f(d_m(x,y)),$$

and f is a real continuous function defined at least in $[0,\pi]$. In the above expression , we write L^x to indicate that the linear functional L is acting on a function of the variable x.

Matrices of the above type arise in connection with the following generalization of well-known scattered data interpolation problems on spheres ([3], [6]): given data $\{d_{\mu i}, 1 \leq \mu \leq r, 1 \leq i \leq n\}$, find an interpolant of the form

$$s(x) = \sum_{\nu=1}^{r}\sum_{j=1}^{n} c_{\nu j} L_{\nu j}^{y}(f(d_m(x,y))),$$

[1]Partially supported by CNPq-Brazil

satisfying

$$L^x_{\mu i}(s) = d_{\mu i}, \quad 1 \le \mu \le r, \quad 1 \le i \le n.$$

This interpolation problem will have a unique solution if and only if the matrix A above is nonsingular. It reduces to regular radial basis interpolation on S^m when $r = 1$ and all linear functionals $L_{1i}, 1 \le i \le n$ are point evaluations. The later is known to be always uniquely solvable for certain choices of the function f. Reference [3] resumes theoretical results in this direction and contains most of the related references.

A preliminary investigation in the general setting was made by Fasshauer in [4] but he only considered the case in which $r = 1$. He showed the nonsingularity of A for two different choices of the function f. For instance, he proved A is positive definite whenever f is the restriction to $[0, \pi]$ of a nonconstant completely monotone function. Hermite interpolation in $\mathbb{R}^n$ based on a method similar to the one above was studied in [11]. Once again, the interpolation matrices there were shown to be nonsingular for the same two classes of functions used in [4]. The method we present here, contains the one in [4] and naturally adapts the one in [11] to spheres.

We study the solvability of the interpolation process when the function f is either positive definite or conditionally negative definite on S^m. The reason for that is that such functions are good choices to solve the problem when Γ contains only point-evaluation functionals. Positive definite functions on spheres were introduced by Schoenberg in [9] and they were shown to be of the form

$$f(t) = \sum_{k=0}^{\infty} a_k P_k^\lambda(\cos t), \quad \lambda = (m-1)/2, \quad a_k \ge 0, \quad f(0) < \infty. \tag{1.1}$$

In the above expression, P_k^λ stands for the usual Gegenbauer or ultraspherical polynomial of degree k as defined in [12]. A conditionally negative definite function on S^m is then a function g of the form $c - f$, where c is a constant and f is positive definite on S^m ([5]). Equivalently,

$$g(t) = g(0) + \sum_{k=1}^{\infty} a_k(1 - p_k^\lambda(\cos t)), \quad \lambda = (m-1)/2, \quad a_k \ge 0, \quad \sum_{k=1}^{\infty} a_k < \infty. \tag{1.2}$$

Here, p_k^λ is the normalized Gegenbauer polynomial $P_k^\lambda / P_k^l(1)$. Closely related is the concept of positive definiteness relative to a set of linear functionals. We say that a continuous function $f : [0, \pi] \to \mathbb{R}$ is *positive definite for* Γ if and only if the $n \times n$ matrix $A_{m,f} = (A_{\mu\nu})^r_{\mu,\nu=1}$, where

$$A_{\mu\nu} = \left(L^x_{\mu i} L^y_{\nu j} f(d_m(x,y)) \right)^n_{i,j=1}$$

is nonnegative definite. We say it is *strictly positive definite for* Γ if and only if the matrix $A_{m,f}$ is positive definite.

The first important fact to be noted is that $A_{m,f}$ is nonnegative definite whenever f is positive definite on S^m and therefore the previous concept generalizes the one given by Schoenberg. In fact, using the Riez representation Theorem, we can write

$$L_{\nu j}(h) = \int_{S^m} h \, d\alpha_{\nu j}, \quad 1 \le \nu \le r, \quad 1 \le j \le n,$$

where $\alpha_{\nu j}$ is a signed regular Borel measure on S^m. Hence, the entries of A may be written in the form

$$L^x_{\mu i} L^y_{\nu j} f(d_m(x,y)) = \int_{S^m}\int_{S^m} \sum_{k=0}^{\infty} a_k P_k^\lambda(\langle x,y\rangle) \, d\alpha_{\mu i}(x) d\alpha_{\nu j}(y).$$

Now, we require the addition theorem for spherical harmonics ([10]):

$$P_k^\lambda(\langle x,y\rangle) = \frac{\omega_{m+1}}{N(m+1,k)} \sum_{l=1}^{N(m+1,k)} S_k^l(x) S_k^l(y).$$

In this formula, $\{S_k^j, 1 \le j \le N(m+1,k)\}$ is an orthonormal set of spherical harmonics of order k in $m+1$ variables, and ω_{m+1} is the surface area of S^m. Using the Dominated Convergence Theorem to interchange the integrals with the summation and then employing the addition formula for spherical harmonics, we obtain

$$\begin{aligned} L^x_{\mu i} L^y_{\nu j} f(d_m(x,y)) &= \sum_{k=0}^{\infty} a_k \frac{\omega_{m+1}}{N(m+1,k)} \int_{S^m}\int_{S^m} \left(\sum_{l=1}^{N(m+1,k)} S_k^l(x) S_k^l(y) \right) d\alpha_{\mu i}(x) d\alpha_{\nu j}(y) \\ &= \sum_{k=0}^{\infty} a_k \frac{\omega_{m+1}}{N(m+1,k)} \sum_{l=1}^{N(m+1,k)} \int_{S^m} S_k^l(x) \, d\alpha_{\mu i}(x) \int_{S^m} S_k^l(y) \, d\alpha_{\nu j}(y). \end{aligned}$$

Thus, if $c^t = (c_{11}, \ldots, c_{1n}, \ldots, \ldots, c_{r1}, \ldots, c_{rn})$ is a real vector,

$$c^t A_{m,f} c = \sum_{k \in K} a_k \frac{\omega_{m+1}}{N(m+1,k)} \sum_{l=1}^{N(m+1,k)} \left(\sum_{\mu=1}^{r} \sum_{i=1}^{n} c_{\mu i} \int_{S^m} S_k^l \, d\alpha_{\mu i} \right)^2 \ge 0.$$

To complete our discussion, we now show that the problem of whether f is positive definite for Γ or not depends only on the set $K_{m,f} := \{k : a_k > 0\}$ and not on the actual values of the a_k.

Theorem 1.1. *Let f_1 and f_2 be positive definite functions on S^m. If $K_{m,f_1} = K_{m,f_2}$ then f_1 is strictly positive definite for Γ if and only if f_2 is so.*

Proof: We need to prove that for $c^t \in \mathbb{R}^{rn}$, $c^t A_{m,f_1} c = 0$ if and only if $c^t A_{m,f_2} c = 0$. From the above discussion, we see that $c^t A_{m,f_j} c = 0$ if and only if

$$\sum_{\mu=1}^{r} \sum_{i=1}^{n} c_{\mu i} L_{\mu i} S_k^l = 0, \quad 1 \le \mu \le r, \quad 1 \le l \le N(m+1,k), \quad k \in K,$$

and therefore the result follows. □

2. Positive definiteness and polynomial interpolation

In this section, we first study a connection between strict positive definiteness for a set Γ and Hermite interpolation in $\mathbb{R}^n$. Using these results, we show how one can find functions that are positive definite for any set of linear functionals. Hereafter, we write H_K^{m+1} to denote the space

$$\left(\sum_{k\in K}\Pi_k^{m+1}\right)\cap H^{m+1},$$

in which Π_k^{m+1} denotes the space of all polynomials of degree $\leq k$ in $m+1$ variables and H^{m+1} denotes the space of all harmonic polynomials in $m+1$ variables. The above notation is borrowed from [8], where the space H_K^{m+1} was introduced. Theorem 2.1 below, relates the interpolation method discussed here to Hermite polynomial interpolation in $\mathbb{R}^n$. It is an extension of Theorem 2.10 in [8].

Theorem 2.1. *Let f be a function as in (1.1) and let $\Gamma = \{L_{\mu i}, 1\leq \mu\leq r,\ 1\leq i\leq n\}$ be a linearly independent set of continuous linear functionals defined on $C(S^m)$. The following statements are equivalent:*

i) $K_{m,f}$ induces strict positive definiteness for Γ on S^m;

ii) There are no nonzero linear functionals of the form $L(h)=\sum_{\nu=1}^r\sum_{j=1}^n c_{\nu j}L_{\nu j}(h)$ annihilating H_K^{m+1};

iii) The space $\Gamma(H_K^{m+1}) := \{(L_{11}(p),\ldots,L_{1n}(p),\ldots,L_{r1}(p),\ldots,L_{rn}(p)) : p\in H_K^{m+1}\}$ is of dimension rn;

iv) If g is a real function defined in $\{1,\ldots,r\}\times\{1,\ldots,n\}$, then there exists a $p\in H_K^{m+1}$ such that $L_{\nu j}(p)=g(\nu,j)$, $1\leq\nu\leq r$, $1\leq j\leq n$.

Proof: We first prove (i) and (ii) are equivalent. From the introduction, it is very easy to see that $K_{m,f}$ does not induce strict positive definiteness for Γ if and only if there is a nonzero vector $(c_{11},\ldots,c_{1n},\ldots,c_{r1},\ldots,c_{rn})$ such that the linear functional L given by $L(h)=\sum_{\mu=1}^r\sum_{i=1}^n c_{\mu i}L_{\mu i}(h)$ annihilates H_K^{m+1}. Since Γ is linearly independent, $L\not\equiv 0$, and therefore the equivalence follows.

Assume (ii) holds and suppose $\Gamma(H_K^{m+1})$ is of dimension less than rn. The orthogonal complement of $\Gamma(H_K^{m+1})$ in $\mathbb{R}^{rn}$ is of dimension at least one. Hence, there is a nonzero vector $(c_{11},\ldots,c_{1n},\ldots,c_{rn},\ldots,c_{rn})$ in $\mathbb{R}^{rn}$ such that $\sum_{\nu=1}^r\sum_{j=1}^n c_{\nu j}L_{\nu j}(p)=0$, $p\in H_K^{m+1}$, contradicting (ii).

If g is a real function defined on $\{1,\ldots,r\}\times\{1,\ldots,n\}$ such that for no $p\in H_K^{m+1}$, $L_{\nu j}(p)=g(\nu,j)$, $1\leq\nu\leq r$, $1\leq j\leq n$, then the nonzero vector $(g(1,1),\ldots g(1,n),\ldots g(r,1),\ldots,g(r,n))$ is not in

$\Gamma(H_K^{m+1})$ and therefore $\Gamma(H_K^{m+1})$ is not rn-dimensional. This shows (iii) implies (iv).

Finally, suppose there is a nonzero linear functional of the form $L(h) = \sum_{\nu=1}^{r}\sum_{j=1}^{n} c_{\nu j}L_{\nu j}(h)$ annihilating H_K^{m+1}. Let $\lambda_{11},\ldots,\lambda_{1n},\ldots,\lambda_{r1},\ldots,\lambda_{rn}$ be real numbers such that the vector given by $(\lambda_{11},\ldots,\lambda_{1n},\ldots\lambda_{r1},\ldots,\lambda_{rn})$ is not orthogonal to $(c_{11},\ldots,c_{1n},\ldots,c_{r1},\ldots,c_{rn})$. We claim there is no $p\in H_K^{m+1}$ such that $L_{\nu j}(p) = \lambda_{\nu j}$, $1\leq \nu\leq r$, $1\leq j\leq n$. Indeed, the existence of such a p would give us

$$0 = L(p) = \sum_{\nu=1}^{r}\sum_{j=1}^{n} c_{\nu j}L_{\nu j}(p) = \sum_{\nu=1}^{r}\sum_{j=1}^{n} c_{\nu j}\lambda_{\nu,j} \neq 0,$$

a contradiction. This shows (iv) implies (ii). □

The so-called strictly integrally positive definite functions on S^m provides a large class of functions that are strictly positive definite for any set of linearly independent linear functionals on $C(S^m)$. A real continuous function f defined in $[0,\pi]$ is *integrally positive definite* on S^m if and only if

$$\int_{S^m}\int_{S^m} f(d_m(x,y))d\alpha(x)d\alpha(y) \geq 0,$$

for all signed, regular Borel measure α on S^m. If the above inequality is strict whenever $\alpha\not\equiv 0$ then f is said to be *strictly integrally positive definite* on S^m. This notion of positive definiteness reduces to that of Schoenberg when one let the measure α be point evaluated. The strictly integrally positive definite functions on S^m may be characterized as follows.

Lemma 2.2. *A function f is strictly integrally positive definite on S^m if and only if it is positive definite on S^m and $K_{m,f} = \mathbb{N}$.*

Proof: First, assume that f is positive definite on S^m and $K_{m,f} = \mathbb{N}$. If α is a signed regular Borel measure on S^m then using the same procedure of the previous section we obtain

$$\int_{S^m}\int_{S^m} f(d_m(x,y))\ d\alpha(x)d\alpha(y) = \sum_{k\in\mathbb{N}} a_k\frac{\omega_{m+1}}{N(m+1,k)}\sum_{l=1}^{N(m+1,k)}\left(\int_{S^m} S_k^l\ d\alpha\right)^2.$$

Thus, if the left-hand side of the above equality vanishes, the linear functional L given by $L(h) := \int_{S^m} h\ d\alpha$ annihilates the set of all spherical harmonics in $m+1$ variables. Any continuous function defined on S^m can be uniformly approximated by linear combinations of those, *i.e.*, the set of all spherical harmonics in $m+1$ variables is fundamental in $C(S^m)$ ([10]). Hence, the Hahn-Banach Theorem may be applied to conclude that $\alpha = 0$.

Conversely, if f is integrally positive definite on S^m then we already know it is positive definite on S^m. To complete the proof, we assume $\emptyset\neq K_{m,f}\neq\mathbb{N}$, *i.e.*, that $a_{k_0} = 0$ for some k_0, and we reach a contradiction. Fix $j \in \{1, 2, \ldots, N(m+1, k_0)\}$ and consider the nonzero regular Borel measure μ on S^m defined on the family $\mathcal{F}$ of Borel subsets of S^m by the following surface integral

$$\mu(B) = \int_B S_{k_0}^j \, d\omega_{m+1}, \quad B \in \mathcal{F},$$

Integration with respect to this measure gives us

$$\int_{S^m} \int_{S^m} f(d_m(x, y)) \, d\alpha(x) d\alpha(y) = \sum_{k \neq k_0} a_k \frac{\omega_{m+1}}{N(m+1, k)} \sum_{l=1}^{N(m+1,k)} \left(\int_{S^m} S_k^l S_{k_0}^j \, d\omega_{m+1} \right)^2.$$

Each integral inside the parentheses vanishes because the spherical harmonics are pairwise orthogonal with respect to the measure $d\omega_{m+1}$ ([10]). Hence, the right hand side of the previous equation vanishes, a clear contradiction. □

The following theorem improves Theorem 5 in [4], when $r = 1$. In addition, it shows that the interpolation method introduced in [11], can be well adapted to spheres with even more generality.

Theorem 2.3. *The set* $\mathbb{N}$ *induces strict positive definiteness for* Γ *on* S^m *for all* m *and all linearly independent sets of linear functionals defined on* $C(S^m)$.

Proof: Write $\Gamma = \{L_{\mu i}, 1 \leq \mu \leq r, 1 \leq i \leq n\}$ and let $L(h) = \sum_{\nu=1}^{r} \sum_{j=1}^{n} c_{\nu j} L_{\nu j}(h)$ be a functional that annihilates $H_{\mathbb{N}}^{m+1}$. Then, it annihilates the set of all spherical harmonics in $m+1$ variables, and the fundamentality of this set in $C(S^m)$ implies once again that $L \equiv 0$. Thus, the result follows from Theorem 2.1. □

Let $f \in C([0, \infty))$ be a nonconstant completely monotone function on $(0, \infty)$. By Theorem 3.7 in [6], the restriction of f to $[0, \pi]$ has a series representation in the form

$$f(t) = \sum_{k=0}^{\infty} a_k \cos^k t, \quad a_k \geq 0, \quad f(0) < \infty,$$

with $a_k > 0$ for infinitely many odd and infinitely many even k. Using Lemma 1 in [2], we see that f is in fact strictly integrally positive definite on S^m, for all m. Thus, $K_{m,f} = \mathbb{N}$ for all m and f can be used in the previous theorem. The same is true if we take a composition $f \circ g$ of f with a function g of the form

$$g(t) = g(0) + \sum_{k=1}^{\infty} a_k (1 - \cos^k t), \quad g(0) \geq 0, \quad a_k \geq 0, \quad g(1/2) < \infty, \tag{3.1}$$

with $a_k > 0$ for at least one odd k. This follows from Theorem 3.5 in [6]. Finally, if h is conditionally negative definite on S^l with the additional feature that its series representation as in (1.2) is such that $a_{2k+1} > 0$ for at least one k, then $f \circ h$ is strictly integrally positive definite on S^m, $m < l$. This follows from Theorem 3.5 in [6] and from a well known interrelation among Gegenbauer polynomials of different types ([1]).

3. Conditionally negative definite functions

In this section, we obtain results similar to the last one in the previous section, but using conditionally negative definite functions instead. Nonsingularity of the interpolation matrices can not be guaranteed so easily, reason why we prefer not to adopt a nomenclature similar to the one used in the previous sections. The version of Theorem 2.3 in this case, is as follows.

Theorem 3.1. *Let Γ be as in Theorem 2.1. Let g be a function as in (1.2) and suppose that $a_k > 0$ for all $k \geq 1$. Then the matrix $A = (A_{\mu\nu})_{\mu\nu=1}^r$ given in blocks by $A_{\mu\nu} = (L_{\mu i}^x L_{\nu j}^y g(d_m(x,y)))_{i,j=1}^n$ is almost negative definite, i.e., $c^t Ac \leq 0$ whenever $\sum_{\mu=1}^r \sum_{i=1}^n c_{\mu i} L_{\mu i}(1) = 0$, with equality holding if and only if $c = 0$.*

Proof: Let c^t be a vector in the following subspace H of $\mathbb{R}^{rn}$

$$H := \{c^t \in \mathbb{R}^{rn} : \sum_{\mu=1}^{r} \sum_{i=1}^{n} c_{\mu i} L_{\mu i}(1) = 0\}.$$

Then,

$$\begin{aligned} c^t Ac &= \sum_{\mu=1}^{r} \sum_{\nu=1}^{r} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{\mu i} L_{\mu i}^x c_{\nu j} L_{\nu j}^y \left(g(0) + \sum_{k=1}^{\infty} a_k (1 - p_k^\lambda(\langle x, y \rangle)) \right) \\ &= -\sum_{k=1}^{\infty} a_k \sum_{\mu=1}^{r} \sum_{\nu=1}^{r} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{\mu i} L_{\mu i}^x c_{\nu j} L_{\nu j}^y p_k^\lambda(\langle x, y \rangle). \end{aligned}$$

Proceeding as in the introduction, we obtain

$$c^t Ac = -\sum_{k=1}^{\infty} \frac{a_k \omega_{m+1}}{N(m+1,k) P_k^\lambda(1)} \sum_{l=1}^{N(m+1,k)} \left(\sum_{\mu=1}^{r} \sum_{i=1}^{n} c_{\mu i} L_{\mu i}(S_k^l) \right)^2 \leq 0.$$

If $c^t Ac = 0$, we use our hypothesis on the a_k to conclude that the linear functional L given by $L(f) = \sum_{\mu=1}^r \sum_{i=1}^n c_{\mu i} L_{\mu i}(f)$ annihilates the set of all spherical harmonics of order $k \geq 1$ in $m+1$ variables. Since $c \in H$, L annihilates the spherical harmonics of order 0 too. Once again, the fundamentality of the set of all spherical harmonics in $m+1$ variables in $C(S^m)$ gives $c = 0$. □

The theorem does not guarantee nonsingularity of the interpolation matrix A. Since H has dimension at least $nr-1$, it implies that A has at least $nr-1$ negative eigenvalues.

In the next three results, we provide additional hypotheses under which nonsingularity holds. The eigenvalue structure of the interpolation matrix is given in each case. The reader should compare these conditions with those presented in Theorem 3 in [4]. The same remark applies to the conditions appearing in the proof of Theorem 3.4 in [11].

Theorem 3.2. *Let Γ, g, and A be as in the previous theorem. If $L_{\mu i}(1)=0$ for all μ and all i, then A is invertible and all its eigenvalues are negative.*

Proof: Under this additional hypothesis on Γ, the subspace H in the proof of the previous theorem is the whole $\mathbb{R}^{rn}$. Therefore, $c^tAc \leq 0$ holds true, whenever $c^t \in \mathbb{R}^{rn} \setminus \{0\}$. Equivalently, $-A$ is positive definite. The result follows. □

Theorem 3.3. *Let Γ, g, and A be as in Theorem 3.1. If $g(0)>0$ and Γ contains at least one point-evaluation functional then A is invertible. It possesses one positive and $rn-1$ negative eigenvalues.*

Proof: By Theorem 3.1, A has at least $n-1$ negative eigenvalues. We now prove the last eigenvalue is positive. Without loss, we assume that L_{11} is a point-evaluation functional. Let c^t be the vector of $\mathbb{R}^{rn}$ with first component equal to 1 and zeros elsewhere. Then,

$$c^tAc = L_{11}^x L_{11}^y g(d_m(x,y)) = g(0) > 0.$$

By the Courant-Fisher Theorem ([7]), the biggest eigenvalue of A, say α_1, satisfies

$$\alpha_1 = \max_{c\neq 0} \frac{c^tAc}{\langle c, c\rangle} > 0.$$

This proves the theorem. □

Theorem 3.4. *Let Γ, g, and A be as in Theorem 3.1. If Γ contains at least two point-evaluation functionals then A is invertible. Its eigenvalue structure is the same given in the previous theorem.*

Proof: Without loss, assume that L_{11} and L_{12} are point-evaluation functionals. The bloch A_{11} of A has a subbloch B of order 2 corresponding to these two functionals. B is invertible because it corrresponds to regular interpolation on S^m with a strictly conditionally negative definite function. As a matter of fact, B has one positive and one negative eigenvalue ([5]). Next, we write A in the form

$$A = \begin{pmatrix} B & C \\ C^t & D \end{pmatrix} = SXS^t,$$

in which

$$S = \begin{pmatrix} I_2 & 0 \\ C^t B^{-1} & I_{rn-2} \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} B & 0 \\ 0 & (A|B) \end{pmatrix}.$$

In this expression, I_j is the identity matrix of order j and $(A|B)$ is the Schur complement (see [7]) of B in A, i.e., $(A|B) = D - C^t B^{-1} C$. Essentially, the above equality is saying that A is congruent to the (block) diagonal matrix X. Since X is block triangular, its eigenvalues are those of B together with those of $(A|B)$. Hence, X has a positive eigenvalue α. If d is an unitary eigenvector corresponding to α then $d^t X d = \alpha$ and hence

$$d^t S^t A S d = (Sd)^t A (Sd) > 0.$$

Since S is nonsingular, $(Sd)^t$ is a nonzero vector. Therefore, by the Courant-Fisher Theorem once again, A has a positive eigenvalue. □

A useful class of functions that fits in Theorem 3.1 is that which comprises functions f of $C([0,\infty))$ having a nonconstant completely monotone derivative in $(0,\infty)$. Indeed, a combination of Theorem 3.7 in [6] and Lemma 1 in [2] shows that such a f has a series representation as in (3.1) with positive coefficients only. Theorem 3.6 in [6] implies that $f \circ g$ also has this property whenever g is as in (3.1) and $a_k > 0$ for at least one odd k.

Finally, we would like to observe that the conclusions of Theorem 3.2 and 3.3 hold true even when Γ is unitary (*i.e.*, when the interpolation matrix is of order 1). The same cannot be said of Theorem 3.4.

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