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RESUMO

O "view graph" de uma superfície N em $\mathbb{R}^3$ é um grafo mergulhado no espaço ν dos centros ou direções de projeção, cujos vértices correspondem às regiões conexas maximais de ν que dão projeções equivalentes de N. O tamanho do "view graph" de uma superfície algébrica regular por partes N com auto-intersecções transversais, pontos triplos e cross-caps isolados é $O(n^{K \dim \nu} d^{6 \dim \nu})$, onde n e d são o número das "componentes" de N e seus graus maximais, respectivamente, e onde $k = 6$ em geral e $k = 3$ quando N é difeomorfa à fronteira de um poliedro. (Para superfícies sem cross-caps, este limite foi obtido em [17]). Também, para o caso de superfície linear por partes, onde $d = 1$ e $K = 3$, sabe-se que o tamanho do "view graph" é $\mathcal{O}\left(n^{3 \dim \nu}\right)$.

Neste artigo, mostramos que os "view graphs" de tais superfícies podem ser determinadas em tempo $\mathcal{O}\left(n^{K(2 \dim \nu + 1)}\right) \cdot P(d, L)$, por um algoritmo determinístico e em tempo esperado $\mathcal{O}(n^{K \dim \nu + \varepsilon}) \cdot P(d, L)$ por um algoritmo aleatorizado. Aqui, P indica um polinômio, L o tamanho do coeficiente maximal dos polinômios que definem N e ε é uma constante pequena arbitrária. Em termos da complexidade, o algoritmo aleatorizado é quase ótimo – seu tempo de complexidade combinatória ultrapassa o tamanho do "view graph" apenas pela potência ε.

Notes on the complexity of exact view graph algorithms for piecewise smooth algebraic surfaces

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Abstract

The view graph of a surface N in 3-space is a graph embedded in the space $\mathcal{V}$ of centres or directions of projection, whose nodes correspond to maximal connected regions of $\mathcal{V}$ which yield equivalent views of N. The size of the view graph of a piecewise smooth algebraic surface N with transverse selfintersection curves and isolated triple-points and cross-caps is $O(n^{K \dim \mathcal{V}} d^{6 \dim \mathcal{V}})$, where n and d denote the number of "component surfaces" of N and their maximal degree, respectively, and where $K = 6$ in general or $K = 3$ for N diffeomorphic to the boundary of a polyhedron. (For surfaces without cross-caps, this bound has been established in [17].) Also, for the special piecewise linear case, where $d = 1$ and $K = 3$, it is known that the size of the view graph is actually $\Theta(n^{3 \dim \mathcal{V}})$.

It is shown that the exact view graphs of such surfaces can be determined in $O(n^{K(2 \dim \mathcal{V}+1)}) \cdot \mathcal{P}(d, L)$ time by a deterministic algorithm and in $O(n^{K \dim \mathcal{V}+\epsilon}) \cdot \mathcal{P}(d, L)$ expected time by a randomized algorithm. Here $\mathcal{P}$ is some polynomial, L the maximal coefficient size of the defining polynomials of N and ϵ is an arbitrarily small positive constant. Note that the randomized algorithm is, in terms of combinatorial complexity (where d and L are assumed to be constants which do not depend on n), nearly optimal — its combinatorial time complexity exceeds the size of the view graph only by ϵ in the exponent.

Keywords: Geometric algorithms, Complexity, Real bifurcation sets, Visibility properties

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1 Introduction

The computation and complexity of view (or aspect) graphs of surfaces in 3-space has been the subject of a fair number of works in the fields of computational geometry and computer vision. The survey article by Bowyer and Dyer [2] gives a good overview of the works on polyhedral surfaces up to 1990 from a computer-vision point of view. Related works in the fields of combinatorics and computational geometry on visibility problems and the number of topologically distinct views of "polyhedral terrains" (i.e. special polyhedral surfaces given as function graphs) are summarized in the recent book by Sharir and Agarwal [18] (see also the discussion at the end of the present paper). The recent survey [15] concentrates on view graphs of smooth and piecewise smooth algebraic surfaces (the latter include both smooth as well as polyhedral surfaces as special cases); most (algorithmic) works on this subject have appeared after 1990.

In [17] we have studied the complexity of view (or aspect) graphs of piecewise smooth algebraic surfaces M having transverse selfintersection curves ("crease curves") and isolated triple points and of subsets $M' \subset M$ representing the boundaries of semi-algebraic solids. In this work, we have also presented exact symbolic algorithms for computing the view graphs of M and M' and have illustrated the results produced by an implementation of these algorithms for simple example surfaces M and $M' \subset M$.

Much remains to be done in terms of analyzing and improving the running times of these algorithms. The algorithm in [17] for the surfaces M has a polynomial running time, but we did not attempt to give a specific bound for the degree of this polynomial (which would have been far from optimal). For bounding surfaces $M' \subset M$ of solids the worst-case running time of the algorithm in [17] is exponential in the number of unions appearing in the defining formula of the solid.

Meanwhile there have been two new developments in the fields of computational geometry and classification of singularities which form the basis for the algorithms described in the present paper. These algorithms are more efficient and (slightly) more general than the ones in [17]. First (and relevant for the efficiency aspect) there is the work by Chazelle *et al.* [4] and by Sharir and Agarwal [18] in the field of computational geometry and combinatorics which yields algorithms for stratifying semi-algebraic sets which are much more efficient, in terms of combinatorial complexity, than previous algorithms. Given n polynomials in D variables, the algorithms in [4, 18] determine a decomposition of $\mathbb{R}^D$ into $O(n^{2D-3+\epsilon})$ sign-invariant connected cells of "simple shape" for which point-location queries can be answered in $O(\log n)$ time. This decomposition can be determined by a deterministic algorithm in $O(n^{2D+1})$ time and by a randomized algorithm in $O(n^{2D-3+\epsilon})$ (for $D \geq 3$) or $O(n^{2+\epsilon})$ (for $D = 2$) expected time (see Theorems 8.21 and 8.23 in [18] and the introduction in [4]). Second, there is the recent classification by Carter *et al.* [3] of sequences of projections $\mathbb{R}^4 \to \mathbb{R}^3 \to \mathbb{R}^2 \to \mathbb{R}$ of knotted 2-surfaces in 4-space which, as a by-product, yields the codimension 1 views of singular surfaces in 3-space having not only transverse intersection curves and triple-points but also cross-caps.

These results, together with the rather detailed *a priori* knowledge about the topology of the view bifurcation set from [17], enable us to obtain — in a fairly straightforward way — exact view graph algorithms for surfaces M and M' (having additional cross-caps) which have a near-optimal combinatorial time complexity. For the somewhat lengthy and technical derivations of certain essential topological properties of these view bifurcation sets $\mathcal{B}$ and $\mathcal{B}'$ of M and M' we refer to [17].

1.1 Singular surfaces and bounding surfaces of solids

In the present paper we are interested in "visibility problems" for the following two classes of curved surfaces.

(i) Singular algebraic surfaces $M = \bigcup_{i=1}^{n} M_i$, where M and all M_i have curves of transverse selfintersections and isolated triple-points and cross-caps.

(ii) Bounding surfaces M' of closed semi-algebraic solids

$$S := \bigcup_{u=1}^{U} \bigcap_{i=1}^{I_u} \{p \in \mathbb{R}^3 : h_{u,i}(p) \geq 0\},$$

such that $M = \bigcup_u \bigcup_i h_{u,i}^{-1}(0)$ is an algebraic surface with the kind of singular points described in (i). Note that any M' is a semi-algebraic subset of some M.

Note, of course, that piecewise linear surfaces are special cases of the surfaces M and M' above. In terms of space and time complexity of the algorithms below it is, however, convenient to single out a special subclass of the surfaces M' for which sharper complexity bounds can be obtained. Namely, the class (ii)$'$ of bounding surfaces M' of semi-algebraic solids which can be deformed — by a non-linear coordinate change in $\mathbb{R}^3$ — into the bounding surface of a polyhedron. The surfaces in this subclass have $O(m)$ selfintersection curves, m being the number of faces, and no cross-caps — on the other hand, a general surface M' has $O(m^2)$ selfintersection curves and also has isolated cross-caps.

Also note that, for zero-sets, one expects in the *generic case* that the component surfaces $h_i^{-1}(0)$ [resp. $h_{u,i}^{-1}(0)$] are *non-singular* and intersect transversely along intersection curves and in isolated triple-points (this is an easy consequence of the Bertini-Sard theorem). Furthermore, such surfaces — without cross-caps — are typically the ones that are relevant in geometric and solid modelling. By contrast, surfaces given as images of *generic mappings* from 2-space into 3-space typically have transverse selfintersection curves, triple-points as well as cross-caps (this is a classical result of Whitney [21]). This more general class of surfaces will be considered in the present paper.

1.2 Combinatorial and algebraic complexity

The *combinatorial complexity* of the surface M is determined by the number n of component surfaces M_i. The combinatorial complexity of M' does *a priori* depend on the number of unions U and the numbers of intersections $I_1, \ldots, I_U$ required to define the solid S. However, the total number $n := \sum_{u=1}^{U} I_u$ of "basic semi-algebraic sets" $\{p \in \mathbb{R}^3 : h_{u,i}(p) \geq 0\}$ will be, for the purpose of the present paper, a sufficiently fine measure of the combinatorial complexity of S and its boundary M'.

The *algebraic complexity* of the surfaces M and M' is given by the maximal degree d of the component surfaces M_i and $h_{u,i}^{-1}(0)$, respectively.

The time complexity of the algorithms described below depends on n and d as well as on the coefficient sizes of the defining polynomials f_i of the surfaces. So we set $L = \sup_i |f_i|_1$. Note that a polynomial-time algorithm in (n, d, L) has also a polynomial running time in the *bit-complexity* model.

Finally, the following two (symbolic) constants will appear in the exponents of the complexity estimates below. First, the dimension of the "viewing space" $\mathcal{V}$ which is two in the case of parallel and three in the case of central projection. Second, for any surface M or M' having only $O(n)$ selfintersection curves we set $K = 3$, for general surfaces (having $O(n^2)$ selfintersection curves) we set $K = 6$. Note that, for surfaces M' which are diffeomorphic to the boundaries of polyhedra, we have that $K = 3$ (recall the discussion in 1.1 above).

1.3 Classifications of views and the definition of view graphs

Let N be either one of the surfaces M or M', let $S(N)$ denote the singular set of N and let $\mathcal{V} = \mathbb{P}^2$ (resp. $\mathbb{R}^3 \setminus N$) be the "view space" of all directions (resp. centres) of projection. Given $q \in \mathbb{R}^3$ and $\omega \in \mathcal{V}$, parallel (resp. central) projection maps q to the line through q parallel to ω (resp. to the line joining q and ω). Let $p_\omega : N \to \mathbb{P}^2$ denote the restriction to N of the parallel (resp. central) projection from $\mathbb{R}^3$ into the "retinal plane" $\mathbb{P}^2$ along the direction (resp. from the centre) $\omega \in \mathcal{V}$. Define a *view* of N from ω as follows:

$$v_\omega(N) := p_\omega(S(N)) \cup p_\omega(\Sigma_\omega(N)),$$

where Σ_ω denotes the set of critical points of the map p_ω (in which dp_ω fails to have full rank). From "almost all" directions $v_\omega(N)$ is a curve in the retinal plane which is the union of the projected selfintersection curves of N and of the apparent contours of the "faces" of N.

A pair of views of N is *equivalent* if one of them is mapped onto the other by a diffeomorphism of the retinal plane. The actual classifications of views of N use a slightly different equivalence relation: a pair of projections of $N \subset \mathbb{R}^3$ is equivalent if there exist diffeomorphisms of $\mathbb{R}^3$, preserving N, and of the retinal plane mapping one projection onto the other. It is known that equivalence of projections implies equivalence of views, but the converse is only true for equivalence classes (of projections and views) of low codimension. In the present paper we merely have to know all possible codimension ≤ 1 views (the stable and "minimally unstable" views), and for each of them there is exactly one corresponding type of projection of the same codimension. Views of surfaces N with transverse selfintersection curves and isolated triple points can have 6 types of *stable* isolated singular points, corresponding to the codimension 0 orbits under the equivalence relation above, and 19 types of *unstable* codimension 1 singularities, see Rieger [17]. For surfaces with additional cross-caps, we get one additional type of stable singular point (a fold of the cross-cap) and 3 additional codimension 1 singularities, see Carter *et al.* [3].

The *view bifurcation set* $\mathcal{B} \subset \mathcal{V}$ of the family of all parallel or central projections of N consists of all directions or centres of projection $\omega \in \mathcal{V}$ which yield an unstable view (that is, a view containing at least one singularity of codimension ≥ 1). It has been shown in Rieger [17] that the view bifurcation set of any surface with transverse selfintersection curves and isolated triple points is a subset of view space of measure zero — however, surfaces with quadruple points have region-filling view bifurcation sets.

Assuming that the view bifurcation set $\mathcal{B} \subset \mathcal{V}$ of a surface N has measure zero, we can define the *view graph* $G(N)$ of N as follows: $G(N) = (V, E)$ is a graph embedded in $\mathcal{V}$ whose set of vertices V are the connected regions of $\mathcal{V} \setminus \mathcal{B}$ and whose set of edges E are the branches of $\mathcal{B}$ of dimension $\dim \mathcal{V} - 1$ separating adjacent connected regions. Note that all views obtained from within a single connected region of $\mathcal{V} \setminus \mathcal{B}$ are related by a diffeomorphism of the retinal plane. Traversing an edge of the view graph $G(N)$ corresponds to a "catastrophic change" in the view of N.

1.4 Results and organization of paper

Section 2 contains some piecewise smooth algebraic example surfaces and their view graphs. (The examples have been computed with a slightly modified implementation of the algorithms in Rieger [17]. The modifications take care of cross-caps and other singularities on the component surfaces of M. The more substantial modifications described in Section 4 below have not yet been implemented.)

In Section 3 we present a table of necessary and sufficient conditions for a direction or centre of projection ω to lie in the view bifurcation set $\mathcal{B}$ (or $\mathcal{B}'$) of M or M'. In other words, the conditions in this table "recognize" all degenerate views of M (or M'). These conditions are derived from the classification of codimension 1 views of such surfaces in Carter *et al.* [3].

In Section 4 we study the time complexity of algorithms for determining the exact view graphs $G(\cdot)$ of piecewise smooth algebraic surfaces M and of bounding surfaces M' of semi-algebraic solids. We show that both $G(M)$ and $G(M')$ can be determined in $O(n^{K(2\dim\mathcal{V}+1)})\cdot\mathcal{P}(d,L)$ time by a deterministic algorithm or in $O(n^{K\dim\mathcal{V}+\epsilon})\cdot\mathcal{P}(d,L)$ expected time by a randomized algorithm. The combinatorial complexity of the randomized algorithm is only by an arbitrarily small positive constant ϵ greater than the size of the view graph $G(\cdot)$ which is of order $O(n^{K\dim\mathcal{V}}d^{6\dim\mathcal{V}})$. It is known that, in the special case of polyhedra where $d=1$ and $K=3$, the size of the view graph is actually $\Theta(n^{3\dim\mathcal{V}})$, see Platinga and Dyer [16]. So, from a combinatorial point of view, the randomized algorithm is nearly optimal.

The overall structure of the algorithm for M is the same as in Rieger [17], except that, in one substep, the (by now classical) cylindrical algebraic decomposition algorithm by Collins is replaced by the semi-cylindrical decomposition algorithm described in Chazelle *et al.* [4].

For bounding surfaces M' of semi-algebraic solids we describe a new polynomial time algorithm — the algorithm for M' in Rieger [17], by contrast, has a running time which is exponential in the number of unions U in the defining formula of the solid.

In Section 5 we present bounds for the degree of the view bifurcation set and for the number of nodes in the view graph of surfaces having cross-caps, as well as curves of transverse selfintersections and isolated triple-points. The complexity of view graphs of surfaces without such cross-caps have been studied previously by Rieger [17] and Petitjean [14]. Using the necessary and sufficient conditions for unstable views presented in Section 3 we show that the asymptotic complexity of the size of view graphs of surfaces with cross-caps is the same as that of piecewise smooth surfaces without cross-caps, namely $O(n^{K\dim\mathcal{V}}d^{6\dim\mathcal{V}})$.

Finally, in Section 6, we discuss some recent works on visibility problems and view graphs, relate them to the results in the present paper and mention some open problems.

2 A few examples

Perhaps it is the best to begin with some examples of piecewise smooth algebraic surfaces and their view graphs (for parallel projection). For parallel projection, the view bifurcation set $\mathcal{B}\subset\mathbb{P}^2$ of a surface N is the dual of the view graph $G(N)$. If $(a:b:c)$ are homogeneous coordinates in $\mathbb{P}^2$ then the computation of $G(N)$ can be carried out in the affine chart $(1,b,c)$ with the understanding that pairs of anti-podal regions r_+, r_-, i.e. nodes of $G(N)$, are to be identified, unless their closures contain components C of $\mathcal{B}$ at infinity. In this case the nodes r_+ and r_- are either connected by an edge, if C is a subarc of the line at infinity $a = 0$, or not if $\dim C = 0$. All three example surfaces below have certain symmetries which, in turn, induce symmetries of $\mathcal{B}$ and $G(N)$ in the plane (b, c). Below, we use the notational convention that regions in the complement of $\mathcal{B}$, i.e. nodes of $G(N)$, which are related by some reflection in the (b, c)-plane are denoted by the same numbers.

Example 1. The zero-set M of $h = x^2y^2 + x^2z^2 + y^2z^2 - xyz$ is known as Steiner's Roman surface (see Chapter VII of [10] and Chapter 6, §46 of [9]). The surface M has six cross-caps at $(\pm 1/2, 0, 0), (0, \pm 1/2, 0), (0, 0, \pm 1/2)$. There are three lines of double-points (the coordinate axes) connecting pairs of cross-caps and intersecting at the origin in a triple-point. Strictly speaking, Steiner's Roman surface is the semi-algebraic surface (having the same view graph as M) given by

$$M \setminus (\{x^2 > 1/4, y = z = 0\} \cup \{y^2 \geq 1/2, x = z = 0\} \cup \{z \geq 1/4, x = y = 0\}),$$

i.e. only the intervals $[-1/2, 1/2]$ of the coordinate axes belong to the surface. See Fig. 1 for a picture of Steiner's surface. The view graph $G(M)$ of M, shown below in Fig. 2, has 88 nodes. For Steiner's surface $\mathcal{B}$ and $G(M)$ are point-symmetric about the origin. Fig. 3 shows a view of M associated to node number 36 (for space reasons, we do not show the views associated with the other nodes of $G(M)$).

Insert Figures 1 to 3 about here

Example 2. The boundary M' of the union of two solid half-spheres defined by

$$(\{2 - (x+1)^2 - y^2 - z^2 \geq 0\} \cup \{2 - (x-1)^2 - y^2 - z^2 \geq 0\}) \cap \{z \geq 0\}$$

has two triple points lying on three circles of double-points (see Fig. 4). The view graph $G(M')$ of M' has 26 nodes and is equal to the view graph of the surface $M \supset M'$ which is given by the zero-set of

$$(2 - (x+1)^2 - y^2 - z^2)(2 - (x-1)^2 - y^2 - z^2)z.$$

These view graphs are symmetric under reflections in the lines $b = 0$ and $c = 0$ and are shown in Fig. 5. Fig. 6 shows the views of M and M' associated to node number 1 in the view graph.

Insert Figures 4 to 6 about here

In general, the graph $G(M')$ is smaller than the graph $G(M)$ — the next example is more typical in this respect.

Example 3 (Rieger [17]). The boundary M' of the solid half-bean defined by

$$\{1 - x^2 - 10(y + x^2)^2 - 10z^2 \geq 0\} \cap \{x \geq 0\}$$

is contained in the zero set M of $(1 - x^2 - 10(y + x^2)^2 - 10z^2)x$, which is the union of a bean-shaped quartic surface and the plane $x = 0$. The view graph $G(M)$ has 51 nodes and $G(M')$ has 27 nodes. Here $G(M)$ is symmetric under reflections in both $b = 0$ and $c = 0$, $G(M')$ is merely symmetric under a reflection in $c = 0$. Figures 7 and 8 show the view graphs $G(M)$ and $G(M')$, for pictures of the view bifurcation sets of M and M' and of the views of M' we refer the reader to [17].

Insert Figures 7 and 8 about here

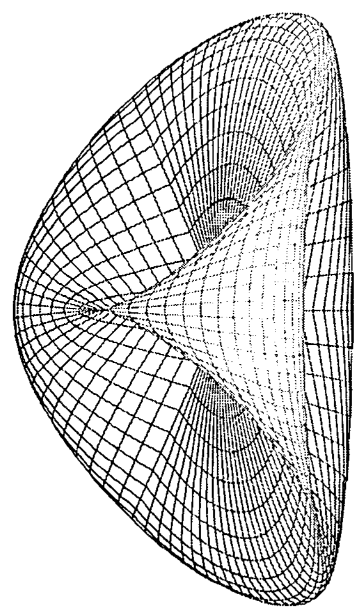

Figure 1: Steiner's Roman surface.

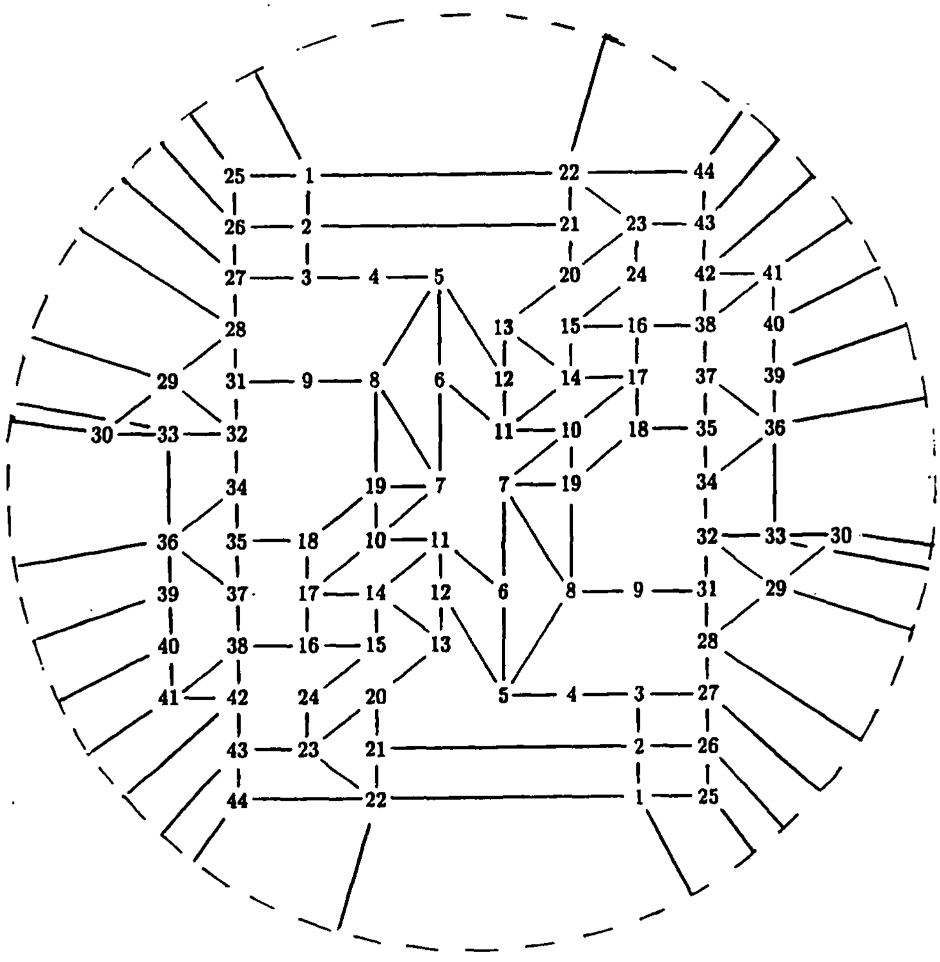

Figure 2: The parallel projection view graph of Steiner's Roman surface: the dashed circle indicates the line at infinity $a = 0$ and edges cutting $a = 0$ connect equally numbered anti-podal nodes.

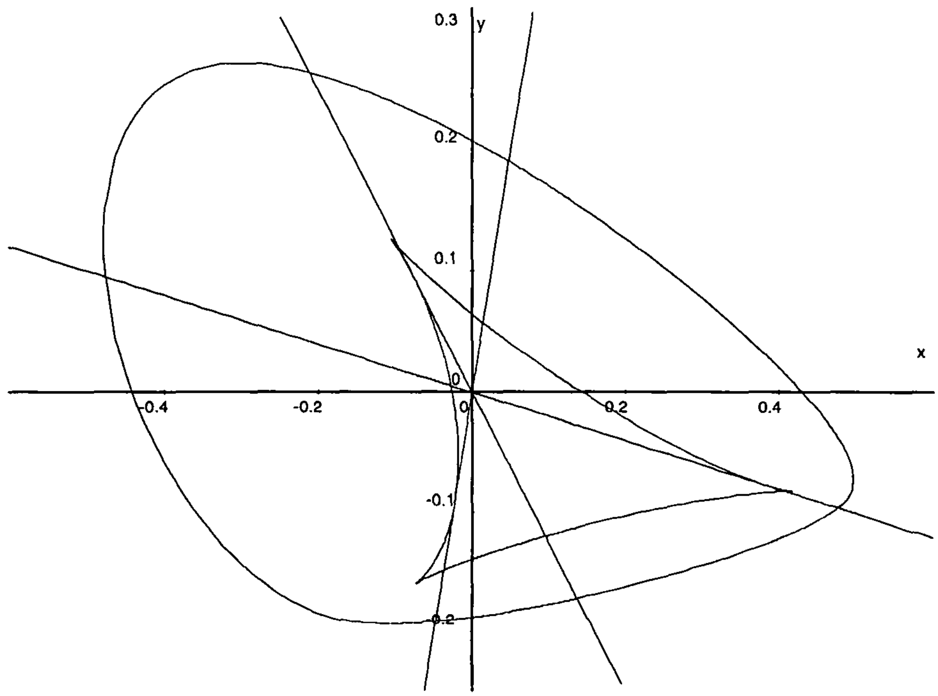

Figure 3: A view of Steiner's Roman surface from node number 36.

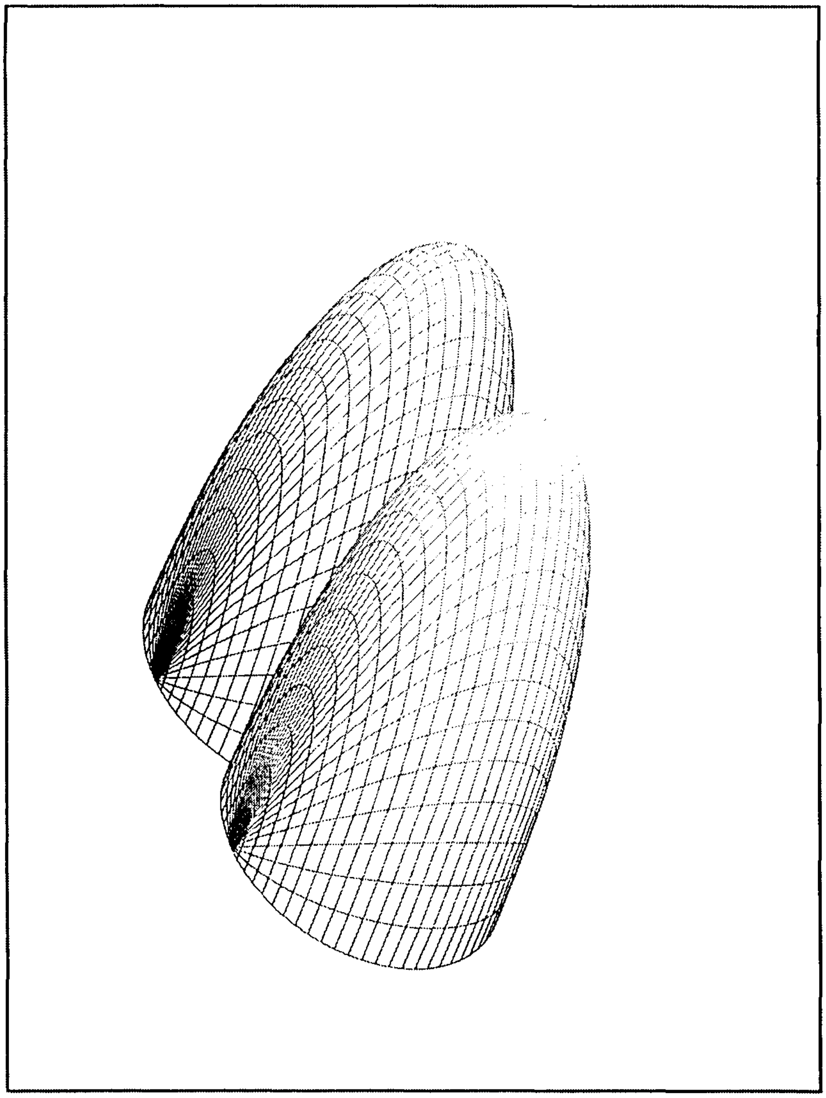

Figure 4: The union of two solid half-spheres.

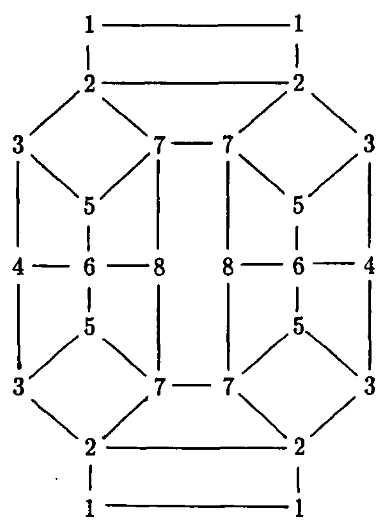

Figure 5: The parallel projection view graph of the bounding surface of two solid half-spheres.

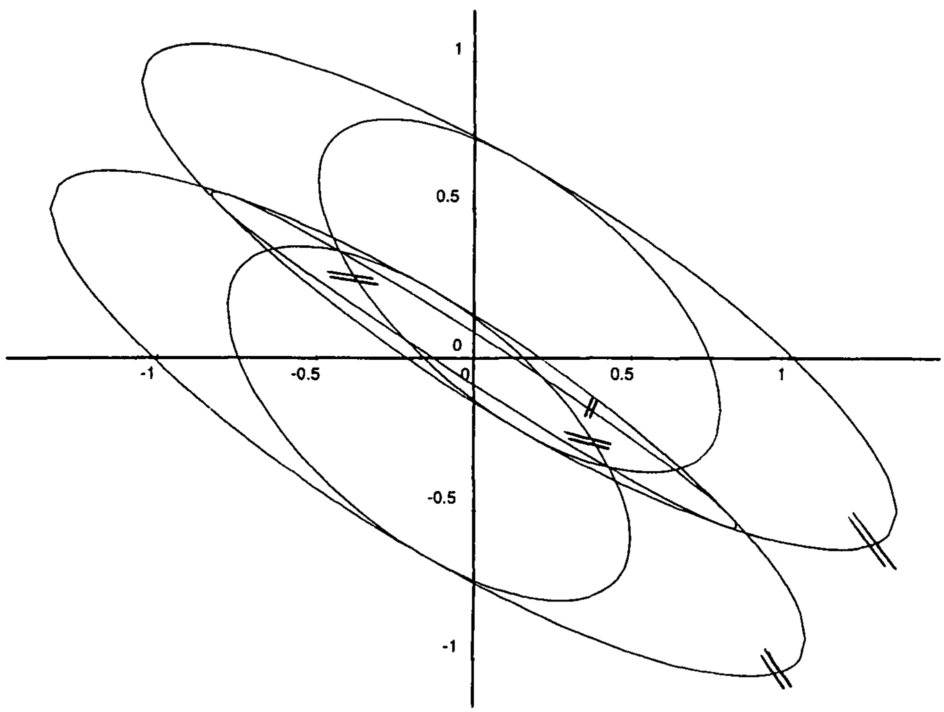

Figure 6: A view of the bounding surface M' of two solid half-spheres from node number 1: crossed-out curves belong to view of $M \setminus M'$.

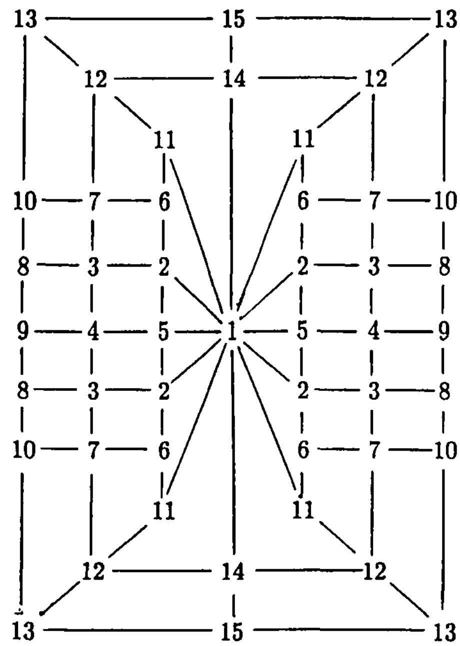

Figure 7: The parallel projection view graph of the union of a bean-shaped surface and a plane.

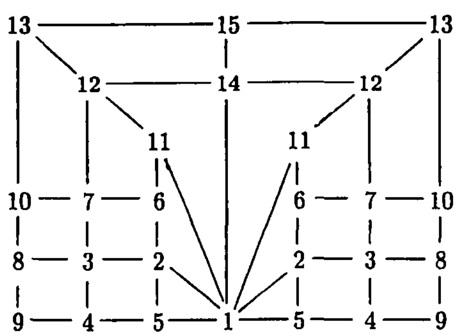

Figure 8: The parallel projection view graph of the bounding surface of a solid half-bean.

3 Defining conditions of view bifurcation set

The 22 types of codimension 1 views (and more degenerate views in the closures of the codimension 1 equivalence classes) can be "recognized" by the conditions in Table 1, which will be defined below. Roughly speaking, these are the conditions for detecting the 19 types of degenerate views of piecewise smooth surfaces with selfintersection curves and triple-points already described in Rieger [17] plus the extra conditions for detecting the 3 types of codimension 1 views of a cross-cap.

Table 1: Unstable view types

$j =$	view type ("name")	intersection multiplicity	additional conditions
1	8 (lip/beak)	([3])	$D_{\text{reg}} = 0$ *
2	7 (swallowtail)	([4])	—
3	5+5+5 (triple-fold crossing)	([2], [2], [2])	—
4	6+5 (cusp fold crossing)	([3], [2])	—
5	5++5 (tacnodal fold crossing)	([2], [2])	$N^{\parallel} = 0$
6	III_2 (semi-lip/beak)	([2, 1])	$C\Sigma^{\parallel} = 0$ *
7	IV (semi-cusp)	([3, 1])	—
8	VII_1 (crease-cusp)	([2, 2])	—
9	1+1+1 (triple-crease crossing)	([1, 1], [1, 1], [1, 1])	—
10	1++1 (tacnodal crease crossing)	([1, 1], [1, 1])	$\pi_C^{\parallel} = 0$
11	6+1 (cusp crease crossing)	([3], [1, 1])	—
12	5++1 (tacnodal fold-crease crossing)	([2], [1, 1])	$\pi_{\Sigma C}^{\parallel} = 0$
13	5+5+1 (fold fold crease crossing)	([2], [2], [1, 1])	—
14	5+1+1 (fold crease crease crossing)	([2], [1, 1], [1, 1])	—
15	5+II (fold semi-fold crossing)	([2], [2, 1])	—
16	II+1 (semi-fold crease crossing)	([2, 1], [1, 1])	—
17	S(5+Y) (fold vertex crossing)	([2], [1, 1, 1])	—
18	S(1+Y) (crease vertex crossing)	([1, 1], [1, 1, 1])	—
19	$\text{S}(\text{Y}_\Sigma)$ (semifold-vertex)	([2, 1, 1])	—
20	$\text{W}_{\Sigma^{1,1}}$ (cross-cap-cusp)	([3])	$\mathbf{W} = \mathbf{0}$
21	W+5 (cross-cap fold crossing)	([2], [2])	$\mathbf{W} = \mathbf{0}$
22	W+1 (cross-cap crease crossing)	([2], [1, 1])	$\mathbf{W} = \mathbf{0}$

Note the following technical difference between the conditions in Table 1 and the ones in [17]. The surfaces $M = \bigcup_{u,i} h_{u,i}^{-1}(0)$ in [17] are unions of *non-singular* algebraic surfaces $h_{u,i}^{-1}(0)$ which intersect transversely along "crease curves" and in isolated triple-points. At a point on the crease (or at a triple point) $p \in M$ it is therefore sufficient to consider the *orders of contact* of a ray of projection l and a pair (or triple) of *non-singular* surfaces intersecting at p. In the present paper, the "component surfaces" $M_{u,i} := h_{u,i}^{-1}(0)$ of M themselves can have singular points, namely selfintersection curves, triple-points and cross-caps.

This has the following consequences.

- At the singular points of $M_{u,i}$ one has to consider the *intersection multiplicity* of a ray of projection and the variety $h_{u,i}^{-1}(0)$ rather than the contact order — at non-singular points both concepts coincide.

- The conditions $D_{\text{reg}} = 0$ and $C\Sigma^{\parallel} = 0$, marked by an $*$ in rows $j = 1$ and 6 of Table 1, differ from the simpler conditions for unions M of regular component surfaces $M_{u,i}$ used in [17]. These conditions define the loci of parabolic points and of conjugate rays to the tangent lines of the selfintersection curves, respectively (see 3.1 below).

- There are now, for example, three types of triple-points of M: the three "local surface branches" can globally lie on one, two or three different component surfaces.

- For component surfaces $M_{u,i}$ with selfintersection curves, triple-points and cross-caps the conditions in Table 1 correspond, in certain cases, to the vanishing of many more equations than the codimension of the solutions of these equations — by contrast: in the generic case studied in [17], where M is the union of non-singular zero sets, the recognition equations define complete intersections.

- Finally, certain types of codimension 1 views are detected by several of the 22 conditions in Table 1 (see 3.2 below). This is essentially due to the fact that we allow non-generic zero-sets M.

3.1 Exact definitions of conditions in Table 1

Let us now define the conditions in Table 1. Let $S(M)$ denote the closure of the selfintersections of M, and let $T(M)$ and $W(M)$ denote the sets of triple-points and of cross-caps (Whitney umbrellas). The conditions for a point p to lie in one of these sets are as follows — recall that $M_{u,i} = h_{u,i}^{-1}(0)$ and note that the conditions on the R.H.S. must hold for some $1 \leq u \leq U$, $1 \leq i \leq I_u$ or for some k-tuple of distinct indices $1 \leq u_m \leq U$, $1 \leq i_m \leq I_{u_m}$ $(1 \leq m \leq k)$:

- $p \in S(M) \iff$ (i) $p \in M_{u_1,i_1} \cap M_{u_2,i_2}$ or (ii) $p \in S(M_{u,i})$;

- $p \in T(M) \iff$ (i) $p \in M_{u_1,i_1} \cap M_{u_2,i_2} \cap M_{u_3,i_3}$ or (ii) $p \in S(M_{u_1,i_1}) \cap M_{u_2,i_2}$ or (iii) $p \in T(M_{u,i})$;

- $p \in W(M) \iff p \in W(M_{u,i})$.

Furthermore, the conditions for p to lie in the selfintersection locus or in the triple-point or cross-cap set of a single component-surface $M_{u,i}$ are as follows:

- $p \in S(M_{u,i}) \iff h_{u,i}(p) = dh_{u,i}(p) = 0$;
- $p \in T(M_{u,i}) \iff p \in S(M_{u,i})$ and $\mathrm{rank}(d^2 h_{u,i})_p = 0$;
- $p \in W(M_{u,i}) \iff p \in S(M_{u,i})$ and $\mathrm{rank}(d^2 h_{u,i})_p = 1$.

The triple-points of $M_{u,i}$ are therefore given by the vanishing of $h_{u,i}$, of the components of $dh_{u,i}$ and of the entries of the Hessian $d^2 h_{u,i}$. The conditions for the cross-caps are slightly more complicated: if one simply replaces the entries of the Hessian by the 2 by 2 minors then the resulting conditions detect cross-caps as well as triple-points. The vanishing ideal of the cross-caps is the union of $(h_{u,i}, dh_{u,i})$ and the ideal quotient $I : J$, where I is generated by the 2 by 2 minors and J by the entries of $d^2 h_{u,i}$. The system of generators of this vanishing ideal is denoted by **W** in Table 1.

The notation in Table 1, then, is as follows. The numbers in the first column enumerate the 22 types of codimension 1 views whose names appear in the second column. The symbols for the view types refer to the classifications of projections in [17, 3]. In these classifications, two projection-maps are considered to be equivalent if one is mapped onto the other by a diffeomorphism of $\mathbb{R}^3$ preserving the projected surface and a diffeomorphism of the retinal plane. For $j = 1$ to 19 we refer to the normal forms for the projection maps given in Proposition 4.1 of [17]. The normal forms for the three remaining view types ($j = 20$ to 22) are as follows (note that they are given by the restriction of a map from $\mathbb{R}^3$ to $\mathbb{R}^2$ to a singular variety or, for bi-local singularities, by a pair of such maps):

- $\mathrm{W}_{\Sigma^{1,1}} := f_1|_{h_1^{-1}(0)}$, where $f_1 := (x_1 + z_1, y_1)$ and $h_1 := y_1^2 - x_1^2 z_1$;
- W+5 $:= \{f_1|_{h_1^{-1}(0)}, f_2|_{h_2^{-1}(0)}\}$, where $f_1 := (x_1, z_1)$, $h_1 := y_1^2 - x_1^2 z_1$, $f_2 := (x_2, x_2 + y_2^2)$ and $h_2 = z_2$; and
- W+1 $:= \{f_1|_{h_1^{-1}(0)}, f_2|_{h_2^{-1}(0)}\}$, where $f_1 := (x_1, z_1)$, $h_1 := y_1^2 - x_1^2 z_1$, $f_2 := (z_2, z_2 \pm x_2 + y_2)$ and $h_2 := x_2 y_2$.

This small list of singularities ($j = 20$ to 22) is a by-product of the much more extensive classification in [3]; a normal form for a cusp of the cross-cap ($j = 20$) has been first obtained in the thesis of Janet West [20].

The unstable view types are "recognized" by the corresponding conditions in columns three and four of the table. Column three of the table describes the intersection multiplicities, denoted by $([i_1], \ldots, [i_k])$, of a ray of projection $l(t) = p + t \cdot L$, where $L = \omega$ (for $\mathcal{V} = \mathbb{P}^2$) or $L = \omega - p$ (for $\mathcal{V} = \mathbb{R}^3 \setminus M$), with M at a k-tuple of points $p_1, \ldots, p_k$. Note that the intersection multiplicity of $l(t)$ and a *single* component surface $M_{u,i} = h_{u,i}^{-1}(0)$ at p_m is given by the order of the function $h_{u,i} \circ l(t)$ at $t_m = l^{-1}(p_m)$. For points $p_m \in T(M)$ [resp. $p_m \in S(M)$] the intersection multiplicity consists of up to 3 [resp. 2] numbers. For example, at a triple-point p_m of M, $[i_m] = [a, b, c]$ either means that p_m lies on

(i) three distinct component surfaces of M whose intersection multiplicities with $l(t)$ are a, b and c, respectively;

(ii) on a pair of component surfaces, one of them having a selfintersection at p_m, and intersection multiplicities $a+b$ and c (or $a+c$, b or a, $b+c$); or

(iii) on a single component surface having intersection multiplicity $a+b+c$.

Column four consists of additional "geometric" conditions imposed on the surface M at p_m. For $j = 1, 5, 6, 10, 12$ and regular component surfaces $M_{u,i}$, these conditions are given in each case by the vanishing of a single polynomial equation supplementing the conditions for the intersection multiplicities in column three (see Section 3.2 of Rieger [17] for explicit expressions). For singular component surfaces, certain modifications of these conditions are necessary which, except for $j = 1$ and 6, are very minor. Geometrically the conditions express the following: $D_{\text{reg}} = 0$ iff p_1 is a parabolic point, $N^{\|} = 0$ iff the normals at p_1, p_2 are parallel, $C\Sigma^{\|} = 0$ iff the directions of the visual ray L and of the tangent line at p_1 are conjugate, $\pi^{\|}_{C} = 0$ iff the projections of the selfintersection curves at p_1 and p_2 are parallel and $\pi^{\|}_{\Sigma C} = 0$ iff the projections of the fold line at p_1 and of the selfintersection curve at p_2 are parallel.

The modifications for $j = 1$ and 6 are as follows. The condition for a parabolic point of a regular zero-set $M_{u,i} = h^{-1}_{u,i}(0)$ used in [17]

$$D_{u,i} := \det \begin{pmatrix} d^2 h_{u,i} & dh_{u,i} \\ (dh_{u,i})^t & 0 \end{pmatrix} = 0$$

vanishes not only on the parabolic set $P(M_{u,i})$ but also on the singular set $S(M_{u,i})$ of a singular component surface $M_{u,i}$. What makes matters worse, the (unwanted) component $S(M_{u,i})$ creates solutions of the recognition equations having higher dimension than the view bifurcation set that we want to compute. We therefore replace the conditions $h_{u,i} = D_{u,i} = 0$ in [17] by a set of generators, denoted by D_{reg}, of the following ideal quotient:

$$I(D_{\text{reg}}) := (h_{u,i}, D_{u,i}) : J_{h_{u,i}},$$

where $J_{h_{u,i}}$ denotes the Jacobian ideal of $h_{u,i}$ (which is generated by the components of $dh_{u,i}$). Note that $I_2 := (h_{u,i}) \cup J_{h_{u,i}}$ is the ideal of functions vanishing on $S(M_{u,i})$ and that such functions also belong to the ideal $I_1 := (h_{u,i}, D_{u,i})$. The quotient $I_1 : I_2$ is contained in the vanishing ideal of the closure of the set $(P(M_{u,i}) \cup S(M_{u,i})) \setminus S(M_{u,i})$ (and, for $I_1 = \sqrt{I_1}$, is equal to it) and is equal to $I(D_{\text{reg}})$ (the latter equality follows from $I_1 : I_2 = \bigcap_{f \in I_2} I_1 : (f)$ and $I_1 : (h_{u,i}) = k[p]$).

For $j = 6$ there are two cases: (i) the double-curve is the intersection of distinct component surfaces and (ii) is the selfintersection of a single component surface. In the case (i) we use the condition given in [17], in the latter case (ii) we replace the expression for the tangent lines of the double-curves of M, which vanishes identically on $S(M_{u,i})$, by the following. Observe that $d^2h_{u,i}$ has corank 1 along $S(M_{u,i})$ except at isolated points: at cross-caps and triple-points of $M_{u,i}$ the corank is 2 and 3, respectively. Let α be a vector along the kernel direction of the gradient map $dh_{u,i} : \mathbb{R}^3 \to \mathbb{R}^3$ and set $m := \langle \alpha, d^2h_{u,i}(L) \rangle$. In the absence of cross-caps and triple-points on $S(M_{u,i})$ one could take m as the new condition for $C\Sigma^{||}$, but any such singular point would create an (unwanted) excess component in the bifurcation set. Hence, in general, we again have to compute the generators of an ideal quotient $I_1 : I_2$, where I_1 is the ideal generated by m, $h_{u,i}$, $J_{h_{u,i}}$ and $d^2h_{u,i}(L,L)$ and where I_2 is generated by the vanishing of the 2 by 2 minors of $d^2h_{u,i}$.

3.2 Hierarchy of dependencies between defining conditions

For surfaces M which are unions of "generic" non-singular zero-sets $M_{u,i}$ each condition in Table 1 will "recognize" exactly one type of codimension 1 view of M. For singular component surfaces $M_{u,i}$ this is no longer true. In this case there exist strictly antisymmetric dependency relations between certain pairs of conditions A and B, denoted by $A \to B$. That is, A holds automatically if B does but not *vice versa.*

In describing these dependencies we denote each condition by its number j in Table 1 and a subscript indicating its intersection multiplicity. Recall that k-local conditions involve k-tuples of intersection multiplicities $([i_1], \dots, [i_k])$, where $[i_l] = [a, b]$ or $[a, b, c]$ at a double- or triple-point of M. We now have to distinguish multiple points of M which arise from the same or from distinct component surfaces: for example, $[a, b, c]$, $[a+b, c]$ and $[a+b+c]$ denote the intersection multiplicities at triple-points of M cut out by 3, 2 and 1 component surface.

Using this notation, the hierarchy of dependencies amongst the defining conditions is given by the following sequences:

$$\begin{array}{ccccc} 2_{([4])} & \leftarrow & 7_{([3+1])} & \leftarrow & 19_{([2+1+1])} \\ & \nwarrow & & \swarrow & \\ & & 8_{([2+2])} & & \end{array},$$

$$3_{([2],[2],[2])} \leftarrow 13_{([2],[2],[1+1])} \leftarrow 14_{([2],[1+1],[1+1])} \leftarrow 9_{([1+1],[1+1],[1+1])},$$

$$\begin{array}{ccccccc} 4_{([3],[2])} & \leftarrow & 11_{([3],[1+1])} & \leftarrow & 16_{([2+1],[1+1])} & \leftarrow & 18_{([1+1+1],[1+1])} \\ & \nwarrow & & \swarrow & & & \\ & & 15_{([2+1],[2])} & \leftarrow & 17_{([1+1+1],[2])} & & \end{array},$$

$$5_{([2],[2])} \leftarrow 12_{([2],[1+1])} \leftarrow 10_{([1+1],[1+1])}$$

and

$$21_{([2],[2])} \leftarrow 22_{([2],[1+1])}.$$

All the codimension ≥ 1 views recognized by the conditions of some given sequence are simultaneously recognized by the corresponding "terminating condition" on the left. One could therefore represent each sequence by its terminating condition. In practice, however, it is often better to first determine the components of the bifurcation set defined by the rightmost conditions in a sequence and then work from right to left. In this way, we can factor out the components already recognized by other conditions in the sequence.

4 The time complexity of computing view graphs

Applying the recognition conditions in Table 1 to algebraic surfaces M yields the defining equations of algebraic sets $\tilde{\mathcal{B}}_{j,r} \subset \mathcal{V} \times \mathbb{R}^{k_j+2}$, where $k_j = 1$, 2 or 3 for local, bi-local or tri-local singularities of the projection (there are no $\geq$ 4-local codimension 1 views, and the views of higher codimension are all "recognized" by the conditions in Table 1). We can think of the union $\tilde{\mathcal{B}}$ of these algebraic sets as being embedded in $\mathcal{V} \times \mathbb{R}^5$, note that $\mathbb{R}^{k_j+2}$ has at most dimension 5. The restriction of the projection $\pi : \mathcal{V} \times \mathbb{R}^5 \to \mathcal{V}$ to $\tilde{\mathcal{B}}$ yields the view bifurcation set $\mathcal{B}$ of M, which is a closed semi-algebraic subset of $\mathcal{V}$. The view bifurcation set $\mathcal{B}'$ of $M' \subset M$ is a semi-algebraic subset of $\mathcal{B}$ — the "branches" of $\mathcal{B} \setminus \mathcal{B}'$ consist of centres or directions of projection $\omega \in \mathcal{V}$ which yield unstable views of "pieces" of M which do not belong to the boundary M'. On the other hand, $\mathcal{B}$ is a subset of the closed real algebraic set $\hat{\mathcal{B}}$ which is the projection of the complexification of $\tilde{\mathcal{B}}$ into $\mathcal{V}$. This set-up is summarized in the following diagram:

$$\begin{array}{ccccc} & & \tilde{\mathcal{B}} & \subset & \mathcal{V} \times \mathbb{R}^5 \\ & & \downarrow \pi & & \\ \mathcal{B}' & \subset & \mathcal{B} & \subset & \hat{\mathcal{B}} \subset \mathcal{V} \end{array}$$

The following properties of the view bifurcation sets $\mathcal{B}$ and $\mathcal{B}'$ of surfaces M and M' are essential in the algorithms described below. For surfaces with transverse selfintersection curves and isolated triple points, these properties have been established in Rieger [17]. Looking at the proofs in [17] we see that these properties still hold for surfaces with additional cross-caps. Let $\mathcal{P}^\infty$ [resp. $\mathcal{P}^{\mathrm{alg}}$] denote the class of infinitely differentiable [resp. algebraic] surfaces with transverse selfintersection curves and isolated triple-points and cross-caps (where, of course, $\mathcal{P}^{\mathrm{alg}} \subset \mathcal{P}^\infty$), then we have the following:

1. $\mathcal{B}$ (and hence $\mathcal{B}'$) has positive codimension in $\mathcal{V}$ for any $M \in \mathcal{P}^\infty$.

2. $\mathcal{B}$ and $\mathcal{B}'$ have no "free boundaries" in $\mathcal{V}$ of codimension 2 for any $M, M' \in \mathcal{P}^{\text{alg}}$. In particular, the boundary of any "component" $\mathcal{B}_{j,r}$ [or $\mathcal{B}'_{j,r}$] of $\mathcal{B}$ [or $\mathcal{B}'$] always lies in some other component $\mathcal{B}_{j',r'}$ [or $\mathcal{B}'_{j',r'}$].

3. There are $O(n^K)$, $K = 6$ or 3, components $\mathcal{B}_{j,r}$. But, for any $M \in \mathcal{P}^{\text{alg}}$, the boundary of $\mathcal{B}_{j,r}$ will lie in $O(1)$ other components which are known *a priori*.

4. Likewise, there are $O(n^K)$, $K = 6$ or 3, components $\mathcal{B}'_{j,r}$. But, for any $M' \in \mathcal{P}^{\text{alg}}$, the boundary of $\mathcal{B}'_{j,r}$ will lie in $O(n)$ other components which are known *a priori*.

Property 1 ensures that the view graph of M (and hence of M') always exists. Property 2 implies that we can "recover" the semi-algebraic sets $\mathcal{B}$ and $\mathcal{B}'$ from the algebraic set $\hat{\mathcal{B}}$ (recall the diagram above) by computing certain cell-decompositions in $\mathcal{V}$ — by contrast, for general semi-algebraic sets $\mathcal{B}$ arising as projections of some algebraic set $\tilde{\mathcal{B}} \subset \mathcal{V} \times \mathbb{R}^5$ one has to compute cell-decompositions of the total space $\mathcal{V} \times \mathbb{R}^5$. Finally, properties 3 [and 4] yield relatively coarse "stratifications" of $\hat{\mathcal{B}}$ [and $\mathcal{B}$] — consisting of $O(n^K)$ [or of $O(n^{K+\dim \mathcal{V}-1})$], as opposed to $O(n^{K \dim \mathcal{V}})$, "branches" — from which $\mathcal{B}$ [and $\mathcal{B}'$] can be constructed by deleting those branches that lie in $\hat{\mathcal{B}} \setminus \mathcal{B}$ [or in $\mathcal{B} \setminus \mathcal{B}'$]. Note that there are $O(n^K)$ components $\hat{\mathcal{B}}_{j,r}$ and $\mathcal{B}_{j,r}$: cutting each $\hat{\mathcal{B}}_{j,r}$ with at most $O(1)$ other components yields $O(n^K)$ branches of $\hat{\mathcal{B}}$; on the other hand, cutting a component $\mathcal{B}_{j,r} \subset \mathcal{V}$ with at most $O(n)$ other components yields $O(n^{\dim \mathcal{V}-1})$ branches of $\mathcal{B}_{j,r}$ and hence a total number of $O(n^{K+\dim \mathcal{V}-1})$ branches of $\mathcal{B}$.

We can now give a rough outline of our algorithms. The view graph algorithms for M [for M'] in the present work, and the ones in Rieger [17], consist of the following very high-level steps 1-3 [1-4]:

1. Compute the (radicals of the) elimination ideals $I(\tilde{\mathcal{B}}_{j,r}) \cap \mathbb{Q}[\omega]$, for $1 \leq j \leq 22$ (or 19 if no cross-caps are present) and $1 \leq r \leq c(j) \sim O(n^K)$. Result: the defining polynomials of real algebraic sets $\hat{\mathcal{B}}_{j,r}$.

2. Determine the connected components of $\mathcal{V} \setminus \hat{\mathcal{B}}$, where $\hat{\mathcal{B}} = \bigcup_{j,r} \hat{\mathcal{B}}_{j,r}$.

3. Decompose $\hat{\mathcal{B}}$ into $O(n^K)$ branches and remove the branches that lie in $\hat{\mathcal{B}} \setminus \mathcal{B}$. Result: the view graph $G(M)$.

4. Decompose $\mathcal{B}$ into $O(n^{K+\dim \mathcal{V}-1})$ branches and remove the branches that lie in $\mathcal{B} \setminus \mathcal{B}'$. Result: the view graph $G(M')$.

Let us now consider some of these steps in more detail. We shall suppress most of the algebraic details of these computations, which are quite technical and have been described in detail in [17], and concentrate on the combinatorial aspects. The combinatorial parts are the only places where the algorithms in [17] differ from the ones in the present paper.

The elimination step 1 uses standard tools from computational algebra and clearly requires $O(n^K) \cdot \mathcal{P}(d, L)$ time (note that the defining polynomials of $\mathcal{B}_{j,r}$ have a constant number of variables and their degree and coefficient size is $O(d)$ and $O(L)$, respectively).

Step 2 is based on a sign-invariant decomposition (of certain planar sections) of $\mathcal{V}$ w.r.t. the defining polynomials of the $O(n^K)$ algebraic sets $\hat{\mathcal{B}}_{j,r}$. In Rieger [17] Step 2 is based on one (for $\mathcal{V} = \mathbb{P}^2$) or several (for $\mathcal{V} = \mathbb{R}^3$) cylindrical algebraic decompositions of $\mathbb{R}^2$. Replacing the cylindrical algebraic decomposition (which is the computational bottle-neck in step 2) by a semi-cylindrical decomposition (as defined by Chazelle *at al.* [4]) of $\mathbb{R}^2$ or $\mathbb{R}^3$ still allows us to determine the regions of $\mathcal{V} \setminus \hat{\mathcal{B}}$ in the same way as in [17]. Using the deterministic algorithm described in [4], step 2 requires $O(n^{K(2\dim\mathcal{V}+1)}) \cdot \mathcal{P}(d, L)$ time, and with the randomized algorithm from [4] the expected running time of step 2 becomes $O(n^{K\dim\mathcal{V}+\epsilon}) \cdot \mathcal{P}(d, L)$.

In step 3 we pick one sample point $\omega' \in S$ in each of the $O(n^K)$ branches of $\hat{\mathcal{B}}$. For $\omega' \in \hat{\mathcal{B}}_{j,r}$ we check whether the specialization of the ideal $I(\tilde{\mathcal{B}}_{j,r})$ to $\omega = \omega'$ has a positive number of real roots, if not we remove S. Step 3 can be carried out in $O(n^K) \cdot \mathcal{P}(d, L)$ time.

Likewise, in step 4 we pick one sample point $\omega' \in S$ in each of the $O(n^{K+\dim\mathcal{V}-1})$ branches of $\mathcal{B}$. However, the decision procedure described in [17] for checking whether ω' belongs to $\mathcal{B}$ has to count the number of real roots of up to $n^{3(U-1)}$ systems of polynomial equations and inequalities. If none of these systems has a positive number of real roots then the branch S has to be removed. The algebraic complexity (fixing n or the number of unions U in the defining formula of the solid) is polynomial, but the combinatorial complexity is not. Below, we shall describe a modified version of step 4 having a polynomial running time.

First, we need the following notation. If $\bigcup_{u=1}^{U} \bigcap_{i=1}^{I_u} \{p \in \mathbb{R}^3 : h_{u,i}(p) \geq 0\}$ is the defining formula of the solid with boundary M' then we set

$$H := \{h_{1,1}, \ldots, h_{U,I_U}\} \subset \mathbb{Q}[p].$$

Recall that a connected subset Y of $\mathbb{R}^3$ is said to be H-invariant if all polynomials in H are sign-invariant on Y (i.e. are either strictly positive, negative or vanish). Denote by S_{reg} the regular part of a branch S of $\mathcal{B}$. The branches S are defined as follows: if

$$X_{j,r} := \mathcal{B}_{j,r} \setminus \{\text{selfintersections of } \mathcal{B}_{j,r}\},$$

then a branch is a connected component of

$$X_{j,r} \cap \bigcup_{(j',r') \in J \times R} \mathcal{B}_{j',r'}$$

of dimension $\dim \mathcal{V} - 1$. The index sets $J \times R$ are defined in Section 5.4 of Rieger [17] and have at most $O(n)$ elements (j', r'). There are $O(n^K)$ components $\mathcal{B}_{j,r}$ of $\mathcal{B}$ each having $O(n^{\dim \mathcal{V}-1})$ branches, hence there are at most $O(n^{K+\dim \mathcal{V}-1})$ branches S of $\mathcal{B}$.

The modified procedure for removing from $\mathcal{B}$ those branches that do not belong to $\mathcal{B}'$, then, consists of the following:

> **begin** {Step 4}
>
> Decompose $\mathbb{R}^3$ into H-invariant cells C and label the cells $C \subset M'$ as boundary-cells;
>
> **for** each branch $S \subset \mathcal{B}$ **do**
>
> > Pick some algebraic representative $\omega' \in S_{\text{reg}}$;
> > $b := FALSE$;
> > **for** each root $(p^m; \lambda_1^m, \ldots, \lambda_{k_j-1}^m)$, $1 \leq m \leq \rho$, "over" ω' **do**
> >
> > > Determine the real algebraic number coordinates of the k_j-tuple of points p^m, $l(\lambda_1^m), \ldots, l(\lambda_{k_j-1}^m)$;
> > > Determine the cells $C_1, \ldots, C_{k_j}$ containing the points above;
> > > **if** all these cells are boundary-cells **then** $b := TRUE$
> >
> > **od**
> >
> > **if** $b = FALSE$ **then** delete S
>
> **od**
>
> **end**

Using the deterministic (resp. randomized) semi-cylindrical stratification algorithm of Chazelle *et al.* [4] yields $O(n^{3+\epsilon}) \cdot \mathcal{P}(d)$ H-invariant cells in $O(n^7) \cdot \mathcal{P}(d, L)$ time (resp. $O(n^{3+\epsilon}) \cdot \mathcal{P}(d, L)$ expected time). For each H-invariant cell C of dimension ≤ 2 we pick a sample point p (with algebraic number coordinates) and check whether $p \in M'$, as follows. Let $\bar{h}_1, \ldots, \bar{h}_t \in H$ denote the polynomials vanishing on the cell C and define $H_u := \{h_{u,1}, \ldots, h_{u,I_u}\} \subset H$, $1 \leq u \leq U$. Also set $\bar{H} := \{\bar{h}_1, \ldots, \bar{h}_t\}$,

$$H^+ := \bigcup_{H_u \cap \bar{H} \neq \emptyset} (H_u \setminus H_u \cap \bar{H})$$

and denote the H_u whose intersection with $\bar{H}$ is empty by H_u^-. We label C as a boundary-cell if (i) for all $h \in H^+$ the algebraic number $h(p)$ has positive sign and (ii) for each H_u^- there exists at least one $h' \in H_u^-$ for which $h'(p)$ has negative sign. Checking whether C belongs to the boundary M' therefore requires $O(n) \cdot \mathcal{P}(d, L)$ time for each cell (recall that $|H| = n$ and note that the signs of these algebraic numbers can be determined in polynomial time). The first line of the above procedure therefore requires $O(n^7) \cdot \mathcal{P}(d, L)$ deterministic or $O(n^{4+\epsilon}) \cdot \mathcal{P}(d, L)$ expected time.

Next, note that the outer loop will be executed at most $O(n^{K+\dim \mathcal{V}-1}) \cdot \mathcal{P}(d)$ times and the inner loop $\mathcal{P}(d)$ times. Recall that the points $l(\lambda_i)$ are either given by $p + \lambda_i \cdot \omega'$ (for $\mathcal{V} = \mathbb{P}^2$) or otherwise by $p + \lambda_i \cdot (\omega' - p)$, where λ_i and the coordinates of p and ω' are algebraic numbers. We can compute the minimal polynomials of the coordinates of the $l(\lambda_i)$ from the minimal polynomials of these algebraic numbers in $\mathcal{P}(d)$ time (using the usual algorithms for adding and multiplying algebraic numbers encoded by their minimal polynomials, see e.g. the article by Loos [11] or Chapter 8.5 of the book by Mishra [13]). Finally note that, again using the algorithms in Chazelle *et al.* [4], we can determine the H-invariant cell containing some given point p or $l(\lambda_i)$ in $O(\log n) \cdot \mathcal{P}(d, L)$ time (note that the algebraic number coordinates of these points and the H-invariant cells are defined by polynomials of size $\mathcal{P}(d, L)$).

Summing up, we see that the above procedure for removing the branches of $\mathcal{B}$ that do not belong to $\mathcal{B}'$ requires $O(n^{K+\dim \mathcal{V}-1} \log n + n^7)$ time using the deterministic algorithm in [4] for computing a H-invariant stratification of 3-space or $O(n^{K+\dim \mathcal{V}-1} \log n)$ expected time using the randomized version.

Looking at the above bounds for the running times of the three (resp. four) steps of the view graph algorithm for M (resp. M') we see that, from a combinatorial point of view, step 2 dominates the asymptotic time complexity. The bounds for the running times of the deterministic and randomized view graph algorithms stated in the introduction now follow.

5 The size of view graphs

It is shown below that the upper bound in [17] of $O(n^{K \dim \mathcal{V}} d^{6 \dim \mathcal{V}})$ for the number of nodes in the view graph of a piecewise smooth surface, which is the union of n non-singular surfaces of degree $\leq d$ intersecting in double-curves and triple-points, remains valid for the surfaces M (and hence M') studied in the present paper. The double-curves and triple-points of the surfaces studied in [17] are cut-out by pairs and triples of regular component surfaces. On the other hand, the surfaces M are unions of singular component surfaces with double-curves, triple-points and cross-caps.

Note that this estimate for the number of nodes $|V|$ in the view graph $G = (V, E)$ yields an upper bound for the size of G, because $|E| \sim O(|V|)$.

This special property of view graphs — which is in contrast to complete graphs having $O(|V|^2)$ edges — boils down to the fact that the edges in E are top-dimensional branches of the bifurcation set $\mathcal{B}$ and there are at most $O(n^{K \dim \mathcal{V}} d^{6 \dim \mathcal{V}}) = O(|V|)$ such branches.

One checks that the degrees of the view bifurcation sets $\mathcal{B}$ of M and of the surfaces studied in [17] are of the same order, namely $O(n^K d^6)$, which yields the desired bound. First, one observes that the presence of double-curves and triple-points on singular component surfaces of M does not change the degree bounds stated in [17] for the components $\mathcal{B}_j$, $1 \leq j \leq 19$, of the view bifurcation set. (Essentially this follows from the fact that the degrees of the double-curve and of the triple-point set of a component surface of M are of order d^2 and d^3, despite the fact that many equations are required to define these sets.)

The result then follows from the following degree bounds for the additional components $\mathcal{B}_{20}$ to $\mathcal{B}_{22}$ of the view bifurcation set of surfaces with cross-caps.

5.1 PROPOSITION. *Let $M = \bigcup_{i=1}^{n} M_i$, $d = \sup_i \deg M_i$, be a piecewise smooth algebraic surface with transverse double-curves and isolated cross-caps and triple-points. Then the degree orders of the components $\mathcal{B}_{20}$, $\mathcal{B}_{21}$ and $\mathcal{B}_{22}$ of the view bifurcation set are nd^3, n^2d^5 and n^3d^5, respectively.*

PROOF. The components of the bifurcation set are unions $\mathcal{B}_j = \bigcup_{r=1}^{c(j)} \mathcal{B}_{j,r}$ whose combinatorial complexity is given by

$$c(j) = \binom{n}{m(j)} \sim O(n^{m(j)}),$$

where $m(j)$ is the maximal number of distinct component surfaces M_i involved in a view singularity of type j. Clearly, $m(j) \leq 1$, 2 and 3 for $j = 20$, 21 and 22, respectively.

The remaining task, then, is to estimate the degree (as a function of d) of a single subset $\mathcal{B}_{j,r}$ ($j = 20, 21, 22$). In doing this, the following lemma will be useful

5.2 LEMMA. *Let $T(M_i)$ and $W(M_i)$ denote the sets of triple-points and cross-caps (Whitney umbrellas) of a degree d surface M_i with transverse double-curves and isolated triple-points and cross-caps. Then the following (asymptotically tight) upper bound holds:*

$$|T(M_i)| + |W(M_i)| \sim \Theta(d^3).$$

PROOF. Consider the following stratification of M_i: take as 0-dimensional strata the triple-points and cross-caps, as 1-dimensional strata the arcs of double-curves in the complement of the 0-strata and as 2-dimensional strata the faces of M_i cut out by the closure of the double-point arcs. Denote the number of i-dimensional strata by e_i. It is convenient to distinguish bounded arcs, which contain at least one triple-point or cross-cap, from unbounded ones — denote the number of bounded and unbounded arcs by e_1^b and e_1^u, repectively.

Now $e_1^b \leq 6e_0$, where equality holds in the worst case where each bounded arc has infinite length and terminates in a triple-point. For the unbounded arcs we note that each contains 4 faces with, in turn, either 1 or 2 unbounded arcs in their closure (note, any face with ≥ 3 arcs in its closure must be cut out by bounded arcs), hence $e_1^u \leq e_2/2$. Therefore:

$$e_1 = e_1^b + e_1^u \sim O(e_0 + e_2).$$

On the other hand, we have that

$$e_0 - e_1 + e_2 = \chi(M_i) = \sum_j (-1)^j b_j(M_i) \leq \sum_j b_j(M_i),$$

and the sum of the Betti numbers b_j is, by a result of Milnor [12], at most $O(d^3)$. Hence, in particular,

$$e_0 = |T(M_i)| + |W(M_i)| \sim O(d^3).$$

Finally, the zero-set of $\prod_{j=1}^3 \prod_{k=1}^d (x_j - k)$ in $\mathbb{R}^3$ has degree $3d$ and d^3 triple-points. This example shows that the degree of our bound is exact. □

Proof of Proposition (Conclusion). Let $l(t) = p + t \cdot L$, where $L = \omega$ (for $\mathcal{V} = \mathbb{P}^2$) or $L = \omega - p$ (for $\mathcal{V} = \mathbb{R}^3 \setminus M$), be a ray through a cross-cap $p \in M_i = h_i^{-1}(0)$. The intersection multiplicity of l and M_i at p is given by the order of $h_i \circ l(t)$ at $t = 0$. One checks that, at a cross-cap p, this order is at least two.

So, for ω to lie in $\mathcal{B}_{20,i}$, p has to be a cross-cap of M_i and l an asymptotic line. The latter means that $h_i \circ l(t)$ has order ≥ 3 at $t = 0$ which — at a cross-cap p where the order is automatically ≥ 2 — corresponds to the single condition

$$\left.\frac{d^2(h_i \circ l(t))}{dt^2}\right|_{t=0} = d^2 h_i\big|_p(L, L) = 0.$$

Hence

$$\mathcal{B}_{20,i} = \bigcup_{p \in W(M_i)} \{\omega \in \mathcal{V} : d^2 h_i\big|_p(L, L) = 0\}$$

is the union of $|W(M_i)| \sim O(d^3)$ quadrics in $\mathcal{V}$, which implies that $\deg \mathcal{B}_{20,i} \sim O(d^3)$.

For ω to lie in $\mathcal{B}_{21,r}$ there must exist — not necessarily distinct — surfaces M_i, $M_j \in \{M_1, \dots, M_n\}$ such that $p \in M_i$ is a cross-cap and the ray $l(t) = p + t \cdot L$ has at least 2-point contact with M_j at $l(\lambda)$, $\lambda \neq 0$. This boils down to the following (where cl$\{\cdot\}$ denotes the closure)

$$\mathcal{B}_{21,r} = \bigcup_{p \in W(M_i)} \mathrm{cl}\{\omega \in \mathcal{V} : \exists \lambda \neq 0 : h_j(p + \lambda \cdot L) = dh_j\big|_{p+\lambda \cdot L}(L) = 0\}.$$

The degrees of h_j, $dh_j(L) \in k[\lambda; \omega]$ are $O(d)$ — hence the union $\mathcal{B}_{21,r}$ of these $O(d^3)$ algebraic sets has degree at most $O(d^5)$.

A similar argument shows that the set

$$\mathcal{B}_{22,r} = \bigcup_{p \in W(M_i)} \mathrm{cl}\{\omega \in \mathcal{V} : \exists \lambda \neq 0 : h_j(p + \lambda \cdot L) = h_k(p + \lambda \cdot L) = 0\},$$

corresponding to bi-local projections in which a cross-cap and a selfintersection curve appear superimposed, has degree at most $O(d^5)$. □

6 Additional remarks and open problems

For the special case of *opaque views* of "polyhedral terrains" — that is for graphs of piecewise linear functions in two variables having $O(n)$ edges, vertices and faces — there is a bound for the number of nodes in the view graph, namely $O(n^{3\dim \mathcal{V}-1+\epsilon})$, which is by about a factor of n sharper than the one for general polyhedra (see Theorems 8.31 and 8.33 of Sharir and Agarwal [18] and their paper [1]). Note that the size of the view graph for opaque views is less than or equal to the size of the view graph for transparent views (actually, the former can be obtained from the latter by contracting certain edges). De Berg *et al.* [5] have obtained a lower bound of $\Omega(n^{3\dim \mathcal{V}-1}\alpha(n))$ (α being the functional inverse of Ackermann's function) for the number of distinct opaque views of polyhedral terrains, which means that the above upper bound is almost tight (actually, for parallel projection there is a slightly sharper upper bound of $O(n^5 2^{c\sqrt{\log n}})$ due to de Berg *et al.* [5] and Halperin and Sharir [8]). Replacing the piecewise linear functions by piecewise algebraic ones — but maintaining the restriction that the surfaces are function graphs with $O(n)$ edges, vertices and faces — one can combine the combinatorial results in [18] with our algebraic estimates to obtain an $O(n^{3\dim \mathcal{V}-1+\epsilon}d^{6\dim \mathcal{V}})$ bound for the nodes in the view graphs of opaque views of such surfaces. In principal, one could compute the view graph of opaque views of "polyhedral terrains", and more generally of semi-algebraic function graphs, by supplementing our algorithm by a postprocessing step which merges those nodes in the view graph which correspond to distinct transparent but not to distinct opaque views. The resulting algorithm would have the same asymptotic complexity as the algorithm described in the present paper and hence would not be optimal for function graphs (terrains), its time complexity would be by about a factor of n too high.

Petitjean [14] has obtained exact formulas, as opposed to asymptotic bounds, for the degrees of the complexified view bifurcation sets of algebraic surfaces with double-curves and triple-points using techniques from enumerative geometry. These techniques also apply to surfaces with additional cross-caps. The transition from the complex to the real case is, however, very difficult: there exist no non-trivial lower bounds for the number of connected regions in the complement of $\mathcal{B}$ in terms of d.

For the case of parallel projections of polyhedra, Gigus *et al.* [7] have presented an algorithm that determines the view graph $G = (V, E)$, as well as an explicit description of a view for each node, in $O(|V| \log |V| + n^4 \log n)$ time. The design of such an "output sensitive" algorithm, whose running time depends on the actual size $|V|$ of the view graph, for curved (piecewise smooth) algebraic surfaces is highly desirable but seems to be a formidable task. Note that, for (piecewise smooth) algebraic surfaces M of high degree d but small actual size of $|V|$, the real bifurcation set $\mathcal{B}$ of M cuts out only a small number of regions, but the complexification of $\mathcal{B}$ is nevertheless an algebraic variety of very high degree (namely $O(n^K d^6)$). It just happens that the high-degree polynomials at the very end of the view graph computation have relatively few *real* roots — but the polynomials up to that point could have lots of real roots. Obtaining an output-sensitive algorithm in such a situation would be of general interest in computational mathematics and would be a major breakthrough in computational real algebraic geometry.

For polyhedra some recent works have considered the computation of finite-resolution view graphs (see Shimshoni and Ponce [19]), which take into account that details of a view that are smaller than some size-threshold cannot be detected by cameras of limited spatial resolution. An extension of this work to curved semi-algebraic surfaces seems possible but not very enlightening. Contrary to the claim by some authors (see the paper [6] based on a panel discussion on aspect graphs in computer vision) that there exist no mathematical techniques for computing finite-resolution view graphs of curved surfaces, one should note that the finite-resolution partition of view space of (piecewise smooth) algebraic surfaces can be defined by Tarski sentences and hence can be effectively computed. (Recall that Tarski sentences are Boolean formulas with quantifiers and with polynomial (in-)equalities as predicates.) However, the boundaries of the regions of this partition are no longer view bifurcation sets (with well-understood topological properties) but rather unrestricted semi-algebraic sets. The exact computation of this finite-resolution partition might therefore be prohibitive, perhaps one has to be satisfied with approximations.

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