UNIVERSIDADE DE SÃO PAULO

SECOND ORDER CONDITIONS FOR LOCAL CONTROLLABILITY

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NOTAS

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Abstract

This paper presents second order sufficient conditions for small time local controllability of affine smooth control systems, based on a very simple observation that can also lead to higher order conditions; a second order obstruction for a certain class of systems is also discussed, as is the gap between the obstruction and the sufficient condition. The new results are used for deciding local controllability of systems for which a direct application of the usual criteria is not conclusive.

Sumário

Neste artigo são apresentadas condições suficientes de segunda ordem para a controlabilidade local em tempo pequeno de sistemas de controle afins, analíticos ou infinitamente diferenciáveis, baseadas numa observação muito simples que pode levar também a condições de ordem superior; são discutidas uma obstrução de segunda ordem para uma certa classe de sistemas e as situações intermédias entre a obstrução e a condição suficiente. Os resultados obtidos são usados para decidir a controlabilidade local de sistemas para os quais a aplicação dos critérios usuais não é conclusiva.

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Abstract

This paper presents second order sufficient conditions for small time local controllability of affine smooth control systems, based on a very simple observation that can also lead to higher order conditions; a second order obstruction for a certain class of systems is also discussed, as is the gap between the obstruction and the sufficient condition. The new results are used for deciding local controllability of systems for which a direct application of the usual criteria is not conclusive.

Keywords: Non linear control systems, affine systems, local controllability, geometric methods, Lie brackets.

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1 Basic results and definitions

The control systems considered here are analytic or smooth affine systems, of the form:

$$\dot{x} = X(x) + \sum_{i=1}^{s} u_i X^i(x), \quad |u_i| \leq \alpha_i \tag{1}$$

where X and X^i are analytic or smooth (C^∞) vector fields on an analytic (smooth) connected manifold M, and $\alpha_i \in \mathbf{R}^+$. As the results are local, M can always be identified with $\mathbf{R}^n$, with n the dimension of M.

Let $\mathcal{U}$ be the set of admissible controls, piecewise constant maps from a subinterval of a fixed interval $[0,T] \subset \mathbf{R}$ into $\mathbf{R}^s$, with the topology induced by the topology of $L^1([0,T], \mathbf{R}^s)$.

The set $A(p,T)$ of the points attainable from p in up to time T as the set of points p' such that there exists a continuous piecewise C^1 map $c : [0, t^*] \mapsto M$ verifying:

- $c(0) = p$, $c(t^*) = p'$ with $0 < t^* \leq T$;
- There exist $t_0, t_1, \ldots, t_m$, with $0 = t_0 < t_1 < \ldots < t_m = t^*$, such that in $]t_{j-1}, t_j[$ the map c is an integral curve of $X(x) + \sum_{i=1}^{s} u_i X^i(x)$ for some constants u_i with $|u_i| \leq \alpha_i$.

A control system will be said to be small time locally controllable (STLC) at p if $A(p,T)$ contains p in its interior for any positive T.

Very general results on local controllability at a point were presented in [5] and[6], and subsequently generalized in [3] and [4]; the resulting sufficient conditions for the systems of the form (1) to be STLC are recalled below.

A vector $v \in T_pM$ is a regular variation of order α [4] at p if there exist positive real numbers $\overline{\gamma}$, $\overline{c}$, $\overline{\varepsilon}$ and a continuous map $\eta(\gamma, c, \varepsilon)$, $\eta : \; [0, \overline{\gamma}] \times [0, \overline{c}] \times [0, \overline{\varepsilon}] \longrightarrow \mathcal{U}$ such that (in a convenient chart):

$$X_{-\varepsilon-\varepsilon\gamma} Y_\varepsilon(p) = p + \varepsilon^\alpha c v + o(\varepsilon^\beta), \quad \beta > \alpha$$

where Y is the vector field corresponding to the control $u = \eta(\gamma, c, \varepsilon)$. The variational cone $\mathcal{K}$ at $P \in M$ is the convex hull of all variations at p.

Let $\mathcal{L}(\xi)$ be the free Lie algebra generated by the $s+1$ indeterminates $\xi_0, \xi_1, \ldots, \xi_s$, and $\mathcal{I}$ the ideal generated by $\xi_1, \ldots, \xi_s$. Substituting X for ξ_0 and X^i for ξ_i, a vector field Λ_X on M is associated to every element Λ of $\mathcal{L}(\xi)$; Λ_X is the evaluation of Λ.

Given $\Lambda \in \mathcal{L}(\xi)$, $|\Lambda|_i$ is the number of times ξ_i appears in Λ; an admissible set of weights $w = (w_0, w_1, \ldots, w_s)$ is a collection of $s+1$ non negative real numbers with $0 \leq w_0 \leq w_i$.

The w-weight of a bracket Λ is given by:

$$\|\Lambda\|_w = \sum_{i=0}^{s} w_i |\Lambda|_i$$

and an element in $\mathcal{L}(\xi)$ is w-homogeneous if it is a linear combination of brackets of the same w-weight. The length of the bracket Λ is its weight for the standard choice $w_0 = w_1 = \ldots = w_s = 1$, or equivalently $\sum_{i=0}^{s} |\Lambda|_i$.

Denote by V_i^w the set of vector fields that are linear combinations of evaluations of elements of $\mathcal{I}$ with w-weight not bigger than i:

$$V_i^w = \text{span}\{\Lambda_X : \Lambda \in \mathcal{I}, \|\Lambda\|_w \leq i\}$$

An w-homogeneous element Λ in $\mathcal{L}(\xi)$ is said to be w-neutralized if:

$$\Lambda_X \in V_i^w, \quad i < \|\Lambda\|_w$$

The obstructions to local controllability appear from evaluating the elements of the set $\mathcal{B}_S^w$, where:

$$\begin{aligned} \mathcal{B} &= \text{span}\{\Lambda \in \mathcal{I} : \ |\Lambda|_0 \text{ is odd }, |\Lambda|_i \text{ is even for every } i \geq 1\} \\ \mathcal{B}_S^w &= \{\Lambda \in \mathcal{B},\ \Lambda \text{ symmetric on the } \xi_i,\ i \geq 1, \text{ with same weight}\} \end{aligned}$$

It is proved in [4] that, given a bracket Λ, if all brackets with w-weight not bigger than $\|\Lambda\|_w$ can be w-neutralized, then the evaluation of Λ at p is a regular variation, even more, the evaluation at p of $\text{ad}_{\xi_0}^j(\Lambda)$, $j > 0$, is a regular variation.

Let $r(w)$ be the biggest integer such that all obstructions of w-weight not bigger than $r(w)$ are w-neutralized, and:

$$\mathcal{V}^w = \{Z : Z = \text{ad}_X^j(Y), \quad Y \in V_{r(w)}^w, \quad j \geq 0\}$$

where $\text{ad}_X^0(Y) = Y$ and $\text{ad}_X^{j+1}(Y) = [X, \text{ad}_X^j(Y)]$.

Theorem 1 ([4]) *If $\mathcal{K} = T_pM$ then the system is STLC at p; in particular, a sufficient condition for small time local controllability at a point p of a control system is:*

$$T_pM = \sum_w \mathcal{V}^w(p)$$

Example 1 Consider the analytic system on $\mathbf{R}^3$ defined by:

$$X = (z - ax^2 - 2bxy - cy^2)\frac{\partial}{\partial z}, \quad X^1 = \frac{\partial}{\partial x}, \quad X^2 = \frac{\partial}{\partial y}$$

with $\alpha_1 = \alpha_2 = 1$; the coefficients a, b and c are not all zero.

To use theorem 7.2 in [6], where all input vector fields have the same weight, the obstruction that it is necessary to neutralize is:

$$[X^1, [X^1, X]](0) + [X^2, [X^2, X]](0) = -2(a+c)\frac{\partial}{\partial z}$$

and therefore only if $a+c=0$ is local controllability guaranteed at the origin. Using different weights for the two input vector fields, both $[X^1, [X^1, X]](0)$ and $[X^2, [X^2, X]](0)$ are obstructions, and then $a = c = 0$ is needed.

On the other hand, the geometric methods presented in [2] show the system is locally controllable at the origin if the quadratic form $ax^2 + 2bxy + cy^2$ is indefinite, i.e. $b^2 - ac > 0$, and not locally controllable if it is positive or negative definite, $b^2 - ac < 0$; of course $a = c = 0 \Longrightarrow a + c = 0 \Longrightarrow b^2 - ac > 0$.

A similar example is also considered in [1], and a different argument is used to prove the non existence of local controllability, that the direction field defined on the surface H, where X, X^1 and X^2 are linearly dependent, by the plane spanned by the input vector fields has a focus at the origin. In theorem 3 of [1] and in the subsequent example a necessary assumption for that argument to be valid is missing: the input vector fields have to define a two dimensional involutive distribution. That situation is dealt with in [2] with the correct assumptions.

2 Second order sufficient conditions and obstructions

A very simple observation allows the extension of the above sufficient condition: define a new control system Σ_A, with dynamics given by:

$$\dot{x} = X(x) + \sum_{i=1}^{s} u_i Y^i(x), \quad Y^i = \sum_{j=1}^{s} a_{ij} X^j \qquad |u_i| \le \alpha_i \tag{2}$$

If all real numbers a_{ij} are sufficiently close to zero, small time local controllability of the above system (2) implies that of the original system (1); it is straightforward to show that a regular variation for system Σ_A is also a regular variation for the original system.

As A can be chosen to be the identity, the criteria based on considering all systems of the above form (2) are an extension of the previous sufficient condition of theorem 1.

This approach leads to very general sufficient conditions, but second order conditions present a clearer picture and are indicative of its power and limitations.

In what follows, unless explicitly stated and for simplicity sake, it will be assumed that $M = \mathbf{R}^n = \mathbf{R}^s \times \mathbf{R}^k$ and:

$$X(x) = \sum_{j=1}^{k} \varphi_j(x) \frac{\partial}{\partial x_{s+j}}, \qquad X^i(x) = \frac{\partial}{\partial x_i} \quad i = 1, \ldots, s \tag{3}$$

with $\varphi_j(0) = 0$ and $d\varphi_j(0) = 0$.

The second derivative of φ_i defines a bilinear form B^i on $\mathbf{R}^s$ and also a quadratic form $Q^i(v) = B^i(v,v) = H_{\varphi_i}(v)$, where H_{φ_i} is the Hessian of φ_i.

Define $\xi_i \in \mathbf{R}^s$ by $\xi_i = (a_{i1}, \ldots, a_{is})$; a straigtforward computation gives:

$$Z^{il} = [Y^i, [Y^l, X]] = \sum_{j=1}^{k} B^j(\xi_i, \xi_l) \frac{\partial}{\partial x_{s+j}}$$

and therefore system (3) is STLC if it is possible to find s vectors $\xi_i \in \mathbf{R}^s$ such that $\sum_{i=1}^{s} [Y^i, [Y^i, X]] = 0$, or equivalently:

$$\sum_{i=1}^{s} Q^j(\xi_i) = \sum_{i=1}^{s} H_{\varphi_j}(\xi_i) = 0, \quad j = 1, \ldots, s$$

with the vector fields Z^{il} spanning $\mathbf{R}^k$.

Proposition 1 *If one of the forms $Q^i = H_{\varphi_i}$ is definite for some i, the system is not locally controllable.*

Proof: Assume $Q^k > 0$ and define $\psi(x) = x_n$. Then $L_{X^i}\psi \equiv 0$ and $L_X\psi = \varphi_k$, and so $L_Y\psi = \varphi_k$ for any admissible vector field Y; as the condition $Q^k > 0$ implies $\varphi_k \geq 0$ on a neighbourhood of the origin, it follows that ψ increases along admissible trajectories issuing from the origin, at least for a small time, and the system cannot be locally controllable.

Remarks:

- This generalizes to multi-input systems, of the particular form (3), the second order obstruction $[X^1,[X^1,X]]$ presented in [5] for single input systems.
- The same reasoning leads to the following more general result: if the first non zero derivative of some φ_i corresponds to a definite form the system is not locally controllable.
- The form does not need to be definite: if it is semidefinite in all variables and definite in the variables $(x_1,\ldots,x_s)$ the conclusion is still the same.

In the codimension 1 case, where $k=1$, the technique used in [2] can be extended in an obvious way to prove the following:

Theorem 2 *Let $M=\mathbf{R}^n$ and:*

$$X(x)=\varphi(x)\frac{\partial}{\partial x_n}, \qquad X^i(x)=\frac{\partial}{\partial x_i} \quad i=1,\ldots,n-1$$

If the origin is an isolated critical point of φ on $\{x_n=0\}$ and $\varphi(0)=0$, the system is not locally controllable at the origin if the critical point is a maximum or minimum and it is STLC otherwise.

Example 2 Consider the analytic system on $\mathbf{R}^3$ defined by:

$$X=(x^2+ay^4)\frac{\partial}{\partial z}, \quad X^1=\frac{\partial}{\partial x}, \quad X^2=\frac{\partial}{\partial y}$$

with $\alpha_1=\alpha_2=1$. The criteria based on Lie brackets analysis are not conclusive, but since $\varphi(x,y)=x^2+ay^4$ has an isolated critical point at the origin if $a\neq 0$, where it is zero, and the critical point is not a maximum nor a minimum if $a<0$, the system is then STLC at the origin if $a<0$ and not locally controllable if $a>0$.

Thus in the codimension one case the geometric analysis of theorem 2 can give an answer in situations where the other criteria do not apply, but its extension to higher codimensions is difficult, and will not be pursued here.

In the codimension two case ($k=2$) the approach described above leads to a sufficient condition for small time local controllability but also to a class of systems for which the method is not conclusive.

Proposition 2 *Let $s = k = 2$; if both forms Q^1 and Q^2 are indefinite, i.e. if:*

$$\begin{aligned}([X^1,[X^2,X]]_3)^2 - [X^1,[X^1,X]]_3[X^2,[X^2,X]]_3 &> 0 \\ ([X^1,[X^2,X]]_4)^2 - [X^1,[X^1,X]]_4[X^2,[X^2,X]]_4 &> 0\end{aligned}$$

and their zero directions alternate in the projective line $\mathbf{RP}^2$, the system is small time locally controllable.

Proof:

The condition on the zero directions implies the existence of ξ and η such that:

$$Q^1(\xi) = Q^2(\xi) > 0, \quad Q^1(\eta) = Q^2(\eta) < 0$$

If $Q^1(\xi) = -k^2 Q^1(\eta)$, then:

$$Q^1(\xi) + Q^1(k\eta) = 0, \quad Q^2(\xi) + Q^2(k\eta) = 0$$

On the other hand:

$$Q^i(\xi + k\eta) = Q^i(\xi) + Q^i(k\eta) + 2B^i(\xi, k\eta) = 2kB^i(\xi,\eta), \quad i = 1,2$$

and the linear dependence condition

$$\begin{vmatrix} Q^1(\xi) & Q^2(\xi) \\ B^1(\xi,k\eta) & B^2(\xi,k\eta) \end{vmatrix} = \frac{Q^1(\xi)}{2}\begin{vmatrix} 1 & 1 \\ 2kB^1(\xi,\eta) & 2kB^2(\xi,\eta) \end{vmatrix} = 0$$

is equivalent to $Q^1(\xi + k\eta) = Q^2(\xi + k\eta)$; as ξ and η are uniquely defined up to multiplication for a nonzero real number, this means that the vectors $\xi + k\eta$ and ξ, or $\xi + k\eta$ and η, are parallel, which is impossible.

The new system, defined by:

$$\dot{x} = X(x) + u_1 Y^1(x) + u_2 Y^2(x), \quad Y^1 = \xi_1 X^1 + \xi_2 X^2, \quad Y^2 = k\eta_1 X^1 + k\eta_2 X^2$$

and the original system are STLC at the origin since the second order obstruction can be neutralized:

$$\begin{aligned}&[Y^1,[Y^1,X]](0) + [Y^2,[Y^2,X]](0) = \\ &= (Q^1(\xi) + Q^1(k\eta))\frac{\partial}{\partial x_3} + (Q^2(\xi) + Q^2(k\eta))\frac{\partial}{\partial x_4} = 0\end{aligned}$$

and the vector fields $Y^1(0)$, $Y^2(0)$, $[Y^1,[Y^1,X]](0)$ and $[Y^1,[Y^2,X]](0)$ are linearly independent.

Next example shows that the condition on the zero directions of the two forms cannot be ignored:

Example 3 Consider the analytic system on $\mathbf{R}^4$ defined by:

$$X = xy\frac{\partial}{\partial z} + (10xy - 3x^2 - 3y^2)\frac{\partial}{\partial w}, \quad X^1 = \frac{\partial}{\partial x}, \quad X^2 = \frac{\partial}{\partial y}$$

with $\alpha_1 = \alpha_2 = 1$. The quadratic forms involved are:

$$Q^1(v) = 2v_1v_2, \quad Q^2(v) = 2(10v_1v_2 - 3v_1^2 - 3v_2^2)$$

and to be able to reach a conclusion using our second order condition it is necessary to to find ξ and η such that:

$$Q^1(\xi) + Q^1(\eta) = 0, \quad Q^2(\xi) + Q^2(\eta) = 0$$

The first condition implies that ξ and η belong to quadrants of different parity, but as the maximum of Q^2 in the second quadrant is smaller than the symmetric of that maximum in the first quadrant, the second condition cannot be fulfilled.

This example is not exceptional; in fact the possibility of applying this method, i.e. finding a system of the form (2) which can be proved to be STLC using theorem 1, implies the condition on the zero directions of the two quadratic forms involved.

Assume there exist ξ and η such that:

$$Q^1(\xi) + Q^1(\eta) = 0, \quad Q^2(\xi) + Q^2(\eta) = 0$$

with $Q^1(\xi)$ and $Q^2(\xi)$ not both zero. Then, considering a convenient multiple of one of the forms if necessary, and this does not change the zero directions, it can be assumed that:

$$Q^1(\xi) = Q^2(\xi) > 0, \quad Q^1(\eta) = Q^2(\eta) < 0$$

The form $Q^1 - Q^2$ has zero directions defined by ξ and η, therefore it is indefinite or identically zero; the second alternative implies the collinearity of $[Y^1,[Y^1,X]](0)$, $[Y^2,[Y^2,X]](0)$ and $[Y^1,[Y^2,X]](0)$ and thus only the first alternative can be considered.

If the zero directions are not separated there exists a third vector ζ such that $Q^1(\zeta) = Q^2(\zeta)$: the two graphs of Q^1 and Q^2 on the projective line $\mathbf{RP}^2$ intersect transversally at $[\xi]$ and $[\eta]$, since $Q^1 - Q^2$ is indefinite, and non separation implies a third intersection. Then $Q^1 - Q^2$ has three different zero directions and it is not identically zero; the contradiction proves the zero directions are separated.

The criteria are not conclusive, but that does not mean the system involved is not STLC, just that a more detailed study will be needed.

The results of proposition 2 can be extended to all codimension two systems of the form (3):

Proposition 3 *Consider the codimension two case, with $k = 2$ and s arbitrary; if there exists a plane P such that the restrictions of both forms Q^1 and Q^2 to P are indefinite and their zero directions alternate in the projective line $\mathbf{RP}^2$, the system is small time locally controllable.*

Proof:

As before, there exist ξ and η such that:

$$Q^1(\xi) = Q^2(\xi) > 0, \quad Q^1(\eta) = Q^2(\eta) < 0$$

and:

$$Q^1(\xi) + Q^1(\eta) = 0, \quad Q^2(\xi) + Q^2(\eta) = 0$$

Consider the system Σ' defined by:

$$\dot{x} = X(x) + v_1 Y^1(x) + v_2 Y^2(x), \quad Y^1 = \sum_{i=1}^{s} \xi_i X^i, \quad Y^2 = \sum_{i=1}^{s} \eta_i X^i$$

The obstructions of length not bigger than three can be neutralized:

$$[Y^1,[Y^1,X]](0) + [Y^2,[Y^2,X]](0) =$$
$$= (Q^1(\xi) + Q^1(\eta))\frac{\partial}{\partial x_{s+1}} + (Q^2(\xi) + Q^2(\eta))\frac{\partial}{\partial x_{s+2}} = 0$$

and so $Y^1(0)$, $Y^2(0)$, $[Y^1,[Y^1,X]](0)$, $[Y^2,[Y^2,X]](0)$ and $[Y^1,[Y^2,X]](0)$ are regular variations of Σ'.

As remarked before, it follows from the definition of regular variations that these vectors are also variations of the original system, and together with the input vector fields X^i, also regular variations, span the tangent space at the origin. Therefore the original system is STLC.

Example 4 Consider the analytic system on $\mathbf{R}^5$ defined by:

$$X = (z^2 - x^2 - y^2)\frac{\partial}{\partial w_1} + (x^2 + y^2 + z^2 + 2axy + 2bxz + 2cyz)\frac{\partial}{\partial w_2}$$

and

$$X^1 = \frac{\partial}{\partial x}, \quad X^2 = \frac{\partial}{\partial y}, \quad X^3 = \frac{\partial}{\partial z}$$

with $\alpha_1 = \alpha_2 = 1$.

It is clear that there is a second order obstruction when using the sufficient condition of theorem 1. On the other hand, the quadratic forms involved are:

$$Q^1(v) = 2(v_3^2 - v_1^2 - v_2^2), \quad Q^2(v) = 2(v_1^2 + v_2^2 + v_3^2 + 2av_1v_2 + 2bv_1v_3 + 2cv_2v_3)$$

and the planes $v_1 = 0$ or $v_2 = 0$ satisfy the conditions of proposition 3, and the system is STLC, if $|c| > 1$ or $|b| > 1$, respectively.

Remark: The sufficient condition of proposition 3, and the general method, are valid for any system such that its second order jet has the form (3), i.e. $j^2(Z) = X$ and $j^2(Z^i) = X^i$ for $i = 1, \ldots, s$; the third and higher order terms do not affect the values at the origin of the vector fields involved: Z^i, $[Z, Z^i]$ and $[Z^j, [Z^i, Z]]$.

For a general system of the form (1) proposition 3 is no longer a sufficient condition: the linear parts of the drift and input vector fields provide new regular variations, but can also lead to obstructions at the level considered.

Assume the control system has the general form (1), let the weights be given by $w_0 = w_1 = w_2 = 1$, and define:

$$\begin{aligned} \mathcal{W}^1 &= \{Z : Z = \mathrm{ad}_X^j(Y), \quad Y \in V_2^w, \quad j \geq 0\} \\ \mathcal{W}^2 &= \{Z : Z = \mathrm{ad}_X^j(Y), \quad Y \in V_3^w, \quad j \geq 0\} \end{aligned}$$

Proposition 4 ([4]) First order sufficient condition: *A general affine control system is STLC at $p \in M$ if $T_pM = \mathcal{W}^1(p)$.*

If the first order condition is not applicable, the method discussed previously leads to the following:

Theorem 3 Second order sufficient condition: *Let $n_1, \ldots, n_k$ form a basis of a complementary space of $\mathcal{W}^1(p)$ in T_pM, and define the quadratic forms on $\mathbf{R}^s$:*

$$Q^r(v) = \sum_{i,j=1}^{s} a_{ij}^r v_i v_j, \quad a_{ij}^r = ([X^i,[X^j,X]](p), n_r)$$

A general affine control system is STLC at $p \in M$ if:

- *There exist s linearly independent vectors $\xi_i \in \mathbf{R}^s$ such that:*

$$\sum_{i=1}^{s} Q^j(\xi_i) = 0, \quad j = 1, \ldots, k$$

- $T_pM = \mathcal{W}^2(p)$.

Proof:
Define, as before, a new control system Σ_ξ, with dynamics given by:

$$\dot{x} = X(x) + \sum_{i=1}^{s} u_i Y^i(x), \quad Y^i = \sum_{j=1}^{s} (\xi_i)_j X^j \qquad |u_i| \leq \alpha_i \tag{4}$$

Since the vectors $\xi_i \in \mathbf{R}^s$ are linearly independent, small time local controllability of the above system implies that of the original system (1); in particular the subspaces $\mathcal{W}^1(p)$ and $\mathcal{W}^2(p)$ are the same for both systems.

It was shown before that:

$$[Y^i,[Y^i,X]](p) = \sum_{j,l=1}^{k} (\xi_i)_j (\xi_i)_l [X^j,[X^l,X]](p)$$

and therefore the obstruction $\sum_{i=1}^{s}[Y^i,[Y^i,X]](p)$ being w-neutralizable is equivalent to:

$$\left(\sum_{i=1}^{s}[Y^i,[Y^i,X]](p), n_r\right) = \sum_{i=1}^{s} Q^r(\xi_i) = 0, \quad r = 1, \ldots, k$$

Thus, for system Σ_ξ, all vectors in $\mathcal{W}^2(p) = T_pM$ are regular variations; therefore Σ_ξ and the original system are small time locally controllable at p.

Remark: As there are no obstructions of length 4, it is immediate to extend theorem 3. Let:

$$\mathcal{W}^3 = \{Z : Z = \mathrm{ad}_X^j(Y), \quad Y \in V_4^w, \quad j \geq 0\}$$

Then the last requirement of the theorem can be:

- $T_pM = \mathcal{W}^3(p)$.

instead of $T_pM = \mathcal{W}^2(p)$. Of course the condition will no longer be dependent solely on the 2-jets of the vector fields involved, only the obstructions to be neutralized are of order two in that sense.

Example 5 Consider the analytic system on $\mathbf{R}^{10}$ defined by:

$$X = x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_5} + x_4\frac{\partial}{\partial x_6} + \left(\frac{x_1^2}{2} + 2x_1x_2\right)\frac{\partial}{\partial x_7} + $$
$$+ \left(\frac{x_2^2}{2} + 2x_1x_2\right)\frac{\partial}{\partial x_8} + x_7\frac{\partial}{\partial x_9}$$
$$X^1 = \frac{\partial}{\partial x_1}, \quad X^2 = \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3} + \frac{x_1^2}{2}\frac{\partial}{\partial x_{10}}$$

with $\alpha_1 = \alpha_2 = 1$.

Then $\mathcal{W}^1(0) = \mathbf{R}^6 \times \{0\}$ and proposition 4 does not prove local controllability; a basis for the complement of $\mathcal{W}^1(0)$ is given by $n_1 = e_7$, $n_2 = e_8$, $n_3 = e_9$, $n_4 = e_{10}$, and the corresponding quadratic forms are:

$$Q^1(v) = v_1^2 + 4v_1v_2, \quad Q^2(v) = v_2^2 + 4v_1v_2, \quad Q^3(v) = 0, \quad Q^4(v) = 0$$

The existence of ξ_1 and ξ_2 such that the first condition of theorem 3 is verified follows from the fact that the zero directions of Q^1, namely $v_1 = 0$ and $v_1 + 4v_2 = 0$, separate the zero directions of Q^2, $v_2 = 0$ and $v_2 + 4v_1 = 0$. Here $\xi_1 = (1,1)$ and $\xi_2 = (k, -k)$, with $k^2 = 5/3$, are a possible choice.

To be able to reach a conclusion it is also necessary that $T_pM = \mathcal{W}^3(0)$; this follows from:

$$[X^1, [X^1, X]](0) = \frac{\partial}{\partial x_7}, \quad [X^2, [X^2, X]](0) = \frac{\partial}{\partial x_8}$$
$$[X, [X^1, [X^1, X]]](0) = -\frac{\partial}{\partial x_9}, \quad [X^1, [X^1, X^2]](0) = \frac{\partial}{\partial x_{10}}$$

and therefore the system is STLC.

Note that

$$[X^1,[X^1,X]](0)+[X^2,[X^2,X]](0)=\frac{\partial}{\partial x_7}+\frac{\partial}{\partial x_8}$$

is an obstruction to the use of theorem 1.

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