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DIFFERENTIAL SYSTEMS**

JAUME LLIBRE
REGILENE OLIVEIRA

Nº 444

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NOTAS DO ICMC

SÉRIE MATEMÁTICA

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JAUME LLIBRE AND REGILENE OLIVEIRA

ABSTRACT. The authors in [7] shown numerically the existence of a limit cycle surrounding the unstable node that system (1) has in the positive quadrant for specific values of the parameters. System (1) is one of the Belousov–Zabotinsky dynamical models. The objective of this paper is to prove that system (1), when in the positive quadrant Q has an unstable node or focus, has at least one limit cycle and, when $f = 2/3$, $q = \epsilon^2/2$ and $\epsilon > 0$ sufficiently small this limit cycle is unique.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

One of the most studied chemical oscillation systems is the Belousov-Zhabotinsky (BZ) reaction, which was elucidated by 20 chemical equations to explain the reaction mechanism and was simplified to three-variable differential equations. Many works about physical and chemical mechanism, numerical simulation and experimental research on BZ reaction appeared, see for instance [1, 3, 4, 9] . After the 1990s, the slow-fast oscillation was found in many chemical reactions the reason was that the catalyst could make the reaction process involve in different time scales with large gap. But most of the researches on the slow-fast oscillation in chemical reaction were limited to numerical simulation and experimental investigation.

One of the BZ dynamical models given as a slow-fast system is the following

$$(1) \qquad \epsilon \dot{x} = x(1 - x) + \frac{f(g - x)}{q + x} y, \qquad \dot{y} = x - y,$$

where the parameters f and q are positive and $\epsilon > 0$ is sufficiently small. As usual the dot denotes derivative with respect to the time t.

In [7] the authors shown numerically the existence of a limit cycle surrounding the unstable node that this system has in the positive quadrant $Q = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$, for the values of the parameters $f = 2/3$, $\epsilon = 1/25$ and $q = \epsilon^2/3$.

The objective of this paper is to prove that system (1), when in the positive quadrant Q has an unstable node or focus, then first it has at least one limit cycle, and second, that when $f = 2/3$, $q = \epsilon^2/2$ and $\epsilon > 0$ sufficiently small this limit cycle is unique. More precisely, our main results are the following two theorems:

Theorem 1. *The BZ differential system* (1) *when for f, q and $\epsilon > 0$ has an unstable focus or node in the first quadrant Q, then it has at least one limit cycle in Q.*

Theorem 2. *The BZ differential system* (1) *for $f = 2/3$, $q = \epsilon^2/3$ and $\epsilon > 0$ sufficiently small has a unique stable limit cycle in the quadrant Q, which is the unique limit cycle of the system.*

Note that Theorem 2 provides an analytic proof of the existence and uniqueness of the limit cycle detected numerically by Leonov and Kutnetsov in their example 1.3 of [7].

Theorems 1 and 2 are proved in sections 2 and 3, respectively.

2. Proof of Theorem 1

With the change of time $t = \epsilon\tau$ system (1) becomes

$$(2) \qquad x' = x(1-x) + \frac{f(g-x)}{q+x}y, \qquad y' = \epsilon(x-y),$$

where the prime denotes the derivative with respect to the new time τ. Now doing other change of time $d\tau = (q+x)ds$ system (2) can be written as

$$(3) \qquad \dot{x} = x(1-x)(q+x) + f(g-x)y, \qquad \dot{y} = \epsilon(x-y)(q+x),$$

where the dot denotes derivative with respect the times s.

Since $q + x > 0$ when $x \geq 0$, the orbits of system (2) and (3) in the quadrant Q are the same, they only are run in different times. So to prove Theorem 1 it is sufficient to show that system (3) has a limit cycle in the quadrant Q.

Due to the fact that system (3) is polynomial we can compactify it in the Poincaré disc $\mathbb{D}$. This disk is the closed disc of radius one and center in the origin of coordinates. The plane $\mathbb{R}^2$ where is defined system (3) is diffeomorphic to the interior of $\mathbb{D}$, and its boundary $\mathbb{S}^1$ corresponds to the infinity of $\mathbb{R}^2$. System (3) is defined in the interior of $\mathbb{D}$. System (3) through a diffeomorphism is defined in the interior of $\mathbb{D}$ and can be extended to the closed disc $\mathbb{D}$ in a unique analytic way in such manner that $\mathbb{S}^1$ boundary of $\mathbb{D}$ is invariant by the extended flow i.e. if an orbit of the extend flow has a point in $\mathbb{S}^1$ the whole orbit is contained in $\mathbb{S}^1$. This extension is called the Poincaré compactification, for more details see Chapter 5 of [2].
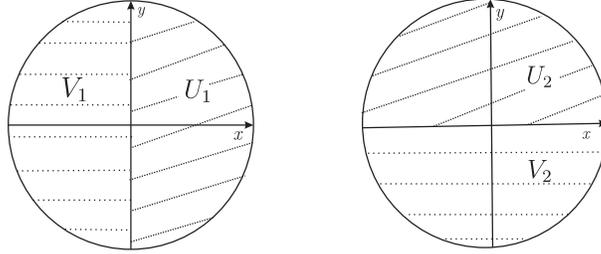

FIGURE 1. The local charts U_i and V_i, for $i = 1, 2$ on the Poincaré disc $\mathbb{D}$.

In order to work with the Poincaré disc we need four local charts. Let

$$U_1 = \{(x,y) \in \mathbb{R}^2 : x > 0\}, \qquad U_2 = \{(x,y) \in \mathbb{R}^2 : x < 0\},$$
$$V_1 = \{(x,y) \in \mathbb{R}^2 : y > 0\}, \qquad V_2 = \{(x,y) \in \mathbb{R}^2 : y < 0\}.$$

We define $\phi_k : U_k \to \mathbb{D}$ and $\psi_k : V_k \to \mathbb{D}$ for $k = 1, 2$ as follows

$$\phi_1(x,y) = \left(\frac{y}{x}, \frac{1}{x}\right) = (u,v), \qquad \phi_2(x,y) = \left(\frac{1}{y}, \frac{x}{y}\right) = (u,v),$$
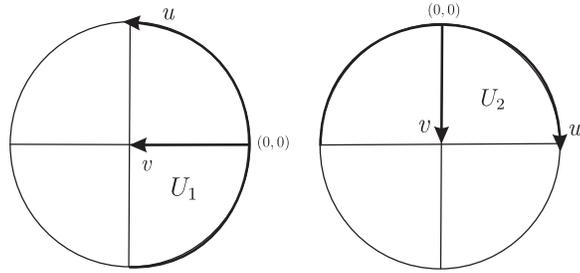
FIGURE 2. The coordinates (u, v) in the local charts U_1 and U_2.

and $\psi_k = -\phi_k$, $k = 1, 2$.

If $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ is a polynomial differential system and d is the maximum of the degrees of P and Q, then the expression of the extended flow on the local chart (U_1, ϕ_1) is

$$(4) \qquad \dot{u} = v^d \left[-uP\left(\frac{y}{x}, \frac{1}{x}\right) + Q\left(\frac{y}{x}, \frac{1}{x}\right) \right], \qquad \dot{v} = -v^{d+1} P\left(\frac{y}{x}, \frac{1}{x}\right).$$

The expression on the local chart (U_2, ϕ_2) is

$$(5) \qquad \dot{u} = v^d \left[P\left(\frac{1}{y}, \frac{x}{y}\right) - uQ\left(\frac{1}{y}, \frac{x}{y}\right) \right], \qquad \dot{v} = -v^{d+1} Q\left(\frac{1}{y}, \frac{x}{y}\right).$$

The expressions in the local charts (V_k, ψ_k), for $k = 1, 2$ are the same than in the chart (U_k, ϕ_k) multiplied by $(-1)^{d-1}$. We note that $v = 0$ corresponds to the infinity $\mathbb{S}^1$ in all the local charts.

We shall study the dynamics of the extended system (3) in the closed first quadrant of the Poincaré disc $\mathbb{D}$, i.e. in $Q_+ = \{(x, y) \in \mathbb{D} : x \geq 0 \text{ and } y \geq 0\}$.

We start studying the infinite singular points on $U_1 \cup \mathbb{Q}_+ \cup \mathbb{S}^1$. From (4) the extended system (3) in the local chart U_1 is

$$(6) \qquad \begin{aligned} \dot{u} &= u + \epsilon v + (q - \epsilon - 1)uv + \epsilon q v^2 + f u^2 v - q(1 + \epsilon)uv^2 - f q u^2 v^2, \\ \dot{v} &= v(1 + (q - 1)v + fuv - qv^2 - fquv^2). \end{aligned}$$

The unique infinite singular point on $v = 0$ is $(0, 0)$ is the origin. Since the eigenvalues of the Jacobian matrix evaluated in this singular point are $1, 1$ we conclude that this is an unstable node (see for instance Theorem 2.15 of [2]).

The extended system (3) in the local chart $U2$ is

$$(7) \qquad \begin{aligned} \dot{u} &= -fuv + fqv^2 - u^3 + (1 + \epsilon - q)u^2 v + q(1 + \epsilon)uv^2 - \epsilon u^3 v - \epsilon q u^2 v^2, \\ \dot{v} &= \epsilon(uv^2 + qv^3 - u^2 v^2 - quv^3). \end{aligned}$$

The origin of the local chart U_2 is an infinity singular point. The linear part of $(0, 0)$ is the matrix zero. In order to obtain it local phase portrait we apply blow up. Making the change of coordinates

$u \to u, v \to W$, where $W = u/v$ and rescaling the common factor u we get the differential system

(8)
$$\dot{u} = - u^2(c + w)(-1 + uw),$$
$$\dot{w} = bc - (b - acu)w + au(1 - c)w^2 - auw^3.$$

So, $(u, w) = (0, c)$ is the unique critical point of system (8) on the straight line $u = 0$. Moreover, the eigenvalues of the Jacobian matrix evaluated in the critical point $(0, c)$ are 0 and $-b$. In order to conclude the topological type of such point we can move the singular point at origin and apply a linear change of coordinates and a rescaling such that system (8) in this new coordinates is written in the normal form of Theorem 2.19 of [2]. Since $c \neq 0$ we get that $(0, c)$ is a saddle–node point. From the blowing down we get the local behavior of all orbits around $(0, 0)$ of system (7). See Figure 3 for details about this process.
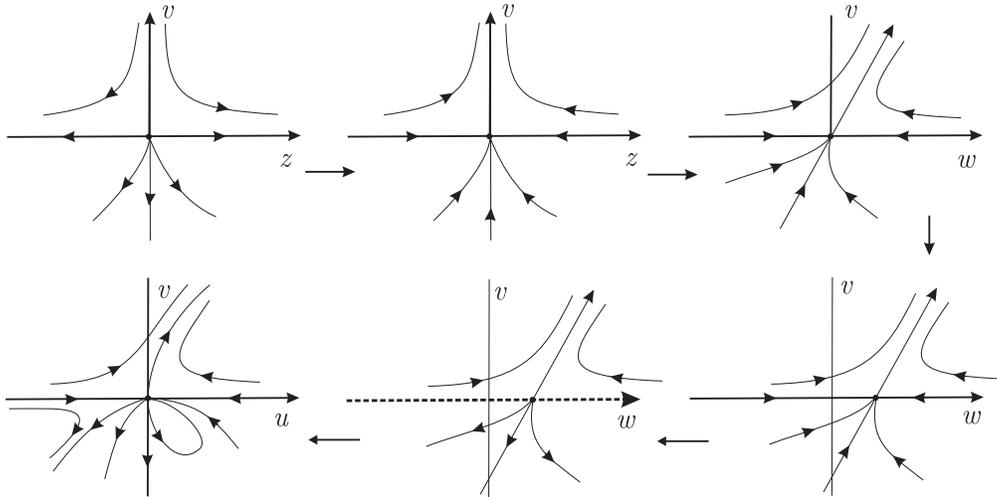

FIGURE 3. The blowing down process to get the local phase portrait at origin of the local chart U_2.

Now, as the flow of system (3) on the y-axis of Q satisfies $\dot{x}|_{x=0} = fqy$ and on the x-axes of Q we have $\dot{y}|_{y=0} = \epsilon x(q + x)$ we conclude that the flow on Q is qualitatively described in Figure 4.
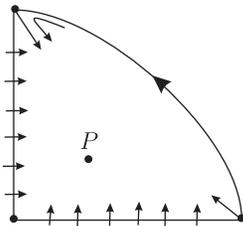

FIGURE 4. The flow in the quadrant Q.

Now it follows from the Poincaré-Bendixon Theorem (see for instance Theorem 1.25 of [2]) that if in the quadrant Q there is a unique singular point p, which is an unstable focus or node than it exists a limit cycle surrounding the point p. So Theorem 1 is proved.

3. Proof of Theorem 2

Now we have that $q = \epsilon^2/2$ and $\epsilon > 0$ is sufficiently small. Therefore following the theory of singularly perturbed differential systems, system (1) is a slow system and system (2) is a fast system. For details on the theory of singularly perturbed differential system see [8, 5, 6, 10, 11]. Then the time t of system (1) in the slow system time and the time τ of system (2) is the fast time.

From system (2) we get that near the straight line $y = c$, where c is a real constant, the motion of the system is fast. Now we shall compute the slow invariant manifold of system (1) (see for instance [8]). Assume that the slow invariant manifold is

(9) $$F(x, y) = y - F_0(x) + \epsilon F_1(x) + O(\epsilon^2) = y - f(x).$$

Then it must satisfy on the orbits $\gamma(t) = (x, y) = (x(t), y(t))$ of system (2) that

$$\frac{dF}{dt}(x, y)\Big|_{y=f(x)} = \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y}\Big|_{y=f(x)} = G_0(x) + \epsilon G_1(x) + O(\epsilon^2) = 0,$$

where $G_0(x) = \dfrac{1}{3}(2F_0(x) + 3x(x - 1))F_0'(x)$. Solving $G_0(x) = 0$ we get that either $F_0'(x) = 0$ or $2F_0(x) + 3x(x - 1) = 0$. Since $F_0'(x) = 0$ implies that the motion takes near $y = c$, and its corresponds to the fast motion, te option of the slow motion is $F_0(x) = 3/2(1 - x)x$. So the slow invariant manifold for $\epsilon = 0$ pass through the origin of coordinates which is a singular point of system (1).

Now we solve $G_1(x) = 0$, and we obtain $F_1(x) = \dfrac{x(6x - 2)}{4(2x - 1)}$. Therefore the slow invariant manifold (9) is defined, in $0 < x << 1/2$, and has the expression

$$y = \frac{3}{2}x(x - 1) + \epsilon\frac{x(6x - 2)}{4(2x - 1)} + O(\epsilon^2).$$

System (1) has three finite singular points, the $(0, 0)$ and

$$p_\pm = \left(\frac{2 - 3\epsilon^2 \pm \sqrt{4 + 108\epsilon^2 + 9\epsilon^4}}{12}, \frac{2 - 3\epsilon^2 \pm \sqrt{4 + 108\epsilon^2 + 9\epsilon^4}}{12}\right).$$

Only the point p_+ is in the interior of the quadrant Q, and its eigenvalues are $\epsilon + O(\epsilon^2)$ and $\frac{1}{3} - 2\epsilon + \sigma(\epsilon^2)$. So p_+ is an unstable node.

It easy to check that $(0, 0)$ and p_- are saddles. So there are no periodic cycles sunrrouding the some of these singularities, because in the region limited by a periodic solution the sum of the indices of the singularities contained in that region must to be 1, and a saddle has index -1. For more details on the (topological) index of a singularity see Chapter 6 of [2].

Due to the slow-fast dynamics if system (1) has a periodic solution $((x(t), y(t))$ of period T sunrrounding the unstable node p_+ it must have a piece of it close to the slow invariant manifold. Then due to the sense that a periodic solution is run, the integral

(10) $$\int_0^T div((x(t), y(t))dt$$

(where $div(x, y) = 1 - 2x - \epsilon + O(\epsilon^2)$ is the divergent of system (1)) is negative near the slow invariant manifold where the periodic orbit passes the major part of the time of its period T.

When the integral (10) is negative the periodic orbit is stable (see for instance Theorem 1.23 of [2]). Due to the fact that two stable periodic orbits cannot be consecutive sunrrounding the unstable node at most there is one periodic solution. So the uniqueness of the periodic solution in the quadrant Q is proved, and consequently, Theorem 2 follows.

Acknowledgements

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`

Instituto de Ciências Matemáticas e Computação, Universidade de São Paulo, Avenida Trabalhador São Carlense, 400, 13566-590, São Carlos, SP, Brazil.

E-mail address: `regilene@icmc.usp.br`