UNIVERSIDADE DE SÃO PAULO

ANALYTIC LINEARIZABILITY OF VECTOR FIELDS DEPENDING ON PARAMETERS AND OF CERTAIN POISSON STRUCTURES

J. BASTO-GONÇALVES
I. CRUZ

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Resumo

Neste manuscrito são provadas versões dos teoremas de Poincaré e Siegel para campos de vectores holomorfos dependendo de parâmetros. Um método alternativo de linearização é também apresentado, permitindo considerar alguns campos de vectores com valores próprios ressonantes; as não linearidades admissíveis são caracterizadas por condições fáceis de verificar. Estes resultados são usados para obter uma versão analítica de um teorema de J. P. Dufour sobre a linearização de certas estruturas de Poisson.

Abstract

A version of the Poincaré and Siegel theorems is proved for holomorphic vector fields depending on parameters. An alternative method is also presented, allowing the linearization of some vector fields with resonant eigenvalues; the admissible non linearities are characterized by conditions that are easy to check. These results are used to obtain an analytic version of a theorem of J.P. Dufour on the linearization of certain Poisson structures.

Analytic linearizability of vector fields depending on parameters and of certain Poisson structures

J. Basto-Gonçalves *
e-mail: jbg@fcma.up.pt

I. Cruz†
e-mail: icruz@fcma.up.pt

Grupo de Matemática Aplicada
Centro de Matemática Aplicada da Universidade do Porto
R. das Taipas, 135
4050 Porto - Portugal

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Let $X(z,w)$ be a vector field on a domain U in $\mathbf{C}^n$ depending on parameters w in the domain $W \subset \mathbf{C}^m$; X is assumed to be holomorphic as a map $X : U \times W \longrightarrow \mathbf{C}^n$.

The vector field X is assumed to have a singular point at the origin in $\mathbf{C}^n$ with linear part A independent of w:

$$X(z,w) = Az + a(z,w), \quad a(0,w) = \frac{\partial a}{\partial z}(0,w) = 0$$

X is said to be biholomorphically equivalent to its linear part if there exists a holomorphic change of coordinates $\zeta = \zeta(z,w)$, depending on the parameters w, preserving the origin, $\zeta(0,w) = 0$, with inverse $z = z(\zeta,w)$ (as a function only of z) also holomorphic, such that in the new coordinates the non linear part is zero:

$$\frac{\partial \zeta}{\partial z}(z(\zeta,w),w)X(z(\zeta,w),w) = A\zeta.$$

If we think of X as a vector field in $U \times W \subset \mathbf{C}^{n+m}$ with null component along $\mathbf{C}^m$ then we have resonances on the linear part of X and the theorems of Poincaré and Siegel on the linearization of vector fields (see [1]) cannot be used.

In section 1 a version of those theorems including parameters is proved. The analogous situation for the C^∞ case was solved by S. Sternberg [11] and later R. Roussarie [10] proved a version with parameters.

The analytic linearizability of real analytic vector fields with all eigenvalues equal, therefore non resonant, and depending on one parameter, was proved in [5]. In section 2 it is proved that the same method can be used for complex holomorphic vector fields depending on parameters for a much larger class of eigenvalues and allowing some non-linearities, when there is resonance, characterized by conditions that are easy to check.

All results are also valid in the real analytic category, with standard adaptations.

A very interesting application of the linearization of vector fields, due to J.P. Dufour [6], is the linearization of a family of Poisson structures. The use of these results allows the proof of an analytic version of his main theorem which holds in the smooth category.

A. Weinstein [12] has proved that every Poisson manifold is locally a product of a *symplectic leaf through the point* and a *transverse Poisson manifold through the point.*

When the original structure P has constant rank, the transverse Poisson manifold is trivial, i.e., the original structure is just a trivial extension of a symplectic manifold. For variable rank the situation is completely different:

there is no equivalent to Darboux's Theorem for Poisson structures, two Poisson structures having the same rank at a given point need not be locally Poisson-diffeomorphic. The main obstacle is precisely the fact that the rank can vary rather unpredictably around a singularity (see [5] for some badly behaved structures). The search for a normal form in Poisson structures is therefore quite a difficult task.

The linearization problem, i.e. the possibility of bringing P to a linear normal form, has been studied by A. Weinstein ([12], [13]), J. Conn ([3],[4]), J.P. Dufour [6] and Cahen *et al.* [2] among others.

In some cases the linearizability of P at x_0 is determined by its 1-jet: if a Lie algebra g is semisimple then any Poisson tensor P for which g is the linear approximation is linearizable in the analytic category [3]; this is still the case in the smooth category if the Lie algebra g is also of compact type [4].

These conditions (semisimplicity and compacity of the associated algebra), however, are not necessary. There are examples of nonsemisimple Lie algebras such that any Poisson tensor associated with them in the above way, is smoothly linearizable: this is a consequence of a theorem proved by J.P. Dufour [6]. In section 3 it is proved that similar results are valid in the analytic category.

In some cases the linearizability of P at x_0 is not determined by its 1-jet: for a given Lie algebra g there are Poisson tensors P, for which g is the linear approximation, that are linearizable in the analytic category and others that are not; k determinacy in this context means that the alternative can be decided by studying the k-jet of the Poisson tensor. In section 4 some results are presented, valid for the same type of Poisson structures considered in section 3.

1 Poincaré and Siegel theorems with parameters

1.1 Statement of the theorems

Let $\lambda = (\lambda_1, \dots \lambda_n) \in \mathbf{C}^n$ be the vector of the eigenvalues of the linear part A of X; the eigenvalues are assumed to be all distinct, but this is not necessary [1]. The approach here will be as close as possible to the one developed in that reference, with the indispensable adaptations for the inclusion of parameters and the treatment of vector fields instead of maps; the results obtained are also valid for that case.

The eigenvalues are said to be resonant of order k if there exists $m = (m_1, \ldots, m_n)$, with m_i non-negative integers and $|m| = m_1 + \cdots + m_n = k \geq 2$, such that for some s:

$$\lambda_s = m \cdot \lambda$$

The vector λ belongs to the Poincaré domain if zero is not in the convex hull of the n points $\{\lambda_1, \ldots \lambda_n\}$ in the complex plane, and to the Siegel domain if zero is in the interior of that set.

Theorem 1 Poincaré theorem with parameters: *If λ belongs to the Poincaré domain and is non-resonant, X is biholomorphically equivalent to its linear part in the neighbourhood of the singular point.*

λ is said to be of type (C, ν) if, for any s:

$$|\lambda_s - \lambda \cdot m| \geq \frac{C}{|m|^\nu}$$

with $m = (m_1, \ldots, m_n)$, m_i non-negative integers and $|m| \geq 2$.

Clearly if λ is of type (C, ν) is non-resonant; if λ belongs to the Poincaré domain and is non-resonant then it is of type (C, ν) for convenient C and ν: every point in the Poincaré domain satisfies not more than a finite number of resonances and has a neighbourhood where no other resonance relation is satisfied [1].

Theorem 2 Siegel theorem with parameters: *If λ is of type (C, ν), X is biholomorphically equivalent to its linear part in the neighbourhood of the singular point.*

1.2 Outline of the proof

Consider the change of coordinates (in z) $H(z, w) = z + h(z, w)$ with inverse $G(z, w) = z - g(z, w)$; assume that $h(0, w) = h_z(0, w) = 0$.

The vector field X can be written in the new coordinates as:

$$\begin{aligned}
X(z, w) &= Az + (a(z, w) - L_A h(z, w)) + R(a, h)(z, w) \\
L_A h(z, w) &= Ah(z, w) - \frac{\partial h}{\partial z}(z, w)Az \\
R(a, h)(z, w) &= A(h(z, w) - g(z, w)) + a(z - g(z, w), w) - a(z, w) + \\
&\quad + \frac{\partial h}{\partial z}(z - g(z, w), w)\,(a(z - g(z, w), w) + A(z - g(z, w)) + \\
&\quad - \frac{\partial h}{\partial z}(z, w)Az
\end{aligned}$$

If h is chosen to be the solution $U(a)(z,w)$ of the homological equation:

$$L_A h(z,w) = a(z,w)$$

then the above change of coordinates gives $X(z,w) = Az + R(a,h)(z,w)$.

Let h_s, H_s and a_s be inductively defined by:

$$h_s = U(a_s), \quad H_s(z,w) = z + h_s(z,w), \quad a_{s+1} = R(a_s, h_s)$$

with $a_0(z,w) = a(z,w)$.

The desired change of coordinates is given by:

$$H = \lim_{s\to\infty} H_s \circ \cdots \circ H_0$$

It is necessary to prove that the sequences h_s, H_s and a_s are well defined, and $\mathcal{H}_s = H_s \circ \cdots \circ H_0$ convergent, in the holomorphic category.

1.3 Order of operators

Let U be a domain in $\mathbf{C}^n$ containing the origin, and $f : U \subset \mathbf{C}^n \longrightarrow B$ a holomorphic (germ at 0) map into a Banach space B with norm $\|\ .\ \|$, such that $f(0) = 0$. For any r such that f is holomorphic on the polydisk $D_r^n = \{z \in \mathbf{C}^n,\ |z| = \max|z_i| < r\}$ and continuous on its closure $\overline{D}_r^n$, we define a norm by:

$$\|f\|_r = \sup_{0\le|z|\le r} \frac{\|f(z)\|}{|z|}.$$

Remark: it follows from the maximum principle that, for f with values in $\mathbf{C}^n$, $\|f\|_r$ is an increasing function of r.

Let Φ be an operator acting on (germs of) maps of the class described above.

Definition 1 *The operator Φ has order d if there exist positive numbers d, α and β such that, for every $\delta \in]0, 1/2[$ and $r \in]0, 1[$,*

$$\|\Phi(f)\|_{re^{-\delta}} \le \|f\|_r^d \delta^{-\alpha},$$

whenever $\|f\|_r \le \delta^\beta$

This relation will be written as $\Phi(f) \prec f^d$, or $g \prec f^d$ if $g = \Phi(f)$. It can be extended to operators of n functions, but only $n = 2$ will be relevant:

Let Θ be an operator acting on pairs of (germs of) maps of the class described above, and $g = \Theta(f_1, f_2)$.

Definition 2 *$\Theta(f_1, f_2) \prec \varphi(f_1, f_2)$, or $g \prec \varphi(f_1, f_2)$, if there exist a polynomial φ and positive numbers α, β_1 and β_2 such that, for every $\delta \in]0, 1/2[$ and $r \in]0, 1[$,*

$$\|\Theta(f_1, f_2)\|_{re^{-\delta}} \leq \varphi(\|f_1\|_r, \|f_2\|_r)\delta^{-\alpha},$$

whenever $\|f_i\|_r \leq \delta^{\beta_i}$, $i = 1, 2$.

The following properties will be useful:

1. if $f_1 \prec f_2$ and $f_2 \prec f_3$ then $f_1 \prec f_3$.
2. if $g \prec \varphi(f_1, f_2)$ and $f_1 \prec \psi(h_1, h_2)$ then $g \prec \varphi(\psi(h_1, h_2), f_2)$.
3. for f with values in $\mathbf{C}^n$ and g in $\mathbf{C}^m$, if there exist positive numbers K, d, β and l such that, for every $\delta \in]0, 1/2[$ and $r \in]0, 1[$, the estimate:

$$\max_{|z| \leq re^{-\delta}} |g(z)| \leq K \max_{|z| \leq r} |f(z)|^d \delta^{-l}$$

is verified, then $g \prec f^d$.

1.4 Technical lemmas and estimates

Lemma 1 *Let $f(z, w)$ be a holomorphic map in D_r^{n+m} and continuous in $\overline{D}_r^{n+m}$, verifying $f(0, w) = f_z(0, w) = 0$. The homological equation*

$$L_A h(z, w) = f(z, w)$$

has a holomorphic solution $h(z, w) = U(f)(z, w)$ such that $h(0, w) = h_z(0, w) = 0$ and:

$$h \prec f$$

or equivalently, the operator U associating the solution of the homological equation to its second member, $h = U(f)$, has order 1.

Proof:

Under the non resonance conditions assumed, the homological equation has a solution as a formal power series [1]; if:

$$f(z, w) = \sum_{|I| \geq 2, |J| \geq 0} f_{I,J}^s z^I w^J e_s, \quad h(z, w) = \sum_{|I| \geq 2, |J| \geq 0} h_{I,J}^s z^I w^J e_s$$

where $I \in \mathbf{Z}^n$ and $J \in \mathbf{Z}^m$ are non negative multi-indices and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbf{C}^n$, then:

$$h_{I,J}^s = \frac{f_{I,J}^s}{\lambda_s - I \cdot \lambda}$$

and in particular $h(0,w) = h_z(0,w) = 0$ if h is holomorphic.

From the Cauchy inequalities:

$$|f^s_{I,J}| \le \frac{M}{r^{|I|+|J|}}, \quad M = \max_{|(z,w)|\le r} |f(z,w)|$$

and from λ being of type (C,ν) it follows that:

$$|h^s_{I,J}| \le \frac{M|I|^\nu}{Cr^{|I|+|J|}}$$

and:

$$\sum_J |h^s_{I,J} w^J| \le \frac{M|I|^\nu}{Cr^{|I|}} \sum_J \frac{|w^J|}{r^{|J|}} \le \frac{M|I|^\nu}{Cr^{|I|}(1-|w|/r)^m}$$

The number of multi-indices I such that $|I| = p$ is bounded by $c_1 p^{n-1}$; the constant c_1 depends on n which is fixed here. Thus:

$$S_p = \sum_{|I|=p} \sum_J |h^s_{I,J} z^I w^J| \le \frac{c_1 M}{C(1-|w|/r)^m} \frac{p^{\nu+n-1}|z|^p}{r^p}$$

Let $l = \nu + n - 1$ and take $|z| \le re^{-\delta}$; noticing that then $|z|/r \le e^{-\delta}$ and $p^l e^{-p\delta/2}$ has maximum $(2l/e)^l \delta^{-l}$ as a function of p, the following estimates are obtained:

$$\begin{aligned}
S_p &\le \frac{c_1 M}{C(1-|w|/r)^m} p^l e^{-p\delta/2} e^{-p\delta/2} \\
&\le \frac{c_1}{C}(2l/e)^l M \frac{1}{(1-|w|/r)^m} \delta^{-l} e^{-p\delta/2} \\
\sum_{p\ge 2} S_p &\le \frac{c_1}{C}(2l/e)^l M \frac{1}{(1-|w|/r)^m} \delta^{-l} \sum_{p\ge 2} e^{-p\delta/2} \\
&\le KM \frac{1}{(1-|w|/r)^m} \delta^{-l} \frac{e^{-\delta}}{1-e^{-\delta/2}}, \quad K = \frac{c_1}{C}(2l/e)^l
\end{aligned}$$

When $\delta \in]0, 1/2[$ and $|w| \le re^{-\delta}$ then:

$$\frac{e^{-\delta}}{1-e^{-\delta/2}} = \frac{1}{e^\delta - e^{\delta/2}} \le \frac{2}{\delta}, \quad \frac{1}{(1-|w|/r)} \le \frac{2}{\delta}$$

and

$$|h(z,w)| \le \sum_{p\ge 2} S_p \le 2^{m+1} KM \delta^{-l-m-1}$$

Therefore h is well defined in the holomorphic category, and as:

$$\max_{|(z,w)|\le re^{-\delta}} |h(z,w)| \le 2^{m+1} K \max_{|(z,w)|\le r} |f(z,w)| \delta^{-l-m-1}$$

the operator U has order 1, $h = U(f) \prec f$.

Lemma 2 *Let $h(z,w)$ be a holomorphic map in D_r^{n+m} and continuous in $\overline{D}_r^{n+m}$, verifying $h(0,w) = h_z(0,w) = 0$. If $H(z,w) = z - h(z,w)$ is a local diffeomorphism as a function of z alone, with inverse $G(z,w) = z - g(z,w)$, and denoting by I the operator associating g to h, then:*

1. $I(h) = g \prec h$
2. $h - g \prec h^2$

Proof:

Consider a point (z_0, w_0) in $D^{n+m}_{re^{-\delta/2}}$ with $\delta \in]0, 1/2[$; taking the Taylor series of h at that point, which has a radius of convergence at least $\rho = r(1 - e^{-\delta/2})$, it follows from the Cauchy inequalities that:

$$\left|\frac{\partial h_s}{\partial z_i}(z_0, w_0)\right| \leq \frac{1}{\rho} \max_{|(z-z_0, w-w_0)| \leq \rho} |h(z,w)| \leq \frac{1}{(1 - e^{-\delta/2})} \max_{|(z,w)| \leq r} \frac{|h(z,w)|}{|(z,w)|}$$

As before, it follows that, for any (z,w) in $D^{n+m}_{re^{-\delta/2}}$:

$$\left|\frac{\partial h_s}{\partial z_i}(z,w)\right| \leq \|h\|_r \delta^{-l}$$

for some positive number l independent of r and δ

The map g is obtained as the limit, if it exists, of the sequence g_i defined by:

$$g_{i+1}(z,w) = h((z - g_i(z,w), w), w), \quad g_1 = h$$

If $\|h\|_r \leq \delta^\beta$ with $\beta > l$, that sequence is in fact convergent, and from:

$$\begin{aligned} \max_{|(z,w)| \leq re^{-\delta}} |g_{i+1}(z,w)| \leq & \max_{|(z,w)| \leq re^{-\delta}} \left|\frac{\partial h_s}{\partial z_i}(z,w)\right| \max_{|(z,w)| \leq re^{-\delta}} |g_i(z,w)| + \\ & + \max_{|(z,w)| \leq re^{-\delta}} |h(z,w)| \end{aligned}$$

it follows that:

$$\|g_{i+1}\|_{re^{-\delta}} \leq \|h\|_r (1 + \|g_i\|_{re^{-\delta}} \delta^{-l}), \quad \|g_1\|_{re^{-\delta}} \leq \|h\|_r$$

and therefore:

$$\|g\|_{re^{-\delta}} \leq \|h\|_r \frac{1}{1 - \|h\|_r \delta^{-l}} \leq \|h\|_r \delta^{-k}$$

for some positive k. Thus $I(h) = g \prec h$.

Now $h(z,w) - g(z,w) = h(z,w) - h(z - g(z,w), w)$ and $|h(z,w) - g(z,w)| \leq \|h\|_r \delta^{-l} |g(z,w)|$ for $|(z,w)| \leq re^{-\delta}$, therefore:

$$\|h - g\|_{re^{-\delta}} \leq \|h\|_r \delta^{-l} \|g\|_{re^{-\delta}} \leq \|h\|_r^2 \delta^{-(l+k)}$$

and $h - g \prec h^2$.

The proof of the lemma above can be generalized to obtain:

Lemma 3 *Let $h(z,w)$ and $g(z,w)$ be holomorphic maps in D_r^{n+m} and continuous in $\overline{D}_r^{n+m}$, verifying $h(0,w) = g(0,w) = 0$. If Θ is the operator defined by $\Theta(h,g)(z,w) = h(z,w) - h(z - g(z,w), w)$, then:*

$$\Theta(h,g) \prec hg$$

Define the operators:

$$\begin{aligned} R_1(h)(z,w) &= A(h(z,w) - I(h)(z,w)) = A(h(z,w) - g(z,w)) \\ R_2(a,h)(z,w) &= a(z - I(h)(z,w), w) - a(z,w) \\ R_3(a,h)(z,w) &= \frac{\partial h}{\partial z}(z - g(z,w), w)\,(a(z - g(z,w), w) + A(z - g(z,w)) + \\ &\quad -\frac{\partial h}{\partial z}(z,w)Az \end{aligned}$$

then:

$$R(a,h) = R_1(h) + R_2(a,h) + R_3(a,h)$$

Lemma 4 *Let $R_1(h)$, $R_2(a,h)$ and $R_3(a,h)$ be as above; then:*

1. $R_1(h) \prec h^2$
2. $R_2(a,h) \prec ah$
3. $R_3(a,h) \prec h(a+h)$

Proof:
$R_1(h) \prec h^2$ is a consequence of $\|R_1(h)\|_{re^{-\delta}} \leq \|A\|\|h - I(h)\|_{re^{-\delta}}$, and $R_2(a,h) \prec ah$ was proved in the previous lemma.

If u and v are defined by:

$$u(z,w) = \frac{\partial h}{\partial z}(z,w)Az, \qquad v(z,w) = A^{-1}a(z - g(z,w), w) - g(z,w)$$

then $R_3(a,h)(z,w) = u(z - v(z)) - u(z)$, and $R_3(a,h) \prec uv$ by lemma 3; on the other hand, $u \prec h$ and $v \prec a + g \prec a + h$, therefore $R_3(a,h) \prec h(a+h)$.

Lemma 5 *The operator $\Phi(a) = R(a, U(a))$ has order 2.*

Proof:

$$R_1(h) \prec h^2,\ h = U(a) \prec a \Longrightarrow R_1(U(a)) \prec a^2$$

$$R_2(a,h) \prec ah,\ h = U(a) \prec a \Longrightarrow R_2(a, U(a)) \prec a^2$$

$$R_3(a,h) \prec h(a+h),\ h = U(a) \prec a \Longrightarrow R_3(a, U(a)) \prec a^2$$

Thus:

$$\Phi(a) = R(a, U(a)) = R_1(U(a)) + R_2(a, U(a)) + R_3(a, U(a)) \prec a^2$$

1.5 Convergence

Construct the following sequences, depending on $r_0 \in]0,1[$ and $\delta_0 \in]0,1/2[$ and $N > 0$:

$$\delta_i = \delta_{i-1}^{3/2}, \qquad M_0 = \delta_0^N \quad M_i = M_{i-1}^{3/2} = \delta_i^N, \qquad r_i = e^{-\delta_i} r_{i-1}$$

Let $r_* = \lim_{i\to\infty} r_i$. The maps h_s, H_s and a_s are well defined and holomorphic in $D_{r_*}^{n+m}$; only convergence of $\mathcal{H}_s = H_s \circ \cdots \circ H_0$ remains to be proved.

Lemma 6 *N and r_0 can be chosen so that $\|a_i\|_{r_i} \le M_i$, for all $i \ge 0$.*

Proof:
Since $R(a, U(a)) \prec a^2$ there exist positive α and β such that, if $N > \beta$:

$$\|a_i\|_{r_i} \le M_i = \delta_i^N \Longrightarrow \|a_{i+1} = R(a_i, U(a_i))\|_{r_{i+1}} \le M_i^2 \delta_i^{-\alpha} = \delta_i^{2N-\alpha}$$

Thus if $N > 2\alpha$, then $\delta_i^{2N-\alpha} \le M_{i+1} = \delta_i^{3N/2}$ and therefore:

$$\|a_0\|_{r_0} \le M_0 = \delta_0^N \Longrightarrow \|a_i\|_{r_i} \le M_i \text{ for all } i \ge 0.$$

Since $a_0 = a$ and $a(0,w) = a_z(0,w) = 0$, a_0 can be written as $a_0(z,w) = z^2 f(z,w)$ with f holomorphic, and there exists a positive constant K such that, for small $|z|$ and $|w|$, $|a_0(z,w)| \le K|z|^2$ and therefore $\|a_0\|_{r_0} \le K r_0$. The value r_0 is chosen so that $K r_0 \le M_0 = \delta_0^N$; then $\|a_0\|_{r_0} \le M_0$.

Lemma 7 *N and δ_0 can be chosen so that $\mathcal{H}_s$ is a Cauchy sequence on $D^{n+m}_{r_0/2}$.*

Proof:
As:

$$\|\mathcal{H}_s(z,w) - \mathcal{H}_{s-1}(z,w)\| \leq \|h_s(\mathcal{H}_{s-1}(z,w))\|$$

the lemma follows from convenient estimates of $\|h_s\|$ on the images of $\mathcal{H}_{s-1}$.

Since $U(a) \prec a$ there exist positive γ and σ such that, if $N > \sigma$:

$$\|a_i\|_{r_i} \leq M_i = \delta_i^N \Longrightarrow \|h_i = U(a_i)\|_{r_{i+1}} \leq M_i \delta_i^{-\gamma} = \delta_i^{N-\gamma}$$

All the above estimates are valid in $|(z,w)| \leq r_*$, and, if N and δ_0 are conveniently chosen, the images of $|(z,w)| \leq r_0/2$ by the maps $\mathcal{H}_s$ are contained in that set:

¿From the above estimates $\mathcal{H}_0(D^{n+m}_{\rho_0}) \subset D^{n+m}_{\rho_1}$, with $\rho_0 = r_0/2$ and $\rho_1 = \rho_0 + \delta_0^{N-\gamma}\rho_0$, and by induction

$$\mathcal{H}_i(D^{n+m}_{\rho_0}) \subset D^{n+m}_{\rho_{i+1}}, \quad \rho_{i+1} = \rho_i + \delta_i^{N-\gamma}\rho_0$$

as long as these sets are contained in $|(z,w)| \leq r_*$.

Now $r_* = r_0 e^{-S}$, where $S = \sum_{i\geq 0}\delta_i = \sum_{i\geq 0}\delta_0^{(3/2)^i}$ and:

$$\rho_* = \lim_{i\to\infty}\rho_i = \frac{r_0}{2}\left(1 + \sum_{i\geq 0}\delta_i^{N-\gamma}\right) = \frac{r_0}{2}\left(1 + \sum_{i\geq 0}\delta_0^{(N-\gamma)(3/2)^i}\right)$$

If $N > 1+\gamma$ then $\sum_{i\geq 0}\delta_0^{(N-\gamma)(3/2)^i} < S$ and

$$2e^{-S} > 1 + S \Longrightarrow r_* = r_0 e^{-S} > \frac{r_0}{2}(1+S) > \rho_*$$

The conditions are then:

$$N > \max(2\alpha, \beta, \sigma, 1+\gamma), \qquad 2e^{-S} > 1 + S \text{ where } S = \sum_{i\geq 0}\delta_0^{(3/2)^i}$$

and this last condition is verified if δ_0 is small enough.

The limit $\mathcal{H} = \lim_{s\to\infty}\mathcal{H}_s$ exists in $D^{n+m}_{r_0/2}$ and it is a holomorphic local diffeomorphism, giving the desired change of coordinates.

2 Holomorphic linearizability of some resonant vector fields

Let again $X(z,w)$ be a holomorphic vector field on a domain U in $\mathbf{C}^n$ depending on parameters w in the domain $W \subset \mathbf{C}^m$, with a singular point at origin in $\mathbf{C}^n$ and linear part A independent of w:

$$X(z,w) = Az + a(z,w), \quad a(0,w) = \frac{\partial a}{\partial z}(0,w) = 0$$

Let $\lambda = (\lambda_1, \dots \lambda_n) \in \mathbf{C}^n$ be the vector of the eigenvalues of the linear part A of X; if:

$$a(x,w) = \sum_{|I|\geq 2, |J|\geq 0} a^i_{IJ} x^I w^J e_i$$

the monomial $x^I w^J e_i$ is resonant if $I \cdot \lambda - \lambda_i = 0$.

The Poincaré-Dulac theorem [1] allows the elimination of the non resonant terms by a formal change of variables, holomorphic under certain conditions, but does not guarantee that the linearization can be performed if the nonlinearity a does not contain any resonant term, as the following example shows:

Example 1 Let $X(x,y) = (-x + y^3, y + x^4 y)$ be a vector field in $\mathbf{C}^2$; the eigenvalues are -1 and 1, therefore resonant, but the non linearity does not contain resonant monomials, of the form $x^{k+1}y^k e_1$ or $x^k y^{k+1} e_2$.

The first step in the Poincaré-Dulac method leads to the change of variables $\xi = x - y^3/4$, $\eta = y$, eliminating the lower order term of the non linearity, but in the new coordinates:

$$X(\xi,\eta) = \left(-\xi - \frac{3}{4}\left(\xi + \frac{\eta^3}{4}\right)^4 \eta^3,\ \eta + \left(\xi + \frac{\eta^3}{4}\right)^4 \eta\right)$$

and the resulting non linearity has now a resonant monomial, $\xi^3 \eta^4 e_2$.

Theorem 3 *Let X be a holomorphic vector field on a neighbourhood $U \times W$ of the origin in $\mathbf{C}^{n+m}$ which, in coordinates (z,w), can be written as:*

$$X(z,w) = Az + a(z,w), \qquad A = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$$

If the non linearity $a(z,w)$ is admissible, i.e. all $I \cdot \lambda - \lambda_i$ are positive integer multiples of a fixed complex number for every monomial $x^I w^J e_i$ in a, there exists a holomorphic change of coordinates $\psi(z,w)$ linearizing X.

Remarks:

- Without loss of generality it may be assumed that all $I \cdot \lambda - \lambda_i$ are positive integers, since a change of coordinates that linearizes a vector field also linearizes any multiple of it.

  In the real analytic category the nonlinearity is admissible if all $I \cdot \lambda - \lambda_i$ are positive integer multiples of a fixed real number.

- If there are no resonances between the eigenvalues and all are integer multiples of a fixed complex number, they all can be taken as positive integers and every non-linearity is admissible:

  The Poincaré-Dulac theorem guarantees the existence of a holomorphic (polynomial even) change of coordinates eliminating all (non resonant) terms up to a certain order. It can be assumed that that change of coordinates has already been performed and $a_i(z, w)$ has only terms of order greater than λ_i in z; this implies that $I \cdot \lambda - \lambda_i$ is a positive integer for all monomials $z^I w^J e_i$ appearing in a.

Proof:
Suppose that ψ exists and let $\xi(y, w)$ denote its inverse (as a function of z alone). Then ξ will satisfy the system of partial differential differential equations:

$$\lambda_1 y_1 \frac{\partial \xi_i}{\partial y_1} + \cdots + \lambda_n y_n \frac{\partial \xi_i}{\partial y_n} = \lambda_i \xi_i + a_i(\xi, w) \quad i = 1, \ldots, n. \tag{1}$$

If the initial value problem composed of (1) and the following initial conditions:

$$\xi(0, w) = 0, \quad \frac{\partial \xi}{\partial y}(0, w) \text{ non singular} \tag{2}$$

has a solution $\xi(z, w)$ which is holomorphic at $0 \in \mathbf{C}^{n+m}$ the proof of the theorem is finished, since (2) implies that ξ is locally invertible, as a function of y alone.

2.1 Change of variables

Let $z = (z_1, \ldots, z_n)$ be defined by:

$$y_1 = z_1^{\lambda_1}, \quad y_i = z_1^{\lambda_i} z_i$$

and ζ be the expression of ξ in the new variables z; then:

$$\begin{aligned}
\zeta(z,w) &= \xi(z_1^{\lambda_1}, z_1^{\lambda_2}z_2, \ldots, z_1^{\lambda_n}z_n, w) \\
\frac{\partial \zeta}{\partial z_1} &= \lambda_1 z_1^{\lambda_1 - 1}\frac{\partial \xi}{\partial y_1} + \sum_{j=2}^{n} \lambda_j z_1^{\lambda_j - 1} z_j \frac{\partial \xi}{\partial y_j}
\end{aligned}$$

and (1) can be rewritten as:

$$z_1 \frac{\partial \zeta_i}{\partial z_1} = \lambda_i \zeta_i + a_i(\zeta, w), \quad i = 1, \ldots, n$$

Introduce new unknown functions η by:

$$\zeta_i = z_1^{\lambda_i}\eta_i, \quad i = 1, \ldots, n$$

Then:

$$\begin{aligned}
\frac{\partial \zeta_i}{\partial z_1} &= \lambda_i z_1^{\lambda_i - 1}\eta_i + z_1^{\lambda_i}\frac{\partial \eta_i}{\partial z_1} \\
z_1 \frac{\partial \zeta_i}{\partial z_1} &= \lambda_i \zeta_i + z_1^{\lambda_i + 1}\frac{\partial \eta_i}{\partial z_1} \\
&= \lambda_i \zeta_i + a_i(\zeta, w), \qquad i = 1, \ldots, n
\end{aligned}$$

and therefore:

$$z_1^{\lambda_i + 1}\frac{\partial \eta_i}{\partial z_1} = a_i(z_1^{\lambda}\eta, w), \quad z_1^{\lambda}\eta = (z_1^{\lambda_1}\eta_1, \ldots, z_1^{\lambda_n}\eta_n) \quad i = 1, \ldots, n.$$

2.2 Reduction to an ordinary differential equation

Thus the system of partial differential equations (1) can be written as a system of non autonomous ordinary differential equations:

$$\frac{d}{dz_1}\eta_i = \alpha_i(z_1, \eta, w), \quad i = 1, \ldots, n. \tag{3}$$

depending on parameters w. That α is holomorphic around the origin is an immediate consequence of a being admissible:

Writing $a_i(z,w)$ as an absolutely convergent power series:

$$a_i(z,w) = \sum_{|I|=2, |J|=0}^{\infty} a_{IJ}^i \, z^I w^J$$

then:

$$\alpha_i(z_1,\eta,w) = \sum_{|I|=2,|J|=0}^{\infty} a^i_{IJ}\, z_1^{I\cdot\lambda-\lambda_i-1}\eta^I w^J.$$

which converges for $z_1 = 1$ and $|(\eta, w)|$ small enough.

Of course this system was obtained using changes of coordinates that are not valid in a full neighbourhood of the origin, but nevertheless this system is well defined even at $z_1 = 0$.

¿From the existence and uniqueness theorem for ordinary differential equations in the complex domain [8], system (3) has a solution which is holomorphic as a function of z_1, the parameters and also the initial conditions; it will be shown that it is possible to select the initial conditions so that the map ξ, corresponding to η, is holomorphic on a neighbourhood of $(y, w) = (0, 0)$ and a solution of the initial value problem.

So let $\eta(z, w)$ be the solution of (3) with initial conditions:

$$\eta(0,\hat{z},w) = (1,\hat{z}), \quad \hat{z} = (z_2,\ldots,z_n). \tag{4}$$

Assume for the moment that the corresponding ξ is holomorphic; ξ satisfies:

$$\xi_i(z_1^{\lambda_1}, z_1^{\lambda_2}z_2,\ldots,z_1^{\lambda_n}z_n,w) = z_1^{\lambda_i}\eta_i(z,w) \qquad i = 1\ldots,n \tag{5}$$

and so $\xi(0, w) = 0$; taking the derivative with respect to z_1:

$$\lambda_1 z_1^{\lambda_1-1}\frac{\partial\xi_i}{\partial y_1} + \sum_{j=2}^{n}\lambda_j z_1^{\lambda_j-1}z_j\frac{\partial\xi_i}{\partial y_j} = \lambda_i z_1^{\lambda_i-1}\eta_i + z_1^{\lambda_i}\frac{\partial\eta_i}{\partial z_1}$$

multiplying by z_1 and writing in terms of the y variables:

$$\lambda_1 y_1\frac{\partial\xi_i}{\partial y_1} + \sum_{j=2}^{n}\lambda_j y_j\frac{\partial\xi_i}{\partial y_j} = \lambda_i z_1^{\lambda_i}\eta_i + z_1^{\lambda_i+1}\frac{\partial\eta_i}{\partial z_1} = \lambda_i\xi_i + a_i$$

shows that ξ is a solution of the partial differential equations (1).

2.3 Conditions for the holomorphy of ξ

Writing $\eta_i(z, w)$ as an absolutely convergent power series:

$$\eta_i(z,w) = \sum_{|I|,|J|=0}^{\infty} b^i_{IJ}\, z^I w^J$$

and taking $\hat{I} = (i_2, \ldots, i_n), |\hat{I}| = i_2 + \ldots + i_n$ and $\hat{z}^{\hat{I}} = z_2^{i_2} \ldots z_n^{i_n}$, then it follows from (5) that $\xi_i(y, w)$ is formally represented by the series:

$$\sum_{|I|,|J|=0}^{\infty} b_{IJ}^i \, y_1^{(i_1 - \hat{I}\cdot\hat{\lambda} + \lambda_i)/\lambda_1} \hat{y}^{\hat{I}} w^J.$$

A monomial $z^I w^J e_i$ will be said to satisfy condition H if there exists a non negative integer k such that:

$$i_1 = \hat{I} \cdot \hat{\lambda} - \lambda_i + k\lambda_1.$$

A holomorphic map is said to satisfy condition H if all monomials with non zero coefficients in its power series centered at the origin satisfy that condition.

A necessary condition for ξ_i to be holomorphic is that η satisfies condition H. In fact this is also a sufficient condition:

Lemma 8 *Suppose that $\sum_{|I|,|J|=0}^{\infty} b_{IJ}^i \, z^I w^J$ converges absolutely in the disc D_r^{n+m} for all $i = 1, \ldots, n$, and condition H is verified. Then ξ is an holomorphic map in a neighbourhood of the origin in $\mathbf{C}^{n+m}$.*

Proof:
¿From the Cauchy inequalities there exists $C \in \mathbf{R}^+$ such that

$$|b_{IJ}^i| \leq \frac{C}{r^{|I|+|J|}}$$

Since condition H holds, $\xi_i(y, w)$ is represented by the formal power series:

$$\xi_i(y, w) = \sum_{k,|\hat{I}|,|J|=0}^{\infty} b_{IJ}^i y_1^k \hat{y}^{\hat{I}} w^J$$

with $i_1 = \hat{I} \cdot \hat{\lambda} - \lambda_i + k\lambda_1 \geq 0$. Then:

$$\sum_{k,|\hat{I}|,|J|=0}^{\infty} |b_{IJ}^i \, y_1^k \, \hat{y}^{\hat{I}} w^J| \leq C \sum_{k,|\hat{I}|,|J|=0}^{\infty} \frac{|y_1^{(i_1 - \hat{I}\cdot\hat{\lambda} + \lambda_i)/\lambda_1} \hat{y}^{\hat{I}} w^J|}{r^{|I|+|J|}}$$

and this last series is convergent if:

$$|y_1| < r^{\lambda_1}, \qquad |y_i| < r^{\lambda_i + 1}, \ i > 1, \qquad |w| < r$$

These conditions define a neighbourhood $\mathcal{N}$ of the origin in $\mathbf{C}^{n+m}$ where the formal power series corresponding to all ξ_i are absolutely convergent. Then ξ is holomorphic in $\mathcal{N}$.

2.4 Holomorphy of ξ

That condition H is verified will be proved by induction on the order in z_1 of the terms in the power series of η.

¿From the initial conditions (4) it follows that:

$$\eta_1(z,w) = 1 + z_1 R_{1,1}(z,w), \quad \eta_i(z,w) = z_i + z_1 R_{i,1}(z,w), \ i > 1$$

and therefore the terms of order zero in z_1 verify condition H: $1 = z^I$, with $I = 0$ and $k = 1$, and similarly $z_i = z^I$ with $i_j = 0$ for all j except $i_i = 1$, and $k = 0$.

Now assume that condition H is verified up to order l, that is, η_i can be written as $\eta_i = \eta_i^l + z_1^{l+1} R_{i,l+1}(z,w)$, where:

$$\eta_i^l = \sum_{r=0}^{l} \sum_{\hat{I},J} b^i_{r\hat{I}J} z_1^r \hat{z}^{\hat{I}} w^J, \quad r = \hat{I}\cdot\hat{\lambda} - \lambda_i + k\lambda_1 \text{ for some } k$$

The differential equations (3) can be written as:

$$z_1 \frac{d}{dz_1}\eta_i = z_1^{-\lambda_i} a_i(z_1^{\lambda}\eta, w), \quad i = 1, \ldots, n \tag{6}$$

and so, as the monomials appearing (i.e. with possibly non zero coefficients) in η_i are exactly the same as the ones appearing in $z_1 d\eta_i/dz_1$, and a_i has no terms of order less than $\lambda_i + 1$, the monomials up to order $l+1$ in η can be determined from the knowledge of the ones up to order l through equation (6): they have to appear in $z_1^{-\lambda_i} a_i(z_1^{\lambda}\eta^l, w)$, to be more precise, if

$$a_i(x,w) = \sum a^i_{IJ} x^I w^J$$

they appear in

$$z_1^{-\lambda_i} \sum a^i_{IJ} (z_1^{\lambda}\eta^l)^I w^J$$

.

All these monomials have the form

$$z_1^{-\lambda_i}(z_1^{\lambda_1}\mathcal{M}_1)^{i_1} \ldots (z_1^{\lambda_n}\mathcal{M}_n)^{i_n} w^J \tag{7}$$

where $\mathcal{M}_i e_i$ denotes monomials of η_i^l, therefore satisfying condition H.

If $z^I w^J e_i$ and $z^K w^L e_j$ satisfy condition H, then so does the monomial $z_1^{-\lambda_k} z_1^{\lambda_i} z^I w^J z_1^{\lambda_j} z^K w^L e_k$ and, by induction, all the products in (7) satisfy condition H, and ξ is holomorphic.

2.5 Biholomorphy of ξ

The partial derivative of ξ_i with respect to y_1 at the origin is the coefficient of $y_1 e_i$ in the series development of ξ; as $y_1 e_i = y^I w^J e_i$, with $I = e_1$ and $J = 0$, the corresponding monomials in η are $1e_1$, if $i = 1$, and $z_1^{\lambda_1 - \lambda_i} e_i$ if $i > 1$.

If $\lambda_1 = \lambda_i$ with $i > 1$, the coefficient of $z_1^{\lambda_1 - \lambda_i} e_i = e_i$ is zero, since $\eta_i = z_i + z_1 R_{i,1}(z, w)$; otherwise $z_1^{\lambda_1 - \lambda_i} e_i$ has to be a product of the form $z_1^{-\lambda_i}(z_1^{\lambda_1}\mathcal{M}_1)^{i_1} \ldots (z_1^{\lambda_n}\mathcal{M}_n)^{i_n}$, with $|I|$ at least 2, as explained before. More precisely,

$$z_1^{\lambda_1 - \lambda_i} = z_1^{-\lambda_i}(z_1^{\lambda_1} z_1^{k_1\lambda_1 - \lambda_1})^{i_1} \ldots (z_1^{\lambda_n} z_1^{k_n\lambda_1 - \lambda_n})^{i_n} = z_1^{(k \cdot I)\lambda_1 - \lambda_i}$$

Since $z_1^{k_j\lambda_1 - \lambda_j}$ has zero coefficient if $k_j = 0$, it is necessary that $k \cdot I = 1$ with with $|I| \geq 2$ and all k_j positive.

As this is impossible, the coefficient of $z_1^{\lambda_1 - \lambda_i} e_i$ has to be zero, and thus:

$$\frac{\partial \xi_1}{\partial y_1}(0, w) = 1, \qquad \frac{\partial \xi_i}{\partial y_1}(0, w) = 0 \; i > 1$$

The reasoning for the other partial derivatives is similar, leading to a Jacobian at $(0, w)$ equal to the identity; thus ξ is a biholomorphic change of coordinates, and the proof is complete.

Example 2 Let $X(x) = Ax + a(x)$ be a vector field in $\mathbf{R}^4$, with $Ax = (x_1, -x_2, x_4, -x_3)$; as the eigenvalues $\lambda = (1, -1, -i, i)$ are resonant and the number of resonant monomials is not finite, classical results can only guarantee the existence of a formal change of coordinates to a normal form containing only resonant monomials besides the linear part.

The non linearity a is admissible (this is independent of parameters) when it contains only monomials of the following types:

$$x_1^{k+l+2} x_2^l x_3^m x_4^m e_1, \; x_1^{k+l} x_2^l x_3^m x_4^m e_2, \; x_1^{k+l+1} x_2^l x_3^{m+1} x_4^m e_3, \; x_1^{k+l+1} x_2^l x_3^m x_4^{m+1} e_4$$

If φ is an analytic map of $\mathbf{R}^3$ into $\mathbf{R}^4$, the non linearity:

$$a(x) = (x_1^2\varphi_1(y), \; \varphi_2(y), \; x_1x_3\varphi_3(y), \; x_1x_4\varphi_4(y)), \quad y = (x_1, x_1x_2, x_3x_4)$$

is then admissible, and theorem 3 guarantees the existence of an analytic change of coordinates that linearizes $X(x) = Ax + a(x)$. As the proof is constructive, it provides a method for the computation of that change of coordinates; for $\varphi(y) = (y_3, 0, 1, 1)$ the corresponding vector field $X(x) = (x_1 + x_1^2 x_3 x_4, -x_2, x_4 + x_1x_3, -x_3 + x_1x_4)$ is linearized by:

$$\psi(x) = \left(x_1 + \frac{1}{2}x_1x_3x_4(\mathrm{e}^{2x_1} - 1), \; x_2, \; x_3\mathrm{e}^{x_1}, \; x_4\mathrm{e}^{x_1}\right)$$

3 Analytic linearizability of some Poisson tensors

3.1 Basic facts about Poisson structures

A Poisson bracket on a smooth (analytic) finite dimensional manifold M is a Lie bracket $\{,\}$ on the space $C^\infty(M)$ ($C^\omega(M)$) which in addition satisfies Leibnitz rule:

1. $\{f,g\} = -\{g,f\}$, $\{f, ag+bh\} = a\{f,g\} + b\{f,h\}$
2. $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} \equiv 0$ (Jacobi identity)
3. $\{f,gh\} = \{f,g\}h + g\{f,h\}$ (Leibnitz rule)

for all $f,g,h \in C^\infty(M)$ ($C^\omega(M)$) and $a,b \in \mathbf{R}$. The pair $(M,\{,\})$ is then called a smooth (analytic) Poisson manifold.

Example 3 A symplectic manifold (M,ω) is a Poisson manifold with bracket given by:

$$\{f,g\} = \omega(X_f, X_g)$$

where X_f stands for the Hamiltonian vector field associated with f.

Example 4 The dual of any Lie algebra (g, [,]) is a Poisson manifold with bracket given by:

$$\{f,g\}(x) = \langle x, [\delta f(x), \delta g(x)]\rangle$$

where $\delta f(x)$ stands for $df(x)$ seen as an element of g instead of an element of $(g^*)^*$. This bracket is known as the Lie-Poisson bracket on g^* and is a linear bracket in the sense that it associates a linear function to any pair of linear functions.

A Poisson manifold can also be defined as a pair (M,P), where P is an alternating contravariant 2-tensor on M which satisfies $[P,P] = 0$, where $[,]$ stands for the Schouten bracket (see for example [9]). The relation between the two structures is given by:

$$\{f,g\} = P(df, dg).$$

and in local coordinates $x = (x_1, ..., x_n)$ on M:

$$P = \sum_{1\leq i<j\leq n} P_{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

where $P_{ij} = \{x_i, x_j\}$.

The rank of the Poisson structure at a point x_0 is the rank of the matrix P_{ij} at x_0; due to skew-symmetry of P, it will always be an even number. If M is a symplectic manifold the corresponding Poisson structure has maximum constant rank, equal to the dimension of M.

A Poisson manifold is locally a product of a *symplectic leaf through the point*, a symplectic manifold of dimension equal to the rank of the structure at the point under consideration, and a *transverse Poisson manifold through the point*, a Poisson manifold having a point of rank zero [12]. This decomposition reduces the local study of Poisson manifolds to a neighbourhood of a null rank point.

Given a Poisson manifold (M, P) and x_0 a null rank point in M, P can be written as:

$$P = \sum_{1 \leq i < j \leq n} \left(\sum_{k=1}^{n} C_{ij}^k x_k + O(2) \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

The numbers C_{ij}^k are the structure constants of a Lie algebra, denoted by $\mathcal{A}(M, P, x_0)$, which can be identified with $T^*_{x_0}M$. As described earlier, the dual of this Lie algebra (i.e., $T_{x_0}M$) together with the Lie-Poisson bracket is a linear Poisson vector space.

In suitable coordinates the Lie-Poisson bracket is given by:

$$P^0 = \sum_{1 \leq i < j \leq n} \left(\sum_{k=1}^{n} C_{ij}^k y_k \right) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$$

and is the *linear approximation to* (M, P) *at* x_0.

The (smooth, analytic) tensor P is (smoothly, analytically) linearizable at x_0 if there is a local (smooth, analytic) Poisson-diffeomorphism transforming P into P^0, a Poisson diffeomorphism being a diffeomorphism preserving the Poisson tensor, and in particular its rank.

3.2 An analytic version of Dufour's theorem

In what follows, an $(n+1)$-dimensional Lie algebra g is said to be of type A if it is the semi-direct product of the 1-dimensional Lie algebra $\mathbf{R}$ and an n-dimensional abelian Lie algebra a, the action of $\mathbf{R}$ on a being by an endomorphism . This means that there exists a basis $\beta = \{t, x_1, ..., x_n\}$ for g and a matrix A, with eigenvalues $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{C}^n$ such that the Lie brackets are given by:

$$[t, e_i] = \sum_{j=1}^{n} a_{ij} e_j$$

and all other brackets are zero. The matrix $A = (a_{ij})$ is assumed to be in Jordan normal form.

Theorem 4 *Let P be an analytic Poisson tensor on a manifold M having only 0- and 2-dimensional symplectic leaves. Let $x_0 \in M$ be a zero rank point such that $\mathcal{A}(M, P, x_0)$ is of type A, with eigenvalues nonresonant and of type (C, ν). Then P is analytically linearizable at x_0.*

Proof:

Most of the proof is adapted from the proof of Theorem 1 in [6] using the theorem of section 1 in place of Lemma 6 of [6], and verifying that the remaining results are valid in the smooth and analytic categories; lemma 14 is different and its proof avoids solving an equation of the type $L_X g = f$ for g, with f analytic but depending on parameters.

Since the theorem is local it can be assumed that U is a neighbourhood of the origin in $\mathbf{R}^{n+1}$, with coordinates $(x, t) = (x_1, ..., x_n, t)$ such that the Poisson bracket $\{\,,\}$ is given by:

$$\{x_i, t\} = \sum_{j=1}^{n} a_{ij} x_j + \phi_i(x, t),$$

and:

$$\{x_i, x_j\} = U_{ij}(x, t),$$

where ϕ_i and U_{ij} are analytic in U and vanish to 2^{nd} order in (x, t) at the origin.

Denoting by X_i the function $\{x_i, t\}$, the vector field:

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$$

is the Hamiltonian vector field associated with the function t.

Given an analytic function f in U, let f^k be its homogeneous part of degree k in x:

$$f^k(x, t) = \sum_{|I|=k} f_I(t) x^I,$$

and similarly, let Y^k be the homogeneous part of degree k of the analytic vector field Y:

$$Y^k(x, t) = \sum_{i=1}^{n} Y_i^k(x, t) \frac{\partial}{\partial x_i}, \quad \text{where} \quad Y(x, t) = \sum_{i=1}^{n} Y_i(x, t) \frac{\partial}{\partial x_i}.$$

Lemma 9 *Up to an analytic change of coordinates, $X^0 \equiv 0$.*

Proof:
Since:

$$\left(\left. \frac{\partial X_i}{\partial x_j} \right|_{(0,0)} \right) = A$$

is nonsingular, due to nonresonancy, the system of equations:

$$X_i(x,t) = 0\,, i = 1, ..., n$$

of which the origin is a solution, defines an unique analytic map θ in a neighbourhood I of $t = 0$ ([7]) such that:

$$X_i(\theta(t), t) = 0\,, \forall t \in I, \quad \theta(0) = 0.$$

The analytic change of coordinates:

$$(x,t) \mapsto (x - \theta(t), t)$$

yields the result.

Lemma 10 *There exists an analytic vector field Z, defined up to a term of the form fX, such that:*

$$U_{ij} = X_i Z_j - Z_i X_j.$$

Proof:
The fact that the rank of P is everywhere less than or equal to two implies:

$$\begin{vmatrix} \{x_i, x_i\} & \{x_i, x_j\} & \{x_i, x_k\} & X_i \\ \{x_j, x_i\} & \{x_j, x_j\} & \{x_j, x_k\} & X_j \\ \{x_k, x_i\} & \{x_k, x_j\} & \{x_k, x_k\} & X_k \\ -X_i & -X_j & -X_k & 0 \end{vmatrix} = 0$$

and therefore:

$$X_i U_{jk} + X_j U_{ki} + X_k U_{ij} = 0 \tag{8}$$

for all i, j, k.

The map:

$$\phi(x,t) = (X(x,t), t).$$

has a nonsingular Jacobian matrix at the origin, and restricting its domain if necessary, it is a local diffeomorphism. Using the change of coordinates given by ϕ, U_{ij} can be expressed in the new variables (X, t).

Evaluating (8) at $(X_i, X_j) = (0,0)$ we get $U_{ij} = 0$, and this together with skew-symmetry produces:

$$U_{ij}(X) = A_{ij}(\hat{X}_{ij})X_i - A_{ji}(\hat{X}_{ij})X_j + C_{ij}(X)X_1^2 - C_{ji}(X)X_2^2 + D_{ij}(X)X_1X_2$$

where $\hat{X}_{ij}$ denotes $(X_1, ..., X_n)$ without the components X_i, X_j; skew-symmetry also implies $D_{ji}(X) = -D_{ij}(X)$. Using (8) again one gets:

1. $A_{ij}(\hat{X}_{ij}) = A_{kj}(\hat{X}_{kj})$ for all i, j, k. In particular each A_{ij} is constant and depends only on j;
2. $C_{ij}(X) = 0$, for all i, j;
3. $D_{ij}(X) + D_{jk}(X) + D_{ki}(X) = 0$, for all i, j, k.

This means that we can write:

$$U_{ij}(X) = A_jX_i - A_iX_j + D_{ij}(X)X_iX_j.$$

The conditions on D_{ij} imply the existence of functions $E_1(X), ..., E_n(X)$ such that:

$$D_{ij}(X) = E_i(X) - E_j(X),$$

(e.g., take $E_1(X) = 0$ and $E_i(X) = D_{i1}(X)$, for $i = 2, ..., n$). This produces:

$$U_{ij}(X) = C_jX_i - C_iX_j + (E_i(X) - E_j(X))X_iX_j,$$

thus it will be enough to take:

$$Z_k = C_k + E_k(X)X_k.$$

If $n = 2$ the above argument does not work, but the Jacobi identity for x_1, x_2 and t implies that $U_{12} = 0$ if $X_1 = X_2 = 0$, and therefore $U_{12}(X) = X_1Z_2 - X_2Z_1$ for convenient Z_1 and Z_2.

Changing Z by taking:

$$Z'_k = Z_k + fX_k$$

makes no difference whatsoever in the commutator $X_iZ_j - Z_iX_j$ and this concludes the proof of the lemma.

Lemma 11 *The Jacobi identity for x_i, x_j and t implies that:*

$$\frac{dX}{dt} - [X, Z] = fX,$$

for some analytic function f.

Proof:

Jacobi identity together with the previous lemma gives:

$$\{X_i Z_j - X_j Z_i, t\} + \{X_j, x_i\} - \{X_i, x_j\} = 0.$$

Using again the previous lemma and rearranging one gets:

$$X_i \left(X(Z_j) - Z(X_j) - \frac{dX_j}{dt} \right) = X_j \left(X(Z_i) - Z(X_i) - \frac{dX_i}{dt} \right)$$

which means that X and $[X, Z] - \frac{dX}{dt}$ are linearly dependent.

Lemma 12 *With an analytic change of coordinates $(\psi(x,t), t)$, X satisfies:*

$$\frac{dX}{dt} = gX,$$

for some analytic function g.

Proof:

Using the previous lemma, the evaluation at the origin gives $Z^0 = 0$; thus Z can be seen as a time-dependent vector field with a critical point at $x = 0$, and there is a well defined flow ϕ_t, with $\phi_t(x, 0) = x$. Then:

$$\frac{d}{dt}(\phi_t^{-1}.X) = \phi_t^{-1}.\left(\frac{dX}{dt} - [X, Z]\right).$$

By definition we have:

$$\frac{d}{dt}\phi_t = Z \circ \phi_t$$

or equivalently:

$$\frac{d}{dt} f \circ \phi_t = Z(f \circ \phi_t).$$

Also:

$$\frac{d}{dt}\phi_t^{-1} = -(d\phi_t^{-1})(Z)$$

i.e.:

$$\frac{d}{dt} f \circ \phi_t^{-1} = -Z(f \circ \phi_t^{-1}).$$

Finally:

$$\begin{aligned}
\frac{d}{dt}(\phi_t^{-1}.X)(f) &= \frac{d}{dt}\left(X(f \circ \phi_t^{-1}) \circ \phi_t\right) \\
&= \left(\phi_t^{-1}.\frac{dX}{dt}\right)(f) + [Z,X](f \circ \phi_t^{-1}) \circ \phi_t \\
&= \phi_t^{-1}.\left(\frac{dX}{dt} + [Z,X]\right)(f).
\end{aligned}$$

The change of coordinates $(\psi(x,t),t)$ is given by the inverse of $(\phi(x,t),t)$.

Lemma 13 *Up to an analytic change of the time-variable by $t \mapsto a(t)$, the linear part (in x) X^1 of X is independent of t.*

Proof:
From:

$$\frac{dX}{dt} = gX$$

we get:

$$\frac{dX^1}{dt} = g^0 X^1.$$

Using the indicated change of coordinates X is replaced by bX, with

$$b = \frac{d}{dt} a^{-1}.$$

Therefore we get:

$$\frac{d(bX)^1}{dt} = \left(\frac{db}{dt} + bg^0\right) X^1,$$

and taking the adequate b produces $(bX)^1$ independent of t.

Now the conditions of Siegel theorem with parameters (theorem 2) are verified, taking t as the parameter, and X can be linearized: in convenient coordinates $X(x,t) = Ax$, with A in Jordan normal form.

The Poisson bracket with t induces a linear map $\mathcal{P}$ on $C^\omega(\mathbf{R}^n \times \mathbf{R})$ by $h \mapsto \{h,t\}$; we are interested on its eigenvalues and eigenvectors.

Let h^k be the homogeneous part of degree k of a time-dependent function h, seen as the value of a symmetric k-linear form B^k of $\mathbf{R}^n$ in $\mathbf{R}$, depending on the parameter t:

$$h^k(x,t) = B^k(t)(x,\ldots,x)$$

As $\{x_i, t\} = X_i = (Ax)_i$, the Poisson bracket with t is then:

$$\{h^k(x,t), t\} = \sum_{i=1}^{k} B^k(t)(x,\ldots,x,Ax,x,\ldots,x)$$

with Ax in the i-th position.

The map:

$$B^k \mapsto \mathcal{P}_A^k(B^k), \quad \mathcal{P}_A^k(B^k)(v_1,\ldots,v_k) = \sum_{i=1}^{k} B^k(t)(v_1,\ldots,Av_i,\ldots,v_k)$$

is a linear map on the symmetric k-linear forms, and its eigenvalues are all possible expressions:

$$\lambda_{i_1} + \ldots + \lambda_{i_k}$$

This is easy to prove if all eigenvalues of A are distinct, and as A is in the Jordan normal form it is then diagonal, and a continuity argument proves the result in the remaining cases.

If e_{i_j} are eigenvectors corresponding to the eigenvalues λ_{i_j}, and they always exist if A is diagonal, the symmetric k-linear form:

$$(v_1,\ldots,v_k) \mapsto \frac{1}{k!}\sum_{\sigma\in S^k} (v_1)_{\sigma(i_1)}\cdots(v_k)_{\sigma(i_k)} \tag{9}$$

is an eigenvector of $\mathcal{P}_A^k$ with eigenvalue $\lambda_{i_1} + \ldots + \lambda_{i_k}$.

All eigenvalues of $\mathcal{P}_A^k$ are obtained from the eigenvectors of A by a similar process.

Denote by $\mathcal{P}_A$ the linear map that A induces in this way on the space of all time dependent symmetric multi-linear maps of $\mathbf{R}^n$ in $\mathbf{R}$. The eigenvalues of $\mathcal{P}$ are the eigenvalues of $\mathcal{P}_A$, and its eigenvectors can be deduced from the corresponding eigenvectors of $\mathcal{P}_A$ as the monomials obtained by taking values at $(x,\ldots,x)$; thus the monomial corresponding to (9) is $x_{i_1}\cdots x_{i_k}$.

The next steps will show that it is now possible to linearize each U_{ij} in such a way that the vector field X remains unchanged.

Lemma 14 *The vector field Z can be chosen to commute with X.*

Proof:

Applying again Lemma 11, now with X linear and time-independent, gives:

$$[Z, X] = fX.$$

for some analytic function f. To prove that X and Z commute it is enough to show that one component of $[Z, X]$ is zero.

From $\{x_i, t\} = \varepsilon_i x_{i-1} + \lambda_i x_i$, with $\varepsilon_i = 1$ if x_i belongs to a Jordan block and 0 otherwise, it follows that:

$$\{\{x_i, x_j\}, t\} = \varepsilon_i\{x_{i-1}, x_j\} + \varepsilon_j\{x_i, x_{j-1}\} + (\lambda_i + \lambda_j)\{x_i, x_j\}$$

If there exists at least one pair (x_i, x_j) such that their Poisson brackets with t verify:

$$\{x_i, t\} = \lambda_i x_i, \quad \{x_j, t\} = \lambda_j x_j$$

then $\{\{x_i, x_j\}, t\} = (\lambda_i + \lambda_j)\{x_i, x_j\}$ and $\{x_i, x_j\}$ is an eigenvector of $\mathcal{P}$ corresponding to the eigenvalue $\lambda_i + \lambda_j$.

Then all monomial terms that appear in $\{x_i, x_j\}$ are of the form

$$c_I(t)x^I, \quad I \cdot \lambda = \lambda_i + \lambda_j$$

where all variables with non zero exponent i_j are eigenvectors of $\mathcal{P}$, thus verifying $\{x_{i_j}, t\} = \lambda_{i_j} x_{i_j}$.

Since moreover $\{x_i, x_j\} = U_{ij}(X) = X_i Z_j - X_j Z_i$, Z_j can be chosen as the factor of X_i in that decomposition, and so it is a linear combination, with time-dependent coefficients, of monomials x^J such that $J \cdot \lambda = \lambda_j$; the non resonance hypothesis implies those monomials are just x_k for which $\lambda_k = \lambda_j$, and as remarked before $\{x_k, t\} = \lambda_j x_k$.

It is easy to see that then the i-th and j-th components of $[X, Z]$ are zero, and the two vector fields commute.

If no such pair (x_i, x_j) exists, A is just one Jordan block:

$$\{x_1, t\} = \lambda x_1, \quad \{x_2, t\} = x_1 + \lambda x_2, \quad \{\{x_1, x_2\}, t\} = 2\lambda\{x_1, x_2\}$$

and again $\{x_1, x_2\}$ is an eigenvector of $\mathcal{P}$, now corresponding to the eigenvalue 2λ:

$$\{x_1, x_2\} = c(t)x_1^2 = \frac{c(t)}{\lambda^2}X_1^2 = X_1 Z_2 - X_2 Z_1$$

and then:

$$Z_1 = 0, \quad Z_2 = \frac{c(t)}{\lambda^2}X_1 = c(t)x_1$$

As before the two vector fields X and Z commute.

Lemma 15 *The change of coordinates given by the inverse of the flow of Z leaves X invariant and linearizes all U_{ij}.*

Proof:

Let ψ_t denote the inverse of the flow of Z; X is invariant because $[X, Z] \equiv 0$. Then:

$$\begin{aligned} \psi_t.U_{ij} &= U_{ij} \circ \psi_t^{-1} \\ &= (X_i Z_j) \circ \psi_t^{-1} - (X_j Z_i) \circ \psi_t^{-1} \end{aligned}$$

whereas the linear approximation to $\{\psi_t.x_i, \psi_t.x_j\}$ at the origin is given by:

$$\sum_{1 \le r \le n} \left(\frac{\partial \psi_t.x_i}{\partial x_r} \frac{\partial \psi_t.x_j}{\partial t} - \frac{\partial \psi_t.x_i}{\partial t} \frac{\partial \psi_t.x_j}{\partial x_r} \right) X_r = \psi_t.X_i \frac{\partial \psi_t.x_j}{\partial t} - \psi_t.X_j \frac{\partial \psi_t.x_i}{\partial t}$$

and now, because ψ_t is the inverse of the flow of Z:

$$\frac{\partial \psi_t.x_j}{\partial t} = Z \circ \psi_t^{-1}$$

concluding the proof of the theorem.

4 Finite determinacy and analytic linearization of some resonant Poisson tensors

In what follows, the $(n+1)$-dimensional Lie algebra g is of type A, and the local coordinates are such that **R** acts on a by a diagonal matrix. The Poisson bracket $\{\ ,\ \}$ is assumed to be given by:

$$\{x_i, t\} = \lambda_i x_i + \phi_i(x, t), \quad \{x_i, x_j\} = U_{ij}(x, t),$$

where ϕ_i and U_{ij} are analytic in U and vanish to 2^{nd} order in x at the origin, in particular, the linear part of the vector field X, as defined in the previous section, is time independent.

Recall that the nonlinearity $\phi(x, t)$ is admissible, in the context of real analytic vector fields, if $I \cdot \lambda - \lambda_i$ are all positive (or all negative) integer multiples of a real number for every monomial $x^I t^J e_i$ in $\phi_i(x, t)$

Let σ be the ratio:

$$\sigma = \frac{\max |\Re \lambda_i|}{\min |\Re \lambda_i|}$$

and denote by d the biggest integer not greater than σ, $d = [\sigma]$.

Theorem 5 *Let P be an analytic Poisson tensor on a manifold M having only 0- and 2-dimensional symplectic leaves. Let $x_0 \in M$ be a zero rank point such that $\mathcal{A}(M, P, x_0)$ is of type A, corresponding to a diagonal matrix, and with eigenvalues in the Poincaré domain.*

If

$$j^d(X)(x,t) = Ax$$

then P is analytically linearizable at x_0.

Proof:

It follows from the hypothesis on the vanishing order of $\phi_i(x,t)$ that all monomials in $\phi_i(x,t)$ are of the type $x^I t^j$ with $|I| > \sigma$.

As the eigenvalues are in the Poincaré domain, all those monomials are non resonant, and as in the proof of theorem 2 the lowest order (in x) of the monomials of the nonlinearity increases in each step, no resonant monomials can appear and the linearization can be carried through in the analytic category.

Now lemma 14 is still essentially valid, as Z can be chosen so that it commutes with $X(x,t) = Ax$ though it is not necessarily linear:

All monomial terms that appear in $\{x_i, x_j\}$ are as before of the form

$$c_I(t)x^I,$$

with $I \cdot \lambda = \lambda_i + \lambda_j$; since moreover $\{x_i, x_j\} = U_{ij}(X) = X_i Z_j - X_j Z_i$, the vector field Z can be chosen by taking Z_j as the factor of X_i in that decomposition. Then Z_j is a linear combination, with time-dependent coefficients, of monomials x^J such that $J \cdot \lambda = \lambda_j$; as A is resonant, those monomials are not all linear, the resonant monomials have to be considered.

Since A is assumed to be diagonal, X commutes with every resonant monomial $c_J(t)x^J e_j$ and therefore commutes with Z.

The same proof, but now using theorem 3 instead of therem 2 for the linearization of X, leads to:

Theorem 6 *Let P be an analytic Poisson tensor on a manifold M having only 0- and 2-dimensional symplectic leaves. Let $x_0 \in M$ be a zero rank point such that $\mathcal{A}(M, P, x_0)$ is of type A, corresponding to a diagonal matrix. If the nonlinear term $\phi(x,t)$ of X vanishes to 2^{nd} order in x at the origin and is admissible, then P is analytically linearizable at x_0.*

If the eigenvalues are in the Poincaré domain, and:

$$j^d(X)(x,t) = Ax + \textit{admissible term}$$

then P is analytically linearizable at x_0.

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