UNIVERSIDADE DE SÃO PAULO

FINITE RELATIVE DETERMINATION ON BOUQUET OF SUBSPACES

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Nº 45

NOTAS

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Resumo

Considere $I_i,\ldots, I_k$ ideais gerados por polinomios lineares e $S=H_t \cup \ldots \cup H_t$ seus zeros comuns, onde os H_i são subespaços de IR^n. Neste trabalho provamos que para um germem f, os conceitos de determinação finita pela direita e determinação finita relativa a G_S, o grupo de difeomorfismos que restritos a S são a identidade e os mesmos.

Finite Relative Determination on Bouquet of Subspaces

LEON KUSHNER

Abstract

Consider $I_1, \ldots, I_k$ ideals generated by linear polynomials and their common zeroes $S = H_1 \cup \cdots \cup H_k$, where H_i are subspaces of $\mathbb{R}^n$. In this paper we prove that for a germ f, the concept of finite determination on the right and finite determination relative to G_S, the group of germs of diffeomorphisms which are identity on S, are the same.

1 Introduction

Let S be a germ of a subset of $\mathbb{R}^n$ containing the origin, we denote by G_S the subgroup of germs of diffeomorphisms which are the identity on S. In particular if I is an ideal of $\varepsilon(n)$, the $\mathbb{R}$-algebra of C^∞ germs, and $S = z(I)$ the common zeroes of I $(0 \in z(I))$, we denote by $\hat{I}$ the ideal of smooth germs which vanish at S. Let f be a germ in $m(n)$, the maximal ideal of $\varepsilon(n)$. In [K] we gave conditions on f and S in such a way that if $g \in m(n)$ is such that $j^k f(0) = j^k g(0)$ and $f - g \in \hat{I}$ then g is in the G_S-orbit of f. In this paper we show that for S being a bouquet of subspaces, those conditions are equivalent to finite determination on the right. We shall denote by $\langle df \rangle$ the ideal of $\varepsilon(n)$ generated by the partial derivatives of f.

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Definition 1 *Let I an ideal of $\varepsilon(n)$, $z(I)$ the common zeroes of I and suppose that $z(I)$ contains the origin, we let $\hat{I} = rad\ I$ the ideal of germs vanishing at the germ of $z(I)$. We say that I is radical if $I = \hat{I}$.*
As in [K], given I a radical ideal, there exists a natural number $k = \mathcal{A}(I)$ such that for $m \geq k$ $\ I \cap m(n)^m = m(n)^{m-k}\left(I \cap m(n)^k\right)$. We give an analogous theorem to Theorem 11 there.

Theorem 2 *Let I be a radical ideal, $k = \mathcal{A}(I)$ and suppose $I \cap m(n)^k$ is finitely generated. Let ℓ be a natural number such that $m(n)^\ell I \subseteq m(n)I\langle df\rangle$. Then f is $(k+\ell-1)$-determined relative to G_S, where $S = z(I)$.*

Proof: Let g be a germ such that $g \equiv f$ on S and $j^{k+\ell-1}g(0) = j^{k+\ell-1}f(0)$. If we define the trivial homotopy $F(x,t) = \mathrm{t}g(x) + (1-t)f(x)$ we get $\dfrac{\partial F}{\partial t} = g - f \in m(n)^{k+\ell} \cap I$ and $\dfrac{\partial f}{\partial x_i} = \dfrac{\partial F}{\partial x_i} + t\left(\dfrac{\partial f}{\partial x_i} - \dfrac{\partial g}{\partial x_i}\right)$.
Since $m(n)^{k+\ell} \cap I \subseteq m(n)^\ell I$ and $Im(n)^{p-1} \subseteq I \cap m(n)^p \ \ \forall p$, we get the usual inclusion $\left(m(n)^{k+\ell} \cap I\right)\varepsilon(n+1) \subseteq m(n)I\left\langle \dfrac{\partial F}{\partial x_i}\right\rangle \varepsilon(n+1) + m(n)\left(I \cap m(n)^{k+\ell}\right)\varepsilon(n+1)$. By Nakayama's lemma we arrive to the inclusion $\left(m(n)^{k+\ell} \cap I\right)\varepsilon(n+1) \subseteq m(n)I\left\langle \dfrac{\partial F}{\partial x_i}\right\rangle \varepsilon(n+1)$.
Hence $\dfrac{\partial F}{\partial t} = \sum_{i=1}^{n} h_i(x,t)\dfrac{\partial F}{\partial x_i}$ with $h_i(x,t) \equiv 0$ for $x \in S$, $t \sim t_0$ (t_0 fixed). We now proceed in the usual way. □

Remarks:

1. If $I = m(n)$ then $k = 1$ and we get that $m(n)^{\ell+1} \subseteq m(n)^2\langle df\rangle$ implies f is ℓ-determined on the right.

2. If $I = \langle x_1, \ldots, x_s\rangle$, then $k = 1$ and we get that $m(n)^\ell I \subseteq m(n)I\langle df\rangle$ implies f is ℓ-determined relative to G_S, $S = \{0\} \times \mathbb{R}^{n-s}$.

Theorem 3 **[K]** *Let f be a m-determined germ relative to G_S, where $S = z(I)$, I radical ideal and $k = \mathcal{A}(I)$. Then we get $I \cap m(n)^{m+1} \subseteq I\langle df\rangle$ for $m \geq k$.*

Corollary 4 *Let f be a germ, I a radical ideal, $k = \mathcal{A}(I)$ and $I \cap m(n)^k$ be finitely generated. Suppose that $m(n)^\ell \subseteq m(n)\langle df\rangle$. Then f is $(k+\ell-1)$-determined relative to G_S, where $S = z(I)$. Hence finite determination on the right implies finite relative determination.*

Proof: Since $m(n)^\ell \subseteq m(n)\langle df\rangle$ we get $m(n)^\ell I \subseteq m(n)I\langle df\rangle$ and by Theorem 2 f is $(k+\ell-1)$-determined relative to G_S. $\square$

Let I be the ideal in $\varepsilon(3)$ generated by $\{xy, xz, yz\}$. It is easy to see that I is a radical ideal and $\mathcal{A}(I) = 2$. Moreover if we set $p_1 = xy$, $p_2 = xz$ and $p_3 = yz$, we get that for $i \neq j$, $i \neq k$, $j \neq k$, the closure of $(z(p_i) \cap z(p_j)) - z(p_k)$ is a plane and does not contain $z(I)$, which is the union of the three axis, and does not satisfy the conditions of Theorem 20 [K], but the conclusion is still true. We give a proof since it will be important for a converse of Corollary 4.

Proposition 5 *With the above notation, if f is m-determined relative to G_S, where $S = z(I)$ are the coordinate axis, then f is finitely determined on the right.*

Proof: Let us suppose that $m(3)^m \cap I \subseteq I\langle df\rangle$, which is equivalent to $m(3)^{m-2}I \subseteq I\langle df\rangle$. We shall show that $m(3)^{2m-3} \subseteq \langle df\rangle m(3)^2$, and hence f is $(2m-4)$-determined on the right.
Any mixed monomial of $m(3)^{2m-3}$ has a factor in I times a monomial of degree $2m-5$, hence for $m \geq 3$ it is contained in the jacobian ideal. Now we give the proof for x^{2m-3}, the other two are similar.

$$x^{m-2}(xy) = xy \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{1j} + xz \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{2j} + yz \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{3j}\,. \qquad (*)$$

If we denote by $\phi = x^{m-2} - \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{1j}$, we get that the zeroes of ϕ contain $\{z = 0\}$ and the zeroes of $x\phi$ contain $\{z = 0\} \cup \{x = 0\}$, hence $x\phi \in \widehat{I} = I$. From $(*)$ we get

$x^{m-1} = x\sum_{j=1}^{3} \frac{\partial f}{\partial x_i} h_{1i} + x\phi$ and hence $x^{2m-3} \in m(3)^2\langle df\rangle$, then $m(3)^{2m-3} \subseteq m(3)^2\langle df\rangle$ and f is $(2m-4)$-determined on the right.

Remarks:

1. Consider the same hypothesis as in the previous proposition but now in the algebra $\varepsilon(4)$. We change our strategy, instead of considering x^{m-2}, we consider p_1 and p_2 arbitrary monomials in $m(4)^{m-2}$, hence by multiplying the first by xy and the second by xz, we get

   $$(*) \qquad p_1 = \sum_{j=1}^{4} \frac{\partial f}{\partial x_1} h_{1j} + \phi_{xy}\,, \quad z(\phi_{xy}) \supseteq \{z=0\}$$

   $$(**) \qquad p_2 = \sum_{j=1}^{4} \frac{\partial f}{\partial x_2} h_{2j} + \phi_{xz}\,, \quad z(\phi_{xz}) \supseteq \{y=0\}$$

   If we multiply $(*)$ by $(**)$ we get $p_1p_2 = \sum_{j=1}^{4} \frac{\partial f}{\partial x_j} h_{3j} + \phi_{xy}\phi_{xz}$ but $z\left(\phi_{xy}\cdot\phi_{xz}\right) = z\left(\phi_{xy}\right) \cup z\left(\phi_{xz}\right) \supseteq \{z=0\} \cup \{y=0\}$ hence $p_1p_2p_3 \in \langle df\rangle m(4)^{m-2} + I\,m(4)^{m-2}$ for $p_3 \in m(4)^{m-2}$ and $p_1p_2p_3 \in \langle df\rangle m(4)^2$. Therefore f is $(3(m-2)-1)$-determined on the right.

2. In the general case $\varepsilon(n)$ we get the same result $(n \geq 4)$.

We recall that if $f : (\mathbb{R}^n, \bar{0}) \to (\mathbb{R}, 0)$ is an analytic germ and $f^{\mathbb{C}} : (\mathbb{C}^n, \bar{0}) \to \mathbb{C}, 0)$ its complexification, f is finitely determined on the right if and only if the common zeroes of the ideal generated by $\left\{\frac{\partial f^{\mathbb{C}}}{\partial z_1}, \ldots, \frac{\partial f^{\mathbb{C}}}{\partial z_n}\right\}$ are at most one point (the origin). We give a partial result in this direction for our case.

Proposition 6 *Let I be an ideal of complex analytic map germs. Then $z\left\langle \frac{\partial f^{\mathbb{C}}}{\partial z_i} \right\rangle \subseteq z(I)$ if and only if for some k, $I^k \subseteq I\left\langle \frac{\partial f^{\mathbb{C}}}{\partial z_i}\right\rangle$.*

Proof: Since $z\left\langle \frac{\partial f^{\mathbb{C}}}{\partial z_i}\right\rangle \subseteq z(I)$ then $z\left(\left\langle \frac{\partial f^{\mathbb{C}}}{\partial z_i}\right\rangle I\right) = z(I)$ and by Hilbert's Nullstellensatz there exists k natural number with $I^k \subseteq I\left\langle \frac{\partial f^{\mathbb{C}}}{\partial z_i}\right\rangle$. For the converse $z(I) = z(I^k) \supseteq z\left(I\left\langle \frac{\partial f^{\mathbb{C}}}{\partial z_i}\right\rangle\right) \supseteq z\left(\left\langle \frac{\partial f^{\mathbb{C}}}{\partial z_i}\right\rangle\right)$. □

If $I = m(n)$, this is a known result [B].

Lemma 7 *Let $\{p_i\}$, $i \leq i \leq r$, be linear polynomials which are linearly independent. Then the ideal I generated by them is radical.*

Proof: Let H be the subspace defined by the common zeroes of I, and $p_{r+1}, \ldots, p_n$ linear polynomials such that $\{p_i\}_{i=1}^n$ is a basis of all linear polynomials in n-variables. Then $\varphi = (p_1, \ldots, p_n)$ is an isomorphism and we have that $f \equiv 0$ on H if and only if $f \circ \varphi^{-1} \equiv 0$ on $\{0\} \times \mathbb{R}^{n-r}$. By Hadamard's lemma we get $f \circ \varphi^{-1}(x_1, \ldots, x_n) = \sum_{i=1}^{r} x_i g_i$ and hence $f = \sum_{i=1}^{r} p_i(g_i \circ \varphi)$. □

Lemma 8 *Let $I_1, \ldots, I_r$ be radical ideals in $\varepsilon(n)$. Then their intersection is also a radical ideal.*

Proof: In general $\mathrm{rad}\left(\bigcap_{i=1}^{r} I_i\right) \subseteq \bigcap_{i=1}^{r} \mathrm{rad}(I_i)$, hence we get $\bigcap_{i=1}^{r} I_i \subseteq \mathrm{rad}\left(\bigcap_{i=1}^{r} I_i\right) \subseteq \bigcap_{i=1}^{r} \mathrm{rad}(I_i) = \bigcap_{i=1}^{r} I_i$. Therefore $\mathrm{rad}\bigcap_{i=1}^{r} I_i = \bigcap_{i=1}^{r} I_i$. □

Remark: Since $z\left(\bigcap_{i=1}^{r} I_i\right) = z\left(\prod_{i=1}^{r} I_i\right)$, we get $\mathrm{rad}\left(\bigcap_{i=1}^{r} I_i\right) = \mathrm{rad}\left(\prod_{i=1}^{r} I_i\right)$.

Example 1 Let $I \subseteq \varepsilon(3)$ be the ideal generated by x and yz, hence $z(I)$ are the y and z axis, not in general position, but I is radical. Let $f \in \hat{I}$ then $f(x,y,z) = xg_1(x,y,z) + yg_2(x,y,z) + zg_3(x,y,z)$, where $g_2(0,y,0) \equiv 0$ and $g_3(0,0,z) \equiv 0$. Hence by Hadamard's lemma we get that $f(x,y,z) = xg_1(x,y,z) + y\left(xh_1(x,y,z) + zh_2(x,y,z)\right) + z\left(xh_3(x,y,z) + yh_4(x,y,z)\right)$, and $f \in I\varepsilon(3)$. □

Notation: Let $I_1 = \langle p_{11}, \dots, p_{1s_1} \rangle$ and $I_2 = \langle p_{21}, \dots, p_{2s_2} \rangle$ where $\{p_{11}, \dots, p_{1s_1}\}$ and $\{p_{21}, \dots, p_{2s_2}\}$ are linearly independent linear polynomials. We denote by $\langle p_{1j} p_{2k} \rangle$ either $p_{1j} \cdot p_{2k}$ if they are linearly independent or p_{1j} if not, and by $\langle I_1 I_2 \rangle$ the ideal generated by $\{\langle p_{1j} p_{2k} \rangle\}$.

Proposition 9 *With the above notation $I_1 \cap I_2$ is generated by $\{\langle p_{1j} p_{2k} \rangle\}$ and $I_1 \cap I_2 = \langle I_1 I_2 \rangle$.*

Proof: We can suppose $I_1 = \langle x_1, \dots, x_t, \dots, x_{s_1} \rangle$ and $I_2 = \langle x_1, \dots, x_t, x_{s_1+1}, \dots, x_{s_1+s_2-t} \rangle$ with $0 \le t \le s_1$. Let $f \in I_1 \cap I_2$ then we write $f = \sum_{i=1}^{s_1} x_i\, q_{1,i} = \sum_{i=1}^{t} x_i\, q_{2,i} + \sum_{j=s_1+1}^{s_1+s_2-t} x_j\, q_{2,j}$, hence q_{2,s_1+s_2-t} vanishes at $x_1 = \cdots = x_{s_1+s_2-t-1} = 0$ and we get $q_{2,s_1+s_2-t} = \sum_{i=1}^{s_1+s_2-t-1} x_i\, q_{2,s_1+s_2-t,i}$, therefore

$$f = \sum_{i=1}^{s_1} x_i\, q_{1,i} = \sum_{i=1}^{t} x_i\, q_{2,i} + \sum_{j=s_1+1}^{s_1+s_2-t-1} x_j\, q_{2,j} + x_{s_1+s_2-t} \left(\sum_{k=1}^{s_1+s_2-t-1} x_1\, q_{2,s_1+s_2-t,k} \right) .$$

Continuing in this way we get that f is in the ideal generated by $\{x_i \mid 1 \le i \le t\}$ and $\{x_i x_j \mid t+1 \le i \le s_1\,,\ s_1+1 \le j \le s_1+s_2-t\}$. Therefore $I_1 \cap I_2 \subseteq \langle I_1 I_2 \rangle$. The other inclusion is obvious. $\square$

Remark: If $p_1, \dots, p_s$ are linear polynomials, we define inductively $\langle p_1 p_2 \cdots p_n \rangle$ and we get $\langle p_1 \langle p_2 \cdots p_n \rangle \rangle = \langle p_1 \langle p_2 \cdots p_n \rangle \rangle = \cdots = \langle p_1 \langle p_2 \langle \cdots \langle p_{n-1} p_n \rangle \rangle \rangle \rangle$ and Proposition 9 is true for k ideals $I_1 = \langle p_{11}, \dots, p_{1s_1} \rangle, \dots, I_k = \langle p_{k_1}, \dots, p_{ks_k} \rangle$.

Corollary 10 *Let $I_1 = \langle p_{11}, \dots, p_{1s_1} \rangle, \dots, I_k = \langle p_{k1}, \dots, p_{ks_k} \rangle$, where $\{p_{i_1}, \dots, p_{is_i}\}$ are linear independent linear polynomials, $I = \bigcap_{i=1}^{k} I_i$ and $I' = \langle I_1 \dots I_k \rangle$. Hence we get the equalities $I = \hat{I} = \hat{I'} = I'$.*

Proposition 11 *Let $h_1, \dots, h_k$ be homogeneous polynomials of degree $s_1, \dots, s_k$ respectively and let s be the maximum of these degrees. Hence if I is the ideal generated by $h_1, \dots, h_k$, we get $\mathcal{A}(I) \le s$.*

Proof: We have to show that $(I \cap m(n)^s)\, m(n)^r = I \cap m(n)^{s+r} \quad \forall r \geq 0$. Let $f \in I \cap m(n)^{s+r}$, hence we have

$$f = g_1 h_1 + \cdots g_k h_k \tag{$*$}$$

and

$$0 = j^{s+r-1} f(0) = h_1 j^{r+s-1-s_1} g_1(0) + \cdots + h_k j^{r+s-1-s_k} g_k(0) \tag{$**$}$$

subtracting $(**)$ from $(*)$ we get $f = \widetilde{g_1} h_1 + \cdots + \widetilde{g_k} h_k$, where $\tilde{g}_i \in m(n)^{r+s-s_i}$. Hence each $\tilde{g}_i$ is a sum of elements of the form $\widetilde{h_i^j}\widetilde{\widetilde{h_i^j}}$ with $\widetilde{h_i^j} \in m(n)^r$ and $\widetilde{\widetilde{h_i^j}}$ is a homogeneous monomial of degree $s - s_i$. Therefore f is a sum of elements of the form $\left(h_i \widetilde{\widetilde{h_i^j}}\right) \widetilde{h_i^j}$, with $h_i \widetilde{\widetilde{h_i^j}} \in I \cap m(n)^s$ and $\widetilde{h_i^j} \in m(n)^r$, hence $f \in (I \cap m(n)^s)\, m(n)^r$. We have shown that $I \cap m(n)^{s+r} \subseteq (I \cap m(n)^s)\, m(n)^r \ \forall r \geq 0$. The other inclusion is obvious. □

Corollary 12 *With the conditions of the previous proposition $I \cap m(n)^s$ is finitely generated.*

Proof: In fact, the proof of Proposition 11 shows that the set $\left\{ h_i \widetilde{\widetilde{h_i^j}} \right\}$ where $\widetilde{\widetilde{h_i^j}}$ is a generator of $m(n)^{s-s_i}$ generates $I \cap m(n)^s$. □

We write Corollary 4 in this case.

Theorem 13 *Let $\{p_{11}, \ldots, p_{1s_1}\}, \ldots, \{p_{k1}, \ldots, p_{ks_k}\}$ be sets of linearly independent linear polynomials and $I_1, \ldots I_k$ the ideals in $\varepsilon(n)$ generated by them. If f is finitely determined on the right, then f is finitely determined relative to G_S, where $S = \bigcup_{i=1}^{k} H_i$ and $H_i = z(I_i)$.*

We state a result of linear algebra that will be helpful to us.

Lemma 14 *Let V and $W_1, \ldots, W_s$ subspaces of $\mathbb{R}^n$, then if cl denotes closure*

$$a)\ cl\left(V \cap \bigcap_{j=1}^{s} W_j^c\right) = \begin{cases} V & \text{if } V \cap W_j \subsetneq V\ \forall j \\ \phi & \text{if } V \cap W_{j_0} = V \text{ for some } j_0 \end{cases}$$

b) If $V \subseteq W_1 \cup \cdots \cup W_s$, then $V \subseteq W_{j_0}$ for some j_0.

Let H_1 and H_2 be subspaces of $\mathbb{R}^n$. By a linear isomorphism we can suppose $H_1 = z\{x_i\}_{i=1}^k$, $H_2 = z\left(\{x_i\}_{i=1}^s \cup \{x_j\}_{j=k+1}^{k+\ell-s}\right)$, where $0 \leq s \leq k$.

Lemma 15 *Let I_1 and I_2 the ideals in $\varepsilon(n)$ generated by $\{x_i\}_{i=1}^k$ and $\{x_i\}_{i=1}^s \cup \{x_j\}_{j=k+1}^{k+\ell-s}$, respectively. Suppose that $m(n)^m\langle I_1 I_2\rangle \subseteq \langle df\rangle\langle I_1 I_2\rangle$, then f is finitely determined on the right.*

Proof: Let $p_j \in m(n)^m$ and $x_k\, x_{\ell-s+k} \in \langle I_1 I_2\rangle$, then we get

$$p_j\,(x_k\, x_{\ell-s+k}) = \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i}\, h_i\right) x_k\, x_{\ell-s+k} + g \qquad (*)$$

If we denote $\phi = p_j - \sum_{j=1}^n \frac{\partial f}{\partial x_i} h_i$, we get that ϕ is zero in the closure of the common zeroes of g minus $\{x_k = 0\} \cup \{x_{\ell-s+k} = 0\}$. Hence ϕ is zero in the union of the closure of sets of the form $\{x_1 = \cdots = x_s = x_{i_1} = \cdots x_{i_\alpha} = x_{j_1} = \cdots x_{j_k} = 0,\ x_k \neq 0,\ x_{\ell-s+k} \neq 0\}$ where $s+1 \leq i_1 < \cdots < i_\alpha \leq k-1$ and $k+1 \leq j_1 < \cdots < j_\beta \leq \ell - s + k - 1$. If $k > 1$ and $\ell > 1$ we get $z(\phi) \supset z(I)$. Let $p_k \in m(n)^m$, hence from $(*)$ we obtain $p_j\, p_k = \left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} h_i\right) p_k + \phi p_k$, since $\phi p_k \in I\langle df\rangle$ we get $m(n)^{2m} \subseteq \langle df\rangle m(n)$ and f is $2m$-determined. In the case $k = 1$ we have two sub-cases, $I_1 = \langle x_1\rangle$ and $I_2 = \langle x_1, x_2, \ldots, x_\ell\rangle$ or $I_1 = \langle x_1\rangle$ and $I_2 = \langle x_2, \ldots, x_{\ell+1}\rangle$. In the first subcase $\langle I_1 I_2\rangle = \langle x_1\rangle$ and in the second $\langle I_1 I_2\rangle = \langle x_1\, x_j\rangle$ $2 \leq j \leq \ell + 1$. Hence $m(n)^m\langle I_1 I_2\rangle \subseteq \langle df\rangle\langle I_1 I_2\rangle$ implies either $m(n)^m \subseteq \langle df\rangle$ or $m(n)^m I_2 \subseteq \langle df\rangle I_2$ respectively and we get that f is $(m+1)$-determined on the right and f is $(2m)$-determined on the right respectively. [P-L]. □

For the case of three subspaces we start with

$$\begin{aligned}
H_1 &= z\left(\{x_i\}_{i=1}^s \cup \{x_{s+j}\}_{j=1}^t \cup \{x_{s+t+k}\}_{k=1}^r \cup \{x_{s+t+r+q}\}_{q=1}^{\ell-(s+t+r)}\right)\\
H_2 &= z\left(\{x_i\}_{i=1}^s \cup \{x_{s+j}\}_{j=1}^t \cup \{x_{\ell+w}\}_{w=1}^u \cup \{x_{\ell+u+\alpha}\}_{\alpha=1}^{m-(s+t+u)}\right)\\
H_3 &= z\left(\{x_i\}_{i=1}^s \cup \{x_{s+t+k}\}_{k=1}^r \cup \{x_{\ell+w}\}_{w=1}^u \cup \left\{x_{\ell+(m-(s+t))+\beta}\right\}_{\beta=1}^{p-(s+r+u)}\right)
\end{aligned}$$

where $0 \leq s \leq \min\{\ell, m, p\}$, $0 \leq t \leq \min\{\ell, m\}$, $0 \leq r \leq \min\{\ell, p\}$, $0 \leq u \leq \min\{m, p\}$.

Lemma 16 *Let H_1, H_2, H_3 as above, I_1, I_2, I_3 the ideals in $\varepsilon(n)$ generated by their monomials respectively. Suppose that $m(n)^m\langle I_1 I_2 I_3\rangle \subseteq \langle df\rangle\langle I_1 I_2 I_3\rangle$, then f is finitely determined on the right.*

Proof: The generators of $\langle I_1 I_2 I_3\rangle$ are $\{x_i\}, \{x_{s+j}\, x_{s+t+k}\}, \{x_{s+j}\, x_{\ell+w}\}, \left\{x_{s+j}\, x_{\ell+(m-(s+t))+\beta}\right\}$, $\{x_{s+t+k}\, x_{\ell+w}\}$, $\{x_{s+t+k}\, x_{\ell+u+\alpha}\}$, $\{x_{\ell+w}\, x_{s+t+r+q}\}$, $\left\{x_{s+t+r+q}\, x_{\ell+u+\alpha}\, x_{\ell+(m-(s+t))+\beta}\right\}$.
The first set of generators are the ones appearing in the three subspaces, the following six in only two subspaces and the last in only one.

Let p monomial in $m(n)^m$, it is clear that $x_i p \in m(n)\langle df\rangle$ for $1 \le i \le s$. Now we want to imitate the previous proof.

a) We pick one x_i being in two ideals but not in the third, let us pick x_{s+1} and we start with $x_{s+1}x_{s+t+1} \in \langle I_1 I_2 I_3\rangle$, then if $p \in m(n)^m$, $p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i + \phi$, where ϕ is zero in $z(I_1') \cup z(I_2') \cup z(I_3')$ and I_i' has as generators the same as I_i except x_{s+1} and x_{s+t+1}. We will get for an arbitrary $q \in m(n)^m$, $pq\, x_{s+1} \in \langle df\rangle m(n)$.

b) We suppose x_{s+1} is only in I_1, hence x_{s+1} does not belong to I_2 neither to I_3. We divide in two subcases.

   i) We have an x_k with $k > s$ and $x_k \in I_2 \cap I_3$ hence $x_{s+1}\, x_k \in \langle I_1 I_2 I_3\rangle$ and $p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i + \phi$, where ϕ is zero in $z(I_1') \cup z(I_2') \cup z(I_3')$ and I_i' has the same generators as I_i except x_{s+1} and x_k. Then $x_{s+1}\phi$ is zero in $z(I_1) \cup z(I_2') \cup z(I_3')$. We continue as in the previous case except if one of the ideals I_2 or I_3 is principal, generated by x_k, in this case our hypothesis $m(n)^m\langle I_1 I_2 I_3\rangle \subseteq \langle df\rangle\langle I_1 I_2 I_3\rangle$ is equivalent to $m(n)^m\langle I_1 I_2\rangle \subseteq \langle df\rangle\langle I_1 I_2\rangle$.

   ii) We don't have such an x_k, hence we consider $x_{s+1}\, x_{\ell+1}\, x_{s+t+1} \in \langle I_1 I_2 I_3\rangle$ and $p = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} h_i + \phi$, where ϕ is zero in $z(I_1') \cup z(I_2') \cup z(_3')$, therefore $x_{s+1}\phi$ is zero in $z(I_1) \cup z(I_2') \cup z(I_3')$. We now proceed as in i).

c) This case is for a monomial x_α of degree m which does not have as a factor any of the generators of I_1, I_2 or I_3. Consider $\{f_j\}_{j=1}^{N}$ the monomials generating $\langle I_1 I_2 I_3\rangle$, hence for x_α monomial in $m(n)^m$ we get, $x_\alpha = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} q_i^\alpha + \phi_j$, and $z(\phi_j)$ contains $z(I'_{1j}) \cup z(I'_{2j}) \cup z(I'_{3j})$, where I'_{ij} is generated by the generators of I_i except the ones appearing in the monomial f_j, (if we cancell all the generators we shall understand $z(I'_{ij}) = \phi$).

If for some j_0 , we get $z(I_1) \cup z(I_2) \cup z(I_3) \subseteq z(I'_{1j_0}) \cup z(I'_{2j_0}) \cup z(I'_{3j_0})$, then for x_β monomial in $m(n)^m$ we get $x_\alpha x_\beta \in \langle df \rangle m(n)$. If not, $z(I_1) \cup z(I_2) \cup z(I_3) \not\subset z(I'_{1j}) \cup z(I'_{2j}) \cup z(I'_{3j})$, $\forall j$. In this case, for $x_{\alpha_1}, \ldots, x_{\alpha_N}$ monomials in the ideal $m(n)^m$, we can write $x_{\alpha_1} \ldots x_{\alpha_N} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} Q_i + \prod_{j=1}^{N} \phi_j$ and $z\left(\prod_{j=1}^{N} \phi_j\right) = \bigcup_{j=1}^{N} z(\phi_j)$. If $\bigcup_{i=1}^{3} z(I'_i) \subseteq \bigcup_{i=1}^{3} \bigcup_{j=1}^{N} z(I'_{ij})$ we finish, if not, we can suppose by Lemma 14, $z(I_1) \not\subset z(I'_{ij})$, $\forall\, 1 \leq i \leq 3$, $1 \leq j \leq N$. Fix $i = 1$, hence $z(I_1) \not\subset z(I'_{1j})$ $\forall j$, this means that $\forall j$, the factors of f_j contain the generators of I_1. Since degree of each f_j is less or equal than 3, we have the following cases:

i) The ideal I_1 is principal. Same as before.

ii) If $I_1 = \langle x, y \rangle$, necessarly I_2 and I_3 are principal. Same as before.

iii) The case $I_1 = \langle x, y, z \rangle$ is impossible. □

Theorem 17 *Let $I_1, \ldots, I_k$ ideals of $\varepsilon(n)$, each of them generated by linearly independent linear monomials and $H_1, \ldots, H_k$ their common zeroes respectively. Then if $m(n)^m \langle I_1 \ldots I_k \rangle \subseteq \langle df \rangle \langle I_1 \ldots I_k \rangle$ hence f is finitely determined on the right.*

Proof: We copy the previous proof as in c). Let $\{f_j\}_{j=1}^{N}$ our generators (of degree less or equal than k) of $\langle I_1 \ldots I_k \rangle$. Let $x_{\alpha_1}, \ldots, x_{\alpha_N}$ be monomials in $m(n)^m$. Hence $x_{\alpha_i} = \sum_{k=1}^{n} \frac{\partial f}{\partial x_i} q_k^i + \phi_j$, where $z(\phi_j) \supset \bigcup_{\ell=1}^{N} z\left(I'_{\ell j}\right)$ and $I'_{\ell j}$ is the ideal generated by the generators of I_ℓ except the linear factors that appear in f_j , with the convention that $z(I'_{\ell j}) = \phi$ if all generators of $I'_{\ell j}$ appear as factors of f_j .

If for some j_0 , $\bigcup_{i=1}^{k} z(I_i) \subseteq \bigcup_{i=1}^{k} z(I'_{ij_0})$, then for $x_\beta \in m(n)^m$ we get $x_\alpha x_\beta \in \langle df \rangle m(n)$ and therefore f is $(2m)$- determined on the right.

In the other case, we consider the equality $x_{\alpha_1}\dots x_{\alpha_N} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} Q_i + \prod_{j=1}^{N} \phi_j$ and $z\left(\prod_{j=1}^{N}\phi_j\right) = \bigcup_{j=1}^{N} z(\phi_j)$. If $\bigcup_{i=1}^{k} z(I_i) \subseteq \bigcup_{i=1}^{k}\bigcup_{j=1}^{N} z(I'_{ij})$, then $\prod_{j=1}^{N} \phi_j \in \hat{I} = I \; (I = \langle I_1 \dots I_k\rangle)$ and for x_β monomial of $m(n)^m$, $x_\beta(x_{\alpha_1}\cdots x_{\alpha_N}) \in \langle df\rangle m(n)$. If this is not the case we can suppose $z(I_1) \not\subset z\left(I'_{ij}\right)$, $1 \le i \le k$, $i \le j \le N$. In particular, for $i = 1$, $z(I_1) \not\subset z\left(I'_{1j}\right)$ $\forall j$, then each of the generators f_j, $1 \le j \le N$, has as factors the variables appearing in I_1. A similar argument as in c) of Lemma 16 finishes the proof.

□

Corollary 18 *Let $I_1, \dots, I_k$ ideals of $\varepsilon(n)$, each of them generated by linearly independent linear monomials and $\langle I_1 \dots I_k\rangle = I$ is generated by $\{p_i\}$ $1 \le i \le N$. Suppose there exists p_j such that X, the closure of the common zeroes of the set $\{p_i\}_{i\neq j}$ minus the common zeroes of p_j contains the common zeroes of I. If a germ f satisfies $m(n)^m I \subseteq \langle df\rangle I$, then f is $(2m)$-determined on the right.*

Proof: Let q_i an arbitrary monomial of degree m, hence as in [P-L] we get $q_i\, p_j = \left(\sum_{\ell=1}^{n} \frac{\partial f}{\partial x_\ell} h_\ell\right) p_j + g_i$ and $q_i = \sum_{\ell=1}^{n} \frac{\partial f}{\partial x_\ell} h_\ell + \phi_i$, where ϕ_i vanishes at X , hence $\phi_i \in \hat{I} = I$ and if q_j is any other monomial of degree m , $q_i\, q_j \in \langle df\rangle m(n)$. Therefore $m(n)^{2m} \subseteq \langle df\rangle m(n)$.

□

Remarks

a) The case $k = 1$ of Corollary 18 was treated in [P-L].

b) The hypothesis of Corollary 18 is equivalent to the existence of $p_j \in \langle I_1 \dots I_k\rangle$, $p_j = x_{j_1}\cdots x_{j_t}$ such that $z(I_1) \cup \dots \cup z(I_k) \subseteq z(I'_1) \cup \dots \cup z(I'_k)$, in other words $z(I_1) \subseteq z(I'_{i_1}), \dots, z(I_k) \subseteq z(I'_{i_k})$, where I'_i is generated by the generators of I_i except the ones factorizing p_j.

c) In the case of Proposition 5, this condition does not verify, since for all $j, z(I'_1) \cup z(I'_2) \cup z(I'_3)$ is a plane and the common zeroes the three axis.

d) From Theorem 17 we get a coarse bound for finite determination on the right. If N is the number of generators of $\langle I_1 \dots I_k\rangle$, we get $m(n)^{m(N+1)} \subseteq \langle df\rangle m(n)$, therefore f is $m(N+1)$-determined on the right.

Adendum

The following theorem was communicated to me by Prof. Brasil Terra Lema.

Theorem 19 *Let A be a commutative ring, I, J, K ideals of A with $I = \langle g_1, \dots, g_k\rangle$. Suppose that $ag_i = 0 \ \ \forall i, \ \ 1 \le i \le k$ and $a \in J^k + K$ implies $a = 0$. Then if $IJ \subseteq IK$ we get $J^k \subseteq K$.*

Proof: Let $m_1, \dots, m_k$ arbitrary elements of J, then $m_i g_i = \sum_{j=1}^{k} d_{ij} g_j$, $\forall i$ with $1 \le i \le k$ ($d_{ij} \in K$). In a matricial notation we can write

$$C \begin{pmatrix} g_1 \\ \\ g_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{where} \qquad C = (\delta_{ij} m_i - d_{ij}) \qquad (*)$$

If we multiply $(*)$ by the adjoint of C we get $(\det C) g_i = 0 \ \forall i$, but $\det C = m_1 \cdots m_k + b$, with $b \in K$, hence by hypothesis $\det C = 0$ and $m_1 \cdots m_k \in K$ and therefore $J^k \subseteq K$. $\square$

Corollary 20 *Let $A = \varepsilon(n)$ $\ J = m(n)^\ell$ or $J = m(n)^\infty$, $K = \langle df\rangle$ and $\hat{I} = \langle g_1, \dots, g_k\rangle$. Suppose that $ag_i = 0 \ \ \forall i, \ \ 1 \le i \le k$ and $a \in m(n)^{\ell k} + K$ $(m(n)^\infty + K)$ implies $a = 0$. Then if $\hat{I} m(n)^\ell \subseteq \hat{I}\langle df\rangle$ $\left(\hat{I} m(n)^\infty \subseteq \hat{I}\langle df\rangle\right)$ hence $m(n)^{\ell k} \subseteq \langle df\rangle$ $(m(n)^\infty \subseteq \langle df\rangle)$ and f is $(\ell k + 1)$-determined (∞-determined).*

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