Instituto de Ciências Matemáticas de São Carlos

ISSN - 0103-2577

ON EQUATIONS OF KP-TYPE

RAFAEL JOSÉ IÓRIO JR.
WAGNER VIEIRA LEITE NUNES

Nº 40

NOTAS DO ICMSC
Série Matemática

São Carlos
Jun./1996

Resumo

Neste trabalho mostramos que o problema de Cauchy associado a equação de Kadomtsev-Petviashvilli generalizada, a saber,

$$(\partial_t u - iP(D)u + a(u)\partial_x u)_x + \gamma\partial_y^2 u = 0$$

onde $P(D)$ é um operador pseudo-diferential cujo símbolo é uma função real, ímpar e de crescimento polinomial, é localmente bem-posto nos espaços de Sobolev $H_\pi^s(R^2)$ e $H^s(R^2)$, para $s > 2$. No caso periódico usamos a Teoria quasi-linear de Kato e no caso não-periódico usamos regularização parabólica. Em ambos os casos provamos que o tempo de existência não depende do expoente de Sobolev, s, desde que $s > 2$.

ON EQUATIONS OF KP-TYPE

RAFAEL JOSÉ IÓRIO JR. (IMPA/CNPQ)†
WAGNER VIEIRA LEITE NUNES (ICMSC/USP)‡

ABSTRACT. We discuss the local Cauchy problem for the generalized Kadomtsev-Petviashvili equation, namely $(\partial_t u - iP(D)u + a(u)\partial_x u)_x + +\gamma\partial_y^2 u = 0$, in the periodic and non-periodic settings.

1. Introduction

In this work we consider the Cauchy problem

$$(\partial_t u - iP(D)u + a(u)\partial_x u)_x + \gamma\partial_y^2 u = 0 \tag{1.1}$$

$$u(0;x,y) = \phi(x,y) \tag{1.2}$$

with periodic and non-periodic boundary conditions. In the periodic case we assume that

$$(\widehat{P(D)\psi})(m,n) = p(m)\widehat{\psi}(m,n) \qquad (m,n) \in Z^2 \tag{1.3}$$

where $p = p(m)$, $m \in Z$ is a real continuous function such that

$$|p(m)| \leq C_k|m|^l \quad |m| >> 1 \quad l \in \{2,3,\dots\} \quad p(0) = 0 \tag{1.4}$$

In the non-periodic case we make a similar assumption, namely

$$(\widehat{P(D)\psi})(\xi,\eta) = p(\xi)\widehat{\psi}(\xi,\eta) \qquad (\xi,\eta) \in \mathbb{R}^2 \tag{1.5}$$

where $p = p(\xi)$, $\xi \in \mathbb{R}$ is a real odd continuous function such that

$$|p(\xi)| \leq C_k|\xi|^l \quad |\xi| >> 1 \quad l \in \{2,3,\dots\} \tag{1.6}$$

In the periodic case, our purpose is to extend recent results of Isaza, Majía and Stallbohm ([5]) and in the non-periodic case we will extend the results of S.Ukay ([14]) and J.C. Saut ([13]) . We will be more precise further on.

The usual Kadomtsev-Petviashvili equation (KP) ($p(m) = m^3$ or $p(\xi) = = \xi^3$ and $a(u) = u$) describes, for instance, the propagation of long nonlinear waves along the x-axis on the surface of a liquid when the variation along the y-axis proceeds slowly. It is a two-dimensional generalization of the Korteweg-de Vries equation (see [11]).

†This work was made while the author was visiting IMECC-Unicamp. Partially supported by FAPESP.

‡Partially supported by CNPq.

The Cauchy problem associated to KP has been solved globally, using the inverse scattering method, in the Sobolev spaces $W^{s,2}(\mathbb{R}^2)\bigcap W^{s,1}(\mathbb{R}^2)$, $s \geq 10$ ([15]). S.Ukai obtained local existence and uniqueness in the Sobolev spaces $H^s(\mathbb{R}^2) = W^{s,2}(\mathbb{R}^2)$, with $s \geq 3$ and some further restrictions. Uniqueness of solutions for (G-KP) ($p(m) = m^3$) was obtained in [16] in the periodic case . Global well-posedness for (KP) in $L^2(\Pi)$, $\Pi = [0,2\pi]\times[0,2\pi]$, was established by J.Bourgain in [2]. His method is based on an analysis of multiple Fourier series and in his case γ must be positive. Finally, the quasi-linear theory of Kato ([6], [7], [8]) was used to show that (KP) is locally well-posed in $H^s(\Pi)$, $s \geq 3$ ([5]). In what follows we show, in particular, that their result can be extended to the case $s > 2$. This also extends the results of [14] and [16]. J.C. Saut obtained a blow-up result for the non-periodic case when $a(u) = u^p$, $p \geq 4$ and $\gamma = -1$ ([13]).

It should be remarked that in the periodic case we will apply the quasi-linear Kato's theory to prove the local well-posedness for $s > 2$. In the non-periodic case the existence and uniqueness result will be established using parabolic regularization (as in the proof of theorem (2.1) of [4]). Continuous dependence can then be established using the Bona-Smith approximations ([3]). We pointed out below why we didn't apply the Kato's theory in the non-periodic case.

Our main result are, as follows:

Theorem 1.1. : *(periodic case)*

1. *Let $\phi \in X^s_\pi$, $s > 2$ and $a \in C^\infty(\mathbb{R})$. Then there exist a $T = T(s, \|\phi\|_s) > 0$ and a unique $u = u(t)$ which satisfies*
$$u \in C([0,T]; X^s_\pi) \cap C^1([0,T]; X^{s-l}_\pi) \tag{1.7}$$
solution of (1.1)-(1.2). Moreover, the map $\phi \to u_\phi$ is continuous in the $\|.\|_s$-norm as usual, i.e., if $\phi_n \in X^s_\pi$, $\phi_n \to \phi$ in X^s_π when $n \to \infty$ and $u_n \in C([0,T_n]; X^s_\pi)\bigcap C^1([0,T_n]; X^{s-l}_\pi)$ is a solution of (1.1) with $u_n(0) = \phi_n$ then for $0 < T' < T$ there exists a $n_0 \in N$ such that, for $n \geq n_0$, $u_n = u_n(t)$ exists on $[0, T']$ and
$$\sup_{0\leq t\leq T'} \|u_n(t) - u(t)\|_s \to 0 \qquad \text{when } n \to \infty \tag{1.8}$$
2. *T can be chosen independent of s in the usual sense, i.e., if u is a solution satisfying (1.7) and $\phi \in X^{s'}_\pi$ for some $s' \neq s$, $s' > 2$, then (1.7) is also true with s replaced by s' and with the same T. In particular, if $\phi \in X^\infty_\pi = \bigcap_s X^s_\pi$ then (1.7) holds with $s = \infty$ and the same T.*

Theorem 1.2. *(non-periodic case)*
Let $\phi \in X_s(\mathbb{R}^2)$, $s > 2$ and $a \in C^\infty(\mathbb{R}; \mathbb{R})$ such that $a(0) = 0$. Then there exists $T > 0$ and unique $u \in C([0,T]; X_s)$ such that (1.1)-(1.2) is satisfied with $\partial_t u$ computed with respect to the topology of $H^{s-l}(\mathbb{R}^2)$. The continuity

of the map $\phi \to u_\phi$ holds exactly as in the periodic case. T can be chosen independent of s.

We will adopt the following definitions and notations: First of all $^\wedge$ and $^\vee$ will denote the Fourier transform and it's inverse in both the periodic and non-periodic settings. No confusion will arise from this. Moreover, in the periodic case we will write,
$\Pi = [0, 2\pi] \times [0, 2\pi]$
$C_\pi^\infty(\mathbb{R}^2) = \{\psi \in C^\infty(\mathbb{R}^2) : \psi \text{ is } 2\pi - \text{periodic in } x \text{ and } y\}$
$M_\pi^\infty = \{\psi \in C_\pi^\infty(\mathbb{R}^2) : \int_0^{2\pi} \psi(x,y)\,dx = 0\,,\ y \in [0, 2\pi]\}$
$(\phi, \psi)_s = \sum_{(m,n)\in Z^2} (1 + m^2 + n^2)^s \widehat{\phi}(m,n)\overline{\widehat{\psi}(m,n)}$, where $\phi, \psi \in C_\pi^\infty$.
$\|\phi\|_s = (\phi, \phi)_s^{\frac{1}{2}}$, where $\phi \in C_\pi^\infty$.
$H_\pi^s(\mathbb{R}^2) = \overline{C_\pi^\infty}^{\|\cdot\|_s}$ = the set of all periodic distributions ϕ such that $\|\phi\|_s^2 =$
$= \sum_{(m,n)\in Z^2} (1 + m^2 + n^2)^s |\widehat{\phi}(m,n)|^2 < \infty$
$X_\pi^s(\mathbb{R}^2) = \overline{M_\pi^\infty}^{\|\cdot\|_s}$ = the closure of M_π^∞ with respect to the H_π^s norm.
In the non-periodic case we will write,
$H^s(\mathbb{R}^2)$ is the usual L^2 type Sobolev spaces the norm of which will be indicated by $\|.\|_s$.
$L_s^2(\mathbb{R}^2) = (H^s(\mathbb{R}^2))^\wedge$
$X_s(\mathbb{R}^2)$ denotes the set of all $f \in H^s(\mathbb{R}^2)$ such that

$$(\partial_x^{-1} f)^\wedge(\xi, \eta) = \frac{\hat{f}(\xi, \eta)}{i\xi} \in L_s^2(\mathbb{R}^2) \tag{1.9}$$

endowed with the norm

$$|f|_s^2 = \|f\|_s^2 + \|\partial_x^{-1} f\|_s^2 \tag{1.10}$$

$\mathcal{F}_1$ (resp. $\mathcal{F}_2$) denote the Fourier transform with respect the variable x (resp. y).
$\mathcal{B}(Y; X)$ = set of all bounded linear operators from Y into X, X, Y Banach spaces.
$\mathcal{B}(X) = \mathcal{B}(X; X)$.
$C = C(.,\ldots,.)$ are constants which are continuous and nondecreasing with respect to their arguments.

This work is organized as follows. In section 2 we will study the linear equation associated to (1.1) in the periodic case and present the proof the theorem (1.1). While in section 3 we will repeat this program in the non-periodic case.

2. The Periodic Case

In this section we will discuss problem (1.1)-(1.2) endowed with periodic boundary conditions.

First of all,

Lemma 2.1. : *If* $\psi \in C^\infty_\pi(\mathbb{R}^2)$ *then*

$$\int_0^{2\pi} (P(D)\psi)(x,y)\,dx = 0 \qquad 0 \le y \le 2\pi \tag{2.1}$$

The same hols for $\psi \in H^s_\pi(\mathbb{R}^2)$ *for* $s \ge k$.

Proof: Let $y \in [0,2\pi]$ be fixed and apply Parseval's identity to obtain

$$\int_0^{2\pi} (P(D)\psi)(x,y)dx = \sum_{m\in Z} p(m)\widehat{\psi}(m,y)\widehat{1}(m) = 0 \tag{2.2}$$

since $\widehat{1}(m) = 0$ for $m \ne 0$, $\widehat{1}(0) = 1$ and $p(0) = 0$. □

Integrating the PDE (1.1) over the interval $[0,2\pi]$ we obtain

$$\int_0^{2\pi} \partial_y^2 u(t;x,y)\,dx = 0 \tag{2.3}$$

so that

$$\int_0^{2\pi} u(t;x,y)\,dx = \alpha(t)y + \beta(t) \tag{2.4}$$

Since $u(t;x,.)$ is periodic we must have $\alpha(t) = 0$ which implies

$$\int_0^{2\pi} dx\, u(t;x,y) = \beta(t) \tag{2.5}$$

We can show that (1.1)-(1.2) is equivalent to

$$\partial_t u - iP(D)u + a(u)\partial_x u + \gamma \int_0^x dx'\, \partial_y^2 u(t;x',y) = g_{u(t)}(y) \tag{2.6}$$

$$u(0) = \phi \tag{2.7}$$

where

$$\begin{aligned} g_\psi(y) &= \frac{\gamma}{2\pi}\int_0^{2\pi} dx \int_0^x dx'\, \partial_y^2\psi(x',y) + h_\psi(\psi,y) \\ &= G_\psi(y) + h_\psi(\psi,y) \end{aligned} \tag{2.8}$$

$$G_\psi(y) = \frac{\gamma}{2\pi}\int_0^{2\pi} dx \int_0^x dx'\, \partial_y^2\psi(x',y) \tag{2.9}$$

and $h_\psi = h_\psi(\psi, y)$ will be chosen later. Therefore, the solution of (2.6)-(2.7) must satisfiy (2.5). For (2.6) we also have,

$$\begin{aligned}\partial_t \int_0^{2\pi} dy \int_0^{2\pi} dx\, u(t;x,y) = i \int_0^{2\pi} dy \int_0^{2\pi} dx\, P(D)u(t;x,y) \\ - \int_0^{2\pi} dy \int_0^{2\pi} dx\, a(u(t;x,y))\partial_x u(t;x,y) \\ - \gamma \int_0^{2\pi} dy \int_0^{2\pi} dx \int_0^{x} dx'\, \partial_y^2 u(t;x',y) \\ + \int_0^{2\pi} dy \int_0^{2\pi} dx\, g_{u(t)}(y) = 0 \qquad (2.10)\end{aligned}$$

by lemma 2.1 and periodicity. This implies $\beta(t) = constant$, i.e.,

$$\int_0^{2\pi} u(t;x,y)\, dx = \beta(0) = \int_0^{2\pi} \phi(x,y)\, dx \qquad y \in [0, 2\pi] \qquad (2.11)$$

We may consider , without loss of generality, $\beta(0) = 0$. In fact, if $\beta(0) \neq 0$ then $u = u(t;x,y)$ is a solution of (1.1)-(1.2) if and only if $u_1(t;x,y) = u(t;x,y) - \frac{\beta(0)}{2\pi}$ is a solution of

$$\begin{aligned}\partial_t u_1 - iP(D)u_1 + a_1(u_1)\partial_x u_1 + \gamma \int_0^x dx'\, \partial_y^2 u_1(t;x',y) = g_{u_1(t)}(y) \\ u_1(0) = \phi_1\end{aligned}$$

where $a_1(r) = a(r + \frac{\beta(0)}{2\pi})$ and $\phi_1(x,y) = \phi(x,y) - \frac{1}{2\pi}\beta(0)$, which is similar to (1.1)-(1.2) and $\beta_1(0) = 0$.

We also observe that $\beta(0) = 0$ is equivalent to

$$\int_0^{2\pi} dx\, \phi(x,y) = 0 \ , \ y \in [0, 2\pi] \qquad (2.12)$$

and so we may consider, without loss of generality, the Cauchy problem (2.6)-(2.7) where the initial data satisfies the constraint (2.12).

2.1. **The Linear Equation.** :

As mentioned in the introduction, this section deals with the linear equation associated to (1.1), that is

$$\partial_t u - iP(D)u + \gamma \int_0^x dx\, \partial_y^2 u = G_{u(t)}(y) \qquad (2.13)$$

$$u(0;x,y) = \phi(x,y) \qquad (2.14)$$

We have

Definition 2.2. *Let $A_\gamma : D(A_\gamma) \subset X^0_\pi \to X^0_\pi$ be given by*

$$D(A_\gamma) = \{\psi \in X^0_\pi : \sum_{(m,n)\in Z^2} |\widehat{(A_\gamma\psi)}(m,n)|^2 < \infty\} \tag{2.15}$$

$$\widehat{(A_\gamma\psi)}(m,n) = \begin{cases} -i\left[p(m) + \gamma\frac{n^2}{m}\right]\widehat{\psi}(m,n) & m \neq 0 \\ 0 & m = 0 \end{cases} \tag{2.16}$$

where $\psi \in D(A_\gamma)$.

Remarks 2.3. *From (2.16) it follows that*

$$A_\gamma\psi = -iP(D)\psi + \gamma\int_0^x dx'\partial_y^2\psi - \frac{\gamma}{2\pi}\int_0^{2\pi} dx \left(\int_0^x dx'\,\partial_y^2\psi\right) \tag{2.17}$$

if $\psi \in X^s_\pi$, $s \geq k$.

Now, we can show

Proposition 2.4. *The linear operator A_γ given by (2.16) is the infinitesimal generator of a strongly continuous unitary group $\{P(t)\}_{t\in R}$ on X^0_π.*

Proof: In view of theorem 10.8 of [12], it is enough to show that:
(i) $\overline{D(A_\gamma)} = X^0_\pi$, which is obvious.
(ii) A_γ is conservative, which it follows from

$$\begin{aligned}\Re(\psi, A_\gamma\psi)_0 &= \Re\left(\sum_{\substack{(m,n)\in Z^2 \\ m\neq 0}} \widehat{\psi}(m,n)i\left(p(m)+\gamma\frac{n^2}{m}\right)\overline{\widehat{\psi}(m,n)}\right) \\ &= 0\end{aligned} \tag{2.18}$$

since $p = p(m)$ is real.
(iii) $I \pm A_\gamma$ is surjective, since that for $\eta \in X^0_\pi$ if we consider $\psi = \psi(x,y)$ given by

$$\widehat{\psi}(m,n) = \begin{cases} \widehat{\eta}(m,n)\left[1 - i\left(p(m)+\gamma\frac{n^2}{m}\right)\right]^{-1} & m \neq 0 \\ 0 & m = 0 \end{cases} \tag{2.19}$$

belongs to $D(A_\gamma)$ and satisfies $\widehat{\psi}(m,n) + \widehat{(A_\gamma\psi)}(m,n) = \widehat{\eta}$, which implies $(I + A_\gamma)\psi = \eta$, i.e., $I + A_\gamma$ is surjective. The case $I - A_\gamma$ can be dealt similarly. □

Remarks 2.5. *It can also be proved that $P(t) = e^{-A_\gamma t}$ defines an unitary group in X^s_π and $u(t) = P(t)\phi$ is the solution of (2.13)-(2.14) in X^{s-l}_π, where $\phi \in X^s_\pi$.*

2.2. The Non-Linear Equation. :

Now we will consider the non-linear periodic Cauchy problem (1.1)-(1.2). Choose h_ψ in (2.8) to be

$$h_\psi(\eta, y) = \frac{1}{2\pi}\int_0^{2\pi} a(\psi(x', y))\partial_{x'}\eta(x'y)\, dx' \tag{2.20}$$

where $\psi, \eta \in X^s_\pi$. We can show that $u = u(t; x, y)$ is the solution of (2.6)-(2.7) if and only if

$$u(t) = P(t)v(t) \tag{2.21}$$

is the solution of

$$\partial_t v + P(-t)(a(P(t)v)\partial_x P(t)v) = P(-t)h_{P(t)v(t)} \tag{2.22}$$

$$v(0) = \phi \tag{2.23}$$

So, we apply the quasi-linear theory of T.Kato to

$$\frac{d}{dt}v + A(t, v)v = 0 \tag{2.24}$$

$$v(0) = \phi \tag{2.25}$$

where the linear operator $A = A(t, \psi)$, which depends on $t \in [0, T]$ and $\psi \in X^s_\pi$, is given by

$$\begin{aligned} A(t,\psi)\eta &= P(-t)(a(P(t)\psi)\partial_x P(t)\eta) \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi} P(-t)(a(P(t)\psi)\partial_{x'} P(t)\eta)\, dx' \end{aligned} \tag{2.26}$$

In order to apply Kato's theory we take $X = X^0_\pi$, $Y = X^s_\pi$ and $W =$ $= \{\eta \in X^s_\pi : \|\eta\|_s < r\}$ where $r > 0$.

Our first step is to prove the following lemma:

Lemma 2.6. : *$A(t, \psi) \in G(X, 1, \beta)$ with $\beta = \beta(r)$, i.e., $-A(t, \psi)$ is infinitesimal generator of a C_0-semigroup in $X = X^0_\pi$ which satisfies*

$$(A(t, \psi)\eta, \eta)_0 \geq -\beta\|\eta\|_0^2, \eta \in C^\infty_\pi \tag{2.27}$$

Proof: Since $P(t)$ is unitary on $X = X^0_\pi$, it suffices to show that (2.27) holds for the operator $a(P(t)\psi)\partial_x$. But

$$\begin{aligned} (a(P(t)\psi)\partial_x w, w)_0 &= -\frac{1}{2}(\partial_x a(P(t)\psi), w^2)_0 \geq -\frac{1}{2}\|\partial_x a(P(t)\psi)\|_\infty \|w\|_0^2 \\ &\geq -\frac{C_s}{2}\sup_{\|\psi\|_s \leq r} \|a(P(t)\psi)\|_s \|w\|_0^2 \\ &\geq -\frac{C_s}{2}\sup_{\|\eta\|_s \leq r} \|a(\eta)\|_s \|w\|_0^2 \end{aligned} \tag{2.28}$$

and so, we take $\beta = \frac{C_s}{2}\sup_{\|\eta\|_s\le r}\|a(\eta)\|_s$. This proves the lemma. □

Lemma 2.7. *Let $S = (1-\triangle)^{s/2} : Y = X^s_\pi \to X = X^0_\pi$. Then S is an isomorphism of Y onto X and*

$$SA(t,\psi)S^{-1} = A(t,\psi) + B(t,\psi) \tag{2.29}$$

$$\|B(t,\psi)\|_{\mathcal{B}(X)} \le \lambda\|\psi\|_Y \tag{2.30}$$

$$\|B(t,\psi') - B(t,\psi)\|_{\mathcal{B}(X)} \le \mu\|\psi' - \psi\|_Y \tag{2.31}$$

for $\psi', \psi \in W$, where $\lambda = \lambda(r)$, $\mu = \mu(r)$ are constants depending only on r and a.

Proof: It suffices to verify that

$$[S, a(P(t)\psi)]\partial_x S^{-1} \in \mathcal{B}(X) \tag{2.32}$$

since $P(t)$ is unitary on X, where $[A,B]$ denotes the commutator of the linear operators A and B.

Using the periodic version of Lemma A.4 of [10] we have

$$\begin{aligned}\|[S, a(P(t)\psi)]\partial_x S^{-1} f\|_0 &\le C_s\|\partial_x a(P(t)\psi)\|_{s-1}\|\partial_x S^{-1} f\|_{s-1}\\ &\le C_s \sup_{w\in W}\|a(w)\|_s\|f\|_0 = \lambda\|f\|_0\end{aligned} \tag{2.33}$$

where $\lambda = \lambda(r) = \sup_{\|w\|_s\le r}\|a(w)\|_s$. So we obtain (2.30). The proof of (3.51) is similar. □

Lemma 2.8. *Let $(t,\psi) \in [0,T]\times W$. Then*

$$Y \subset D(A(t,\psi)) \tag{2.34}$$

$$A(t,\psi)|_Y \in \mathcal{B}(Y;X) \tag{2.35}$$

The map

$$\begin{array}{rccc} A(.,\psi): & [0,T] & \to & \mathcal{B}(Y;X)\\ & t & \mapsto & A(t,\psi)\end{array} \tag{2.36}$$

is strongly continuous for each $\psi \in W$. For each $t \in [0,T]$, the map

$$\begin{array}{rccc} A(t,.): & Y & \to & \mathcal{B}(Y;X)\\ & \psi & \mapsto & A(t,\psi)\end{array} \tag{2.37}$$

is Lipschitz uniformly on t, i.e.

$$\|A(t,\psi') - A(t,\psi)\|_{\mathcal{B}(Y;X)} \le \mu\|\psi' - \psi\|_X \tag{2.38}$$

where $\mu = \mu(r)$ depends only on r and a.

Proof: Since $P(t)$ is unitary on X it follows from (2.26) that (2.34) holds and for $f \in Y$

$$\begin{aligned} \|A(t,\psi)f\|_0 &= \|a(P(t)\psi)\partial_x P(t)f\|_0 \\ &\leq \sup_{w\in W} \|a(w)\|_s \|f\|_1 \leq \mu \|f\|_s \end{aligned} \tag{2.39}$$

where $\mu = \sup_{\|w\|_s \leq r} \|a(w)\|_s$ and so (2.35). Let $t, t' \in [0,T]$ and $f, \psi \in W$. Then

$$\begin{aligned} \|A(t,\psi)f - A(t',\psi)f\|_0 &\leq \|(P(-t) - P(-t'))(a(P(t)\psi)\partial_x P(t)f)\|_0 \\ &+ \|(a(P(t)\psi) - a(P(t')\psi)\partial_x P(t)f\|_0 \\ &+ \|(a(P(t')\psi)\partial_x(P(t) - P(t'))f\|_0 \to 0 \end{aligned} \tag{2.40}$$

when $t' \to t$, so (2.36).

For $t \in [0,T]$ we have

$$\begin{aligned} \|A(t,\psi')f - A(t,\psi)f\|_0 &\leq \|a(P(t)\psi') - a(P(t)\psi)\|_0 \|\partial_x P(t)f\|_\infty \\ &\leq C_s \|a'(P(t)w)\|_\infty \|\psi' - \psi\|_0 \|f\|_s \\ &\leq C_s \sup_{\|w\|_s \leq r} \|a(w)\|_s \|f\|_s \|\psi' - \psi\|_0 \\ &\leq \mu \|\psi' - \psi\|_0 \|f\|_s \end{aligned} \tag{2.41}$$

where $\mu = C_s \sup_{\|w\|_s \leq r} \|a(w)\|_s$, which implies (2.38). □

Proof of theorem 1.1 : In view of the previous lemmas, the quasi-linear theory of T.Kato implies the existence of unique solution $v \in C([0,T]; X_\pi^s) \bigcap \bigcap C^1([0,T]; X_\pi^{s-l})$ of (2.22)-(2.23) which depends continuously on initial data. Then the transformation (2.21) gives us a unique solution

$$u \in C([0,T]; X_\pi^s) \bigcap C^1([0,T]; X_\pi^{s-l}) \tag{2.42}$$

of (2.6)-(2.7) which, of course, depends continuously on initial data. For the second part of the theorem we again follow closely the ideas of T.Kato in [9]. For this reason we present only a brief sketch of the proof. It is enough to consider the case $s' > s$. The case $s' < s$ follows from uniqueness. It is easy to see that if $w = w(t)$ is given by

$$w(t) = \partial_x^2 v(t), \tag{2.43}$$

where $v = v(t)$ is the solution of (2.22)-(2.23), satisfies the following evolution equation

$$\frac{d}{dt} w + A(t)w + C(t)w = f(t) \tag{2.44}$$

where

$$A(t) = \partial_x P(-t)a(u(t))P(t) \tag{2.45}$$

$$C(t) = 2P(-t)a'(u(t))u_x(t)P(t) \tag{2.46}$$

$$f(t) = -P(-t)a''(u(t))u_x^3(t) \tag{2.47}$$

where $u \in C([0,T); X_\pi^s)$ is a known function.

Since $v \in C([0,T); X_\pi^s)$ it follows that $w \in C([0,T); X_\pi^{s-2})$. Furthermore, $w(0) = \phi_{xx} \in X_\pi^{s'-2}$, because $\phi \in X_\pi^{s'}$. Our purpose is to show that $w \in \in C([0,T); X_\pi^{s'-2})$, which will imply $v \in C([0,T); X_\pi^{s'})$ and so that $u = u(t)$ has the same property.

The proofs of the next two lemmas are similar to those of lemmas (3.1) and (3.2) of [9] and will therefore be omitted.

Lemma 2.9. : *The family $\{A(t)\}_{0\le t\le T}$ has a unique evolution operator $\{U(t,\tau)\}_{0\le t\le\tau\le T}$ associated with the spaces $X = H_\pi^p$, $Y = H_\pi^q$ in the sense of* [8], *where*

$$-s \le p \le s-2 \quad 1-s \le q \le s-1 \quad p+1 \le q \tag{2.48}$$

In particular, $U(t,\tau) : H_\pi^r \to H_\pi^r$ for $-s \le r \le s-1$.

Lemma 2.10. : *Let $w \in C([0,T]; H_\pi^{s-2})$ be as before. Then*

$$w(t) = U(t,0)\phi_{xx} + \int_0^t U(t,\tau)[-C(\tau)w(\tau) + f(\tau)]\,d\tau \tag{2.49}$$

Finally, we will prove part 2) of theorem (1.1). We have $w(0) = \phi_{xx} \in \in X_\pi^{s'-2}$ and (2.47) implies that $f \in C([0,T]; H_\pi^{s-1}) \subset C([0,T]; H_\pi^{s'-2})$, if $s' \le s+1$. From (2.46) we can show $t \to C(t) \in \mathcal{B}(H_\pi^{s'-2})$ is strongly continuous in t, if $s' \le s+1$. Since the evolution operator $\{U(t,\tau)\}_{0\le t\le\tau\le T}$ is strongly continuous on $H_\pi^{s'-2}$ to itself, it follows from lemma (2.9) that (2.49) is a integral equation of Volterra type in $H_\pi^{s'-2}$, which can be solved by sucessive aproximations. Thus $w \in C([0,T]; H_\pi^{s'-2})$ if $s' \le s+1$. We may now repeat the above argument to obtain the general case. □

3. Non-Periodic Case

In this section we will consider the Cauchy problem (1.1)-(1.2) in $\mathbb{R}^2$. We will start with the linear equation and then discuss the non-linear problem.

3.1. The Linear Equation. :

In this section we will analyse the Cauchy problem

$$(\partial_t u - iP(D)u)_x + \gamma\partial_y^2 u = 0 \tag{3.1}$$

$$u(0;x,y) = \phi(x,y) \tag{3.2}$$

We begin with some formal computations. Integrate the PDE in (3.1) from $-\infty$ to x to obtain

$$\partial_t u - iP(D)u + \gamma\partial_y^2 F(t;x,y) = g(t;y) \tag{3.3}$$

$$F(t;x,y) = \int_{-\infty}^{x} dx' u(t;x',y) \tag{3.4}$$

where $g(t;y)$ denotes a constant of integration (with respect to x). Taking the Fourier transform of (3.3) we get

$$\hat{u}_t + ip(\xi)\hat{u} - \gamma\eta^2\hat{F}(t;\xi,\eta) = \hat{g}(t;\xi,\eta) \tag{3.5}$$

Using the fact that the Dirac δ function can be written as

$$\delta(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} dx\, e^{-i\xi x} \tag{3.6}$$

it follows that

$$\hat{g}(t;\xi,\eta) = \sqrt{2\pi}(\mathcal{F}_2 g)(t;\eta)\delta(\xi) \tag{3.7}$$

Next, interchanging the order of integration we obtain

$$\begin{aligned}\hat{F}(t;\xi,\eta) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-i\eta y}\int_{-\infty}^{\infty} dx\, e^{-i\xi x}\left(\int_{-\infty}^{x} dx' u(t;x',y)\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy\, e^{-i\eta y}\int_{-\infty}^{\infty} dx' u(t;x',y)\int_{x'}^{\infty} dx\, e^{-i\xi x}\end{aligned} \tag{3.8}$$

But

$$\begin{aligned}\int_{x'}^{\infty} dx\, e^{-i\xi x} &= \int_{-\infty}^{\infty} dx\, H(x-x')e^{-i\xi x} = \\ &= p.v.\left(\frac{e^{-i\xi x'}}{i\xi}\right) + \pi\delta(\xi)\end{aligned} \tag{3.9}$$

wher $p.v.$ denotes the principal value distribuition associated to the function $(\frac{e^{-i\xi x'}}{i\xi})$ and $H(x)$ is the usual Heaviside function. It follows that

$$\hat{F}(t;\xi,\eta) = \frac{\hat{u}(t;\xi,\eta)}{i\xi} + \left[\frac{1}{2}\int_{-\infty}^{\infty} dy\, e^{-i\eta y}\int_{-\infty}^{\infty} dx' u(t;x',y)\right]\delta(\xi) \tag{3.10}$$

Since in the nonlinear case we will be dealing with the solutions in $H^s(\mathbb{R}^2)$ with $s > 2$ we must get rid of the δ functions. Thus if we choose

$$\sqrt{2\pi}(\mathcal{F}_2 g)(t;\eta) = -\gamma\eta^2\frac{1}{2}\int_{-\infty}^{\infty} dy\, e^{-i\eta y}\int_{-\infty}^{\infty} dx' u(t;x',y) \tag{3.11}$$

we obtain

$$g(t;y) = \frac{\gamma}{2}\partial_y^2\int_{-\infty}^{\infty} dx' u(t;x',y) \tag{3.12}$$

With this choice, $u(t;x,y)$ satisfies

$$\hat{u}_t + ip(\xi)\hat{u} + i\frac{\gamma\eta^2}{\xi}\hat{u} = 0 \qquad \xi \neq 0 \tag{3.13}$$

Taking the inverse Fourier transform,

$$u_t - iP(D)u + \gamma\partial_y^2\int_{-\infty}^{x} dx' u(t;x',y) = \frac{\gamma}{2}\partial_y^2\int_{-\infty}^{\infty} dx' u(t;x',y) \tag{3.14}$$

But

$$\int_{-\infty}^{x} dx' u(t;x',y) - \frac{1}{2}\int_{-\infty}^{\infty} dx' u(t;x',y) =$$
$$= \frac{1}{2}\left(\int_{-\infty}^{x} dx' u(t;x',y) - \int_{x}^{\infty} dx' u(t;x',y)\right) \tag{3.15}$$

so that with

$$\partial_x^{-1} f = \frac{1}{2}\left(\int_{-\infty}^{x} dx' f(x',y) - \int_{x}^{\infty} dx' f(x',y)\right) \tag{3.16}$$

(3.13) becomes,

$$u_t - iP(D)u + \gamma\partial_y^2\partial_x^{-1}u = 0 \tag{3.17}$$

Note that ∂_x^{-1} is exactly the antiderivative used in [1]. Moreover,

$$\widehat{(\partial_x^{-1}f)}(\xi,\eta) = \frac{1}{i\xi}\hat{f}(\xi,\eta) \tag{3.18}$$

which justifies the notation introduced in (1.9). These spaces were used by J.C.Saut in ([13]).

Now let

$$A_0 f = -iP(D)f + \gamma\partial_y^2\partial_x^{-1}f$$
$$= \left(i(p(\xi) + \gamma\frac{\eta^2}{\xi}\hat{f}(\xi,\eta)\right)^{\vee} \tag{3.19}$$

Lemma 3.1. *If $\phi \in X^s$ then $A_0\phi \in H^{s-l}$.*

Proof: Indeed,

$$\int_R d\xi \int_R d\eta\, (1+\xi^2+\eta^2)^{s-l}|p(\xi) - \gamma\frac{\eta^2}{\xi}|^2|\hat{\phi}(\xi,\eta)|^2 \leq$$
$$\leq C\int_R d\xi \int_R d\eta\, (1+\xi^2+\eta^2)^{s-l}(|p(\xi)\hat{\phi}(\xi,\eta)|^2 + |\gamma\frac{\eta^2}{\xi}\hat{\phi}(\xi,\eta)|^2)$$
$$\leq C\|\phi\|_s^2 + C|\gamma|\int_R d\xi \int_R d\eta\, (1+\xi^2+\eta^2)^{s-1}|\frac{\hat{\phi}(\xi,\eta)}{\xi}|^2$$
$$\leq C(\|\phi\|_s^2 + \|\partial_x^{-1}\phi\|_s^2) = |\phi|_s^2 \tag{3.20}$$

□

Lemma 3.2. $E_0(t) = e^{-tA_0}$ *(defined in the obvious way using the Fourier transform) is an unitary group both in* $H^s(\mathbb{R}^2)$ *and* $X_s(\mathbb{R}^2)$. *Moreover*

$$\partial_t u = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h} = -A_0 u(t) \tag{3.21}$$

in the topology of $H^s(\mathbb{R}^2)$ *with* $\phi \in D(A_0)$ *where*

$$D(A_0) = \{f \in H^s(\mathbb{R}^2) : |p(\xi) - \gamma \frac{\eta^2}{\xi}||\hat{f}(\xi,\eta)| \in L^2_s(\mathbb{R}^2)\} \tag{3.22}$$

and in the topology of $H^{s-l}(\mathbb{R}^2)$ *in case* $\phi \in X_s(\mathbb{R}^2)$.

Proof: The statements about H^s are easy consequence of the definition given above. It is also easy to verify that $E_0(t)$ also defines an unitary group in X_s. Now, if $\phi \in X_s$,

$$\|\frac{u(t+h) - u(t)}{h} - (-A_0 u(t))\|^2_{s-l} = \|\frac{E_0(h) - 1}{h} + A_0 \phi\|^2_{s-l} =$$

$$= \int_R d\xi \int_R d\eta |\frac{e^{ih(p(\xi) - \gamma\frac{\eta^2}{\xi})} - 1}{h} - i(p(\xi) - \gamma\frac{\eta^2}{\xi})|^2 |\hat{\phi}(\xi,\eta)|^2 \tag{3.23}$$

and the result follows from the dominated convergence theorem. □

Finally, we consider the regularized equation, namely

$$\partial_t u + A_\mu u = 0 \tag{3.24}$$

$$u(0) = \phi \in X_s \tag{3.25}$$

where

$$A_\mu = -\mu\Delta + A_0 \qquad \mu \geq 0 \tag{3.26}$$

Let

$$E_\mu(t) = e^{tA_\mu} \tag{3.27}$$

Since $\phi \in X_s$ iff $\phi \in H^s$ and $\partial_x^{-1}\phi \in H^s$ it is easy to show that $E_\mu(t)$, $\mu > 0$ is infinitely smoothing in X_s. Indeed, we have

Proposition 3.3. *Let* $\phi \in X_s$. *Then*

$$|E_\mu(t)\phi|_{s+\lambda} \leq K_\lambda (1 + (\frac{1}{2\mu t})^\lambda)^{1/2} |\phi|_s \tag{3.28}$$

for all $\lambda \geq 0$, $\mu > 0$, $t > 0$ *where* K_λ *is a constant that depends only on* λ.

Proof: Same as that of Lemma (1.1) in ([4]). □

3.2. The Non-Linear Equation. :.

In this section we will briefly discuss problem (1.1)-(1.2) endowed with the non-periodic boundary conditions and point out how it differs from the situation studied in the periodic case. We will employ the parabolic regularization method. The reason for this is that we have not been able so far to obatin convinient spaces (X and Y) and an isomorphism (S) between these spaces which are needed in the Kato's quasi-linear theory (see Lemma (2.7)).

Consider

$$\partial_t u + A_\mu u + a(u)\partial_x u = 0 \tag{3.29}$$

$$u(0) = \phi \tag{3.30}$$

$\phi \in X_s(\mathrm{IR}^2)$, $s > 2$. We will list a series of lemmas, most of which we will leave unproved since their proofs a rather similar to those given in ([4]).

Lemma 3.4. *For $\mu > 0$ problem (3.29)-(3.30) is equivalent to the integral equation*

$$u(t) = E_\mu(t)\phi - \int_0^t dt' E_\mu(t-t')(\partial_x A(u))(t') \tag{3.31}$$

where $A(\nu) = \int_0^\nu a(r)\,dr$. More precisely, if $\phi \in X_s$ then $u \in C([0,T];X_s)$ solves (3.29)-(3.30) (with $\partial_t u$ computed in H^{s-l}) if and only if u solves (3.31).

Lemma 3.5. *Let*

$$\mathcal{X}_s(T) = \{f \in C([0,T];X_s) : |E_\mu(t) - \phi|_s \leq M,\ \forall t \in [0,T]\} \tag{3.32}$$

where M , $T > 0$ are fixed. Then $\exists T > 0$ sufficiently small such that the map

$$(\mathcal{A}f)(t) = E_\mu(t) - \int_0^t dt' E_\mu(t-t')(\partial_x A(f))(t') \tag{3.33}$$

is a contraction in $\mathcal{X}_s(T)$.

Thus (3.31) has an unique solution in $\mathcal{X}_s(T)$ and therefore (3.29)-(3.30) also has this property. Standard arguments show that this is the unique solution in $C([0,T];X_s)$. Here $T = T(s, |\phi|_s, \mu)$.

Lemma 3.6. *$u \in C((0,T];X_\infty)$ where, $X_\infty = \bigcap_s X_s$ provided with the natural Frechet topology.*

Lemma 3.7. *The existence time can be chosen independently of μ.*

Proof: Since this is the most important part of the proof of well-posedness we look at it in some detail. Due to lemma (3.6) we have
($u := u_\mu$, $\mu > 0$)

$$\begin{aligned}\partial_t\|u_\mu(t)\|_s^2 &= 2(u|\partial_t u)_s = 2(u|-A_\mu u)_s - 2(u|a(u)u_x)_s \\ &= 2(u|\mu\Delta u)_s - 2(u|A_0 u)_s - 2(u|a(u)u_x)_s \\ &\leq -2\int_R d\eta \int_R d\xi\, \overline{\hat{u}(t;\xi,\eta)i(p(\xi)-\gamma\frac{\eta^2}{\xi})\hat{u}(t;\xi,\eta)} \\ &\quad - 2(u|a(u)u_x)_s \qquad (3.34)\end{aligned}$$

Since $\overline{\hat{u}(t;\xi,\eta)} = \hat{u}(t;-\xi,-\eta)$ because u is real-value and $p(\xi)$ is an odd real function, it follows that for each fixed $(t;\eta)$ (a.e) the integrand is odd in ξ so the integral is zero. Therefore

$$\begin{aligned}\partial_t\|u(t)\|_s^2 &\leq -2(u|a(u)u_x)_s \leq 2|(u|a(u)u_x)_s| \\ &\leq \tilde{a}(\|u(t)\|_s))\|u(t)\|_s^2 \leq \tilde{a}(|u(t)|_s))|u(t)|_s^2 \qquad (3.35)\end{aligned}$$

in view of Lemma (A.5) in ([10]) and $\tilde{a}$ is a positive monotone increasing function depending only on a and s (see Lemma A.3 in ([10])).

Next we look at the behavior of $\|\partial_x^{-1}u\|_s^2$. Proceeding as before we obtain,

$$\begin{aligned}\partial_t\|\partial_x^{-1}u(t)\|_s^2 &\leq -2(\partial_x^{-1}u|\partial_x^{-1}(a(u)u_x))_s = -2(\partial_x^{-1}u|\partial_x^{-1}\partial_x(A(u)))_s \\ &\leq 2|(\partial_x^{-1}u|A(u))_s| \leq \|\partial_x^{-1}u\|_s\|A(u)\|_s \leq C_s\tilde{a}(\|u\|_s)\|u\|_s|u|_s \\ &\leq \tilde{a}(|u(t)|_s)|u(t)|_s^2 \qquad (3.36)\end{aligned}$$

Therefore

$$\partial_t|u(t)|_s^2 \leq \tilde{a}(|u(t)|_s)|u(t)|_s^2 \qquad (3.37)$$

so that $|u(t)|_s^2 \leq \rho(t)$ where whenever both sides are defined, where

$$\begin{aligned}\partial_t\rho &= \tilde{a}(\rho^{1/2})\rho \\ \rho(0) &= |\phi|_s^2 \qquad (3.38)\end{aligned}$$

From now on the proof of existence and uniqueness for generalized KP should follow the same lines as those presented in [4]. The continuous dependence can be established using the Bona-Smith approximations as in [3]) and will therefore be omitted.

Out next task is to show that the existence time T may be assumed to be independent of s. We have established already the local well-posedness for

$$\partial_t u - iP(D)u + a(u)\partial_x u + \gamma\partial_x^{-1}\partial_y^2 u = 0 \qquad (3.39)$$

$$u(0;x,y) = \phi(x,y) \qquad (3.40)$$

in $C([0,T_s];X_s)\bigcap C^1([0,T];H^{s-3})$. Now, for $s' \geq s > 2$ and $\phi \in X_{s'}$ it is easy to see

$$T_{s'} \leq T_s \tag{3.41}$$

We will prove that $T_s = T$, for all $s > 2$. In view of (3.41), it is sufficient to show that $T_s = T_{s_0}$ for some $s_0 > 2$.

Note that $u = u(t)$ is a solution of (3.39)-(3.40) if and only if $v(t) = = P(-t)u(t)$ is a solution of

$$\partial_t v + \mathcal{A}(t,v)v = 0 \tag{3.42}$$

$$v(0;x,y) = \phi(x,y) \tag{3.43}$$

where $P(t) = e^{-t(-iP(D)+\gamma\partial_x^{-1}\partial_y^2)}$ and $\mathcal{A}(t,y) = P(-t)(a(P(t)y)\partial_x P(t)$.

We shall prove that the existence time for (3.42)-(3.43) can be chosen independently of s (which is equivalent to show $T_s = T_{s_0}$ for some $s_0 > 4$) and so the same holds for (3.39)-(3.40).

Let $s_0 > 4$, $\phi \in X_s$, $s \geq s_0$. Applying the operator ∂_x^2 to (3.42) yields the following linear evolution equation for $w(t) = \partial_x^2 v(t)$

$$\partial_t w + A(t)w = 0 \tag{3.44}$$

$$w(0) = \partial_x^2\phi \in X_{s-2} \tag{3.45}$$

where $A(t) = \partial_x P(-t)(\partial_x a(u(t)))P(t)\partial_x^{-1} + \partial_x P(-t)a(u(t))P(t) =: A_1(t)+ +A_2(t)$ and $u \in \in C([0,T_{s_0}];X_{s_0})$ is given. We know that $w \in C([0,T];X_{s_0-2})$ since $v \in C([0,T_{s_0}],X_{s_0})$. Moreover $w(0) = \partial_x^2\phi \in X_{s-2}$. It is our purpose to prove that $w \in C([0,T_{s_0}];X_{s-2})$ which will imply $v \in C([0,T_{s_0}],X_s)$ so that the same holds for $u = u(t)$. To this end, we have to study the linear evolution equation (3.44), i.e., the evolution operator $\{U(t,\tau)\}_{0\leq\tau\leq t\leq T_{s_0}}$ associated with the family $\{A(t)\}_{0\leq\tau\leq t\leq T_{s_0}}$.

Lemma 3.8. *The family $\{A(t)\}_{0\leq\tau\leq t\leq T_{s_0}}$ has an unique evolution operator $\{U(t,\tau)\}_{0\leq\tau\leq t\leq T_{s_0}}$ associated with the spaces $X = X_h$ and $Y = X_k$ in the usual sense, where*

$$4 - s_0 \leq h \leq s_0 - 3\,, \qquad 3 - s_0 \leq k \leq s_0 - 2\,, \qquad k \geq h+1 \tag{3.46}$$

In particular, $U(t,\tau) : X_r \to X_r$ for $4 - s_0 \leq r \leq s_0 - 3$.

Proof:

(i) $A(t)$ is an infinitesimal generator of a strongly continuous semigroup in

X_h and for $z \in X_h$ we have

$$\begin{aligned}
(A_1(t)z|z)_{X_h} &= (\Lambda^h \partial_x P(-t)(\partial_x a(u))\partial_x^{-1} P(t)z|\Lambda^h z)_0 + \\
&\quad + (\partial_x^{-1}\Lambda^h \partial_x P(-t)(\partial_x a(u))\partial_x^{-1} P(t)z|\partial_x^{-1}\Lambda^h z)_0 \\
&= (\Lambda^h \partial_x((\partial_x a(u))\partial_x^{-1} q)|\Lambda^h q)_0 + (\Lambda^h(\partial_x a(u))\partial_x^{-1} q|\partial_x^{-1}\Lambda^h q)_0 \\
&= (\Lambda^h(\partial_x^2 a(u))\partial_x^{-1} q|\Lambda^h q)_0 + (\Lambda^h(\partial_x a(u))q|\Lambda^h q)_0 \\
&\quad + (\Lambda^h(\partial_x a(u))\partial_x^{-1} q|\Lambda^h \partial_x^{-1} q)_0 \\
&= (\partial_x^2 a(u)\Lambda^h \partial_x^{-1} q|\Lambda^h q)_0 + ([\Lambda^h, \partial_x^2 a(u)]\partial_x^{-1} q|\Lambda^h q)_0 + \\
&\quad + (\partial_x a(u)\Lambda^h q|\Lambda^h \partial_x^{-1} q)_0 + ([\Lambda^h, \partial_x a(u)]q|\Lambda^h \partial_x^{-1} q)_0 \\
&\quad + (\partial_x a(u)\Lambda^h \partial_x^{-1} q|\Lambda^h \partial_x^{-1} q)_0 + ([\Lambda^h, \partial_x a(u)]\partial_x^{-1} q|\Lambda^h \partial_x^{-1} q)_0 \\
&\geq - \sup_{0 \leq t \leq T_{s_0}} \|a(u(t))\|_{s0}^2 \|q\|_{X_h}^2 = -\beta_1 \|z\|_{X_h}^2 \qquad (3.47)
\end{aligned}$$

where $q = P(t)z$, $P(t)$ is unitary in X_h, commutes with ∂_x, ∂_x^{-1} and $\Lambda = (1-\Delta)^{1/2}$. We estimate the first, the third and the fifth terms as usual while, in order to control the other three, we employ Lemma 2.6 of [9].

$$\begin{aligned}
(A_2(t)z|z)_{X_h} &= (\Lambda^h \partial_x P(-t)a(u)P(t)z|\Lambda^h z)_0 \\
&\quad + (\partial_x^{-1}\Lambda^h \partial_x P(-t)a(u)P(t)z|\partial_x^{-1}\Lambda^h z)_0 \\
&= (\Lambda^h(\partial_x a(u))q|\Lambda^h q)_0 + (\Lambda^h a(u)\partial_x q|\Lambda^h q)_0 + (\Lambda^h a(u)q|\partial_x^{-1}\Lambda^h q)_0 \\
&= (\partial_x a(u)\Lambda^h q|\Lambda^h q)_0 + ([\Lambda^h, \partial_x a(u)]q|\Lambda^h q)_0 \\
&\quad + (a(u)\partial_x \Lambda^h q|\Lambda^h q)_0 + ([\Lambda^h, a(u)]\partial_x q|\Lambda^h q)_0 \\
&\quad + (a(u)\Lambda^h q|\partial_x^{-1}\Lambda^h q)_0 + ([\Lambda^h, a(u)]q|\partial_x^{-1}\Lambda^h q)_0 \\
&\geq - \sup_{0 \leq t \leq T_{s_0}} \|a(u(t))\|_{s0}^2 \|q\|_{X_h}^2 = -\beta_2 \|z\|_{X_h}^2 \qquad (3.48)
\end{aligned}$$

where we used the same ideas as above. Then, by (3.47) and (3.48), we obatin

$$(A(t)z|z)_{X_h} \geq -\beta \|z\|_{X_h}^2 \qquad \beta = \beta_1 + \beta_2. \qquad (3.49)$$

(ii) Let $S = \Lambda^{k-h} : Y \to X$ be an isomorfism from $Y = X_k$ onto $X = X_h$. We have to prove that the operator $B(t) := [S, A(t)]S^{-1}$ is uniformly bounded in $X = X_h$ for $t \in [0, T_{s_0}]$.

We observe that $P(t)$ is unitary in X_h, commutes with ∂_x, ∂_x^{-1} and $\Lambda = (1-\Delta)^{1/2}$ and so it sufficies to show that

$$\begin{aligned}
\Lambda^h \partial_x [\Lambda^{k-h}, \partial_x a(u(t))]\Lambda^{-k}\partial_x^{-1} &+ \Lambda^h \partial_x [\Lambda^{k-h}, a(u(t))]\Lambda^{-k} \\
&=: B_1(t) + B_2(t) \qquad (3.50)
\end{aligned}$$

is uniformly bounded in X_0. For $q \in X_0$ we have

$$\begin{aligned}
\|B_1(t)q\|_{X_0}^2 &= \|\Lambda^h\partial_x[\Lambda^{k-h},\partial_x a(u)]\Lambda^{-k}\partial_x^{-1}q\|_{X_0}^2 \\
&= \|\Lambda^h\partial_x[\Lambda^{k-h},\partial_x a(u)]\Lambda^{-k}\partial_x^{-1}q\|_0^2 \\
&\quad + \|\partial_x^{-1}\Lambda^h\partial_x[\Lambda^{k-h},\partial_x a(u)]\Lambda^{-k}\partial_x^{-1}q\|_0^2 \\
&= \|\Lambda^h\partial_x[\Lambda^{k-h},\partial_x a(u)]\Lambda^{-k}\partial_x^{-1}q\|_0^2 + \|\Lambda^h[\Lambda^{k-h},\partial_x a(u)]\Lambda^{-k}\partial_x^{-1}q\|_0^2 \\
&\leq c \sup_{0\leq t\leq T_{s_0}} \|a(u(t))\|_{s_0}^2\|q\|_{X_0}^2 \ , \quad 0 \leq t \leq T_{s_0}
\end{aligned} \tag{3.51}$$

where we apply Lemma 2.6 of [9]. In the same way

$$\begin{aligned}
\|B_2(t)\|_{X_0}^2 &= \|\Lambda^h\partial_x[\Lambda^{k-h},a(u)]\Lambda^{-k}q\|_{X_0}^2 \\
&= \|\Lambda^h\partial_x[\Lambda^{k-h},a(u)]\Lambda^{-k}q\|_0^2 + \|\partial_x^{-1}\Lambda^h\partial_x[\Lambda^{k-h},a(u)]\Lambda^{-k}q\|_0^2 \\
&= \|\Lambda^h\partial_x[\Lambda^{k-h},a(u)]\Lambda^{-k}q\|_0^2 + \|\Lambda^h[\Lambda^{k-h},a(u)]\Lambda^{-k}q\|_0^2 \\
&\leq c \sup_{0\leq t\leq T_{s_0}} \|a(u(t))\|_{s_0}^2\|q\|_{X_0}^2 \ , \quad 0 \leq t \leq T_{s_0}
\end{aligned} \tag{3.52}$$

so (3.51), (3.52) imply that $B = B(t)$ is uniformly bounded in X_h for $t \in [0, T_{s_0}]$.

(iii) It is not difficult to show that

a) $Y = X_k \subseteq D(A(t))$, $\forall t \in [0, T_{s_0}]$.

b) $A(t) \in \mathcal{B}(Y; X)$, $\forall t \in [0, T_{s_0}]$

c) $A(.) : [0, T_{s_0}] \to \mathcal{B}(Y; X)$ is strongly continuous.

By Kato's linear theory (see [8]) , we obtain an unique evolution operator $\{U(t,\tau)\}_{0\leq t\leq T_{s_0}}$ associated with $\{A(t)\}_{0\leq t\leq T_{s_0}}$. □

It is easy to see that

Lemma 3.9. *We have*

$$w(t) = U(t,0)w(0) \ , \quad 0 \leq t \leq T_{s_0} \tag{3.53}$$

Using this lemma we obtain:

Lemma 3.10. $w \in C([0, T_{s_0}]; X_{s-2})$, $s \geq s_0$.

Proof:

We observe that $w(0) = \partial_x^2\phi \in X_{s-2} \hookrightarrow X_{s_0-2}$. If $s_0 \leq s \leq s_0 + 1$ then $\{U(t,\tau)\}_{0\leq t\leq T_{s_0}}$ is strongly continuous in X_{s-2}. Lemma (3.9) then implies the desire result. □

Finally

$$w \in C([0, T_{s_0}]; X_{s-2}) \Rightarrow v \in C([0, T_{s_0}]; X_s) \Rightarrow u \in C([0, T_{s_0}]; X_s) \tag{3.54}$$

if $s_0 \leq s \leq s_0+1$. If $s > s_0+1$ we will apply the above argument successively. □

References

[1] M.J.Ablowitz, J.Villarroel - *On the Kadomtsev-Petviashvili equation and associated constraints.* Studies in Appl. Math., 85, (1991), pp.195-213.

[2] J.Bourgain - *On the Cauchy problem for the Kadomtsev-Petviashvili equation.* Geom. Func. Anal., 3, (1993), 315-341.

[3] J.L.Bona, R.Smith - *The initial-value problem for the Korteweg-de Vries equation.* Philos. Trans. Roy. Soc. London, Ser. A 278, (1975), pp. 555-600

[4] R.J.Iório Jr. - *KdV, BO and friends in weighted Sobolev spaces.* Functional-Analytic Methods for Partial Differential Equations. Lect. Notes in Math., 1450, ed. T.Ikebe, H.Fujita, S.T.Kuroda, (1990), pp. 104-121.

[5] P.Isaza J., J.Mejía L., V.Stallbohm - *Local solution for the Kadomtsev-Petviashvili equation with periodic conditions.* Manuscripta Math., 75, (1992), pp. 383-393.

[6] T.Kato - *Linear evolution equations of "hyperbolic" type.* J. Fac. Sci. Univ. of Tokio, vol. 17, (1970), pp. 241-258.

[7] T.Kato - *Linear evolution equation of "hyperbolic" type. II.* J. of the Math. of Japan, vol. 25, n.º 4, (1973), pp. 648-666.

[8] T. Kato - *Quasi-linear equations of evolution with applications to partial differential equations.* Lect. Notes in Math., 448, (1975), pp. 25-70.

[9] T.Kato - *On the Korteweg-de Vries equation.* Manuscripta Math., 28, (1979), pp. 89-99.

[10] T.Kato - *On the Cauchy problem for the (generalized) Korteweg-de Vries equation.* Studies in Appl. Math. Suppl. Studies, vol. 8, AP, (1983), pp. 93-128.

[11] B.B.Kadomtsev, V.L.Petviashvili - *On the stability of solitary waves in weakly dispersive media.* Sov. Phys. Dokl., 15, (1970), pp. 539-543.

[12] A.Pazy -*Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, AMS, vol. 44, (1983).

[13] J.C.Saut - *Remarks oh teh generalized Kadomtsev-Petviashili equations.* preprint (1993). to appear in Math. J.Fall, (1993).

[14] S.Ukai - *On the Cauchy problem for the KP-equation.* Lect. Notes in Num. Appl. Anal., 10, (1989), pp. 179-194.

[15] M.V.Wickerhauser - *Inverse scattering for the heat operator and evolution in 2+1 variables.* Comm. Math. Phys., 108, (1987), pp. 67-87.

[16] D.Wu, S.-L. Wen - *A uniqueness theorem for the two-dimensional generalized Korteweg-de Vries equation.* J. Math. Anal. and Appl., 158, (1991), pp. 203-212.

NOTAS DO ICMSC

SÉRIE MATEMÁTICA

039/96 MENEGATTO, VALDIR A.; PERON A.P. - Generalized hermite interpolation on spheres via positive definite and related functions.

038/96 MENEGATTO, VALDIR A. - Interpolation on the complex hilbert sphere using positive definite and conditionally negative definitive kernels.

037/96 MENEGATTO, VALDIR A. - Eigenvalue estimates for distance matrice associated with conditionally negative definite functions on spheres.

036/96 CARTER, J.S.; RIELGER, J.H.; SAITI, M. - A combinatorial description of knotted surfaces and their isotopies.

035/96 CARVALHO, A.N.; DLOTKO, T. - Parabolic problems in H'with fast grawing nonlinearities

034/96 KUSHNER, A.L. - Finite determination on algebraic sets.

033/95 CARVALHO, A.N.; OLIVA, S.M.; PEREIRA, A.L.; BERNAL, A.R. - Attractors for parabolic problem with nonlinear baundary conditions.

032/95 GIBSON, C.G.; HOBBS, C.A.; MARAR, W.L. - On the bifurcation of general two-dimensional spatial motions[1]

031/95 LUCAS, L.A., MANZOLI-NETO, O. and SAEKI, O. - An unknotting theorem for $S^p \times S^q$ EMBEDDED IN S^{P+q+2}

030/95 BIASI, C. and GODOY, S.M.S. - On the aleatory variable $Y_n = 10^n X - [10^n X]$ for large n