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RELATIVE STABILITY ON ALGEBRAIC SETS

BRASIL TERRA LEME
LEON KUSHNER

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Resumo:

Neste trabalho generalizamos a equivalência entre estabilidade relativa provada nos trabalhos de Porto-Loibel [P-L] e Porto [P] para uma classe maior de conjuntos semi algébricos.

Começamos dando condições para que um conjunto semi algébrico coincida com a fechadura de seu interior e trabalhamos com conjuntos satisfazendo esta propriedade (Lema 5 e Proposição 7). No Teorema 9 provamos a primeira parte de nosso teorema principal: estabilidade infinitesimal relativa implica estabilidade relativa, e por tanto determinação finita (infinita) implica estabilidade relativa.

No teorema 16 estudamos polinômios quase homogêneos e damos condições para eles serem infinitesimalmente estáveis. Os teoremas 23 e 27 provam a recíproca de nosso teorema principal via Teoria de Lojasiewicz, como em [P].

Resumé

In this work we generalize the equivalence of relative stability and relative infinitesimal stability proved in the works of Porto-Loibel [P-L] and Porto [P] for a big class of semialgebraic sets.
We start by giving conditions for semialgebraic set to coincide with the closure of its interior and then work with sets satisfying this condition (Lemma 5 and Proposition 7). In theorem 9 we prove the first part of the main theorem: relative infinitesimal stability implies relative stability and hence finite (infinite) determination implies relative stability.
In theorem 16 we study quasi-homogeneus polynomials and we give conditions for them to be infinitesimally stable. In theorems 23 and 27 we prove the converse of our main theorem via Lojasiewicz Theory, as in [P].

Relative Stability on Algebraic Sets

BRASIL TERRA LEME and LEON KUSHNER

Introduction

In [P-L] and [P] it was given an equivalence between relative stability and relative infinitesimal stability of function germs with respect to the group of germs of diffeomorphisms which are the identity on the semialgebraic subset of $\mathbb{R}^n$ defined by $x_1 \leq 0$. In this paper we generalize this equivalence to closed subsets of $\mathbb{R}^n$ which coincide with the closure of their interior. We give conditions for semialgebraic subsets to satisfy this property and some examples are worked for polynomials in two variables and quasihomogeneous polynomials in n-variables.

Notation. We shall work in $\varepsilon(n)$, the local algebra of function germs of $\mathbb{R}^n$ to $\mathbb{R}$ around the origin, with maximal ideal $m(n)$. If f is a function germ, $I = (i_1, \ldots, i_n)$ a multi-index, then $|I| = \sum_{j=1}^{n} i_j$ and $\dfrac{\partial^I f}{\partial x^I} = \dfrac{\partial^{i_1+\cdots+i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$. Also for f we have its jacobian ideal $\left\langle \dfrac{\partial f}{\partial x_i} \right\rangle$, the ideal of $\varepsilon(n)$ generated by its partial derivatives, ∇f the gradient of f and for $f(\overline{0}) = 0$, $z(f)$ will be the germ of common zeroes of f.

The powers of $m(n)$ will be denoted by $m(n)^k$ and $m(n)^\infty = \bigcap_{k=1}^{\infty} m(n)^k$. If S is a subset of $\mathbb{R}^n$ cointaing the origin, int S and $cl\, S$ will denote the interior and the closure of S and G_S the group of germs of diffeomorphisms ϕ such that $\phi(x) = x \ \ \forall x \in S$. Also for $x \in \mathbb{R}^n$, $d(x, S)$ will denote the distance of x to S.

Finally for f a function germ, $j^k f(x)$ will denote the Taylor expansion of f at the point x up to degree k and its called the k-jet of f at x and $J^k(\mathbb{R}^n, \mathbb{R})$ will denote the $\mathbb{R}$-vector space for all k-jets of function maps at the origin.

Definition 1 *Let $S \subseteq \mathbb{R}^n$ be a germ of a closed set such that $\bar{0} \in cl(\text{int } S)$. We say that a germ f is S-stable if given g such that $g(x) = f(x) \ \forall x \in S$ we have a diffeomorphism $\phi \in G_S$ such that $g = f \circ \phi$.*

Examples

1. Let $S = \left\{(x, y) \in \mathbb{R}^2 \mid x^3 - x^2 - y^2 \geq 0\right\}$, then S is closed but $\bar{0} \notin cl(\text{int } S)$.

2. Let $S = \left\{(x, y) \in \mathbb{R}^2 \mid x^4 - x^3 - xy^2 \geq 0\right\}$, in this case $S = cl(\text{int } S)$.

3. $S = \{(x, y, z) \mid z(x^2 + y^2) - x^3 \leq 0\}$, in this case $\bar{0} \in cl(\text{int } S)$ but $cl(\text{int } S) \neq S$.

Notation 2 We let $C_S(\mathbb{R}^n)$ be the $\mathbb{R}$-subalgebra of $\varepsilon(n)$ consisting of germs constant at S. It is a local algebra with maximal ideal $\varepsilon(S, n)$, germs of $C_S(\mathbb{R}^n)$ vanishing at the origin. In fact $\varepsilon(S, n)$ is an ideal of $\varepsilon(n)$.

Remark If $f \in \varepsilon(S, n)$ then $f \in m(n)^\infty$ and if $S = cl(\text{int } S)$ then $\dfrac{\partial^I f}{\partial x^I} \in \varepsilon(S, n) \ \forall I$ multi-index.

Proposition 3 [P-L] *Let $f \in m(n)^\infty$, then there exist $g \in m(n)^\infty$, $g(x) > 0 \ \forall x \neq \overline{0}$ and $h \in m(n)^\infty$ such that $f = gh$.*

Corollary 4 *We have the equality $\varepsilon(S,n) = m(n)^\infty \varepsilon(S,n)$.*

Proof: Let $f \in \varepsilon(S,n) \subseteq m(n)^\infty$, hence $f(x) = 0 \ \forall x \in S$ then $h(x) = 0 \ \ \forall x \in S$ and therefore $h \in \varepsilon(S,n)$, finally $f \in \varepsilon(S,n)m(n)^\infty$. The other inclusion is obvious. □

We want to give conditions such that if S is a semialgebraic set defined by inequalities of the form $P_i(\overline{x}) \leq 0$, $1 \leq i \leq s$, then $S = cl(\text{int } S)$. As examples 1 and 3 show, this is not always the case.

Lemma 5 *Let $P_1(x), \ldots, P_s(x)$ polynomials in $\mathbb{R}[x]$, $x = (x_1, \ldots, x_n)$, then $cl\,\{x \mid P_i(x) < 0,\ 1 \leq i \leq s\} = cl\,(\text{int } \{x \mid P_i(x) \leq 0,\ 1 \leq i \leq s\})$.*

Observe that if $P(x) = -\sum_{i=1}^{n} x_i^2$ then int $\{x \mid P(x) \leq 0\} = \mathbb{R}^n$ and $\{x \mid P(x) < 0\} = \mathbb{R}^n - \{\overline{0}\}$.

Proof: Since $\{x \mid P_i(x) < 0\} \subseteq \{x \mid P_i(x) \leq 0\}$ it is clear that $cl\,\{x \mid P_i(x) < 0\} \subseteq cl\,(\text{int}\,\{x \mid P_i(x) \leq 0\})$. Now let $x_0 \in cl\,(\text{int}\,\{x \mid P_i(x) \leq 0\})$ and suppose $x_0 \notin cl\,\{x \mid P_i(x) < 0\}$, then there exists $\varepsilon > 0$ such that $B_\epsilon(x_0) \cap \{x \mid P_i(x) < 0\} = \emptyset$. Let $x_j \to x_0$, $x_j \in B_\epsilon(x_0)$ and $x_j \in \text{int}\,\{x \mid P_i(x) \leq 0\}$. Obviously there exists $\epsilon_j > 0$ with $B_{\epsilon_j}(x_j) \subset \{x \mid P_i(x) \leq 0\}$ and $B_{\epsilon_j}(x_j) \subset B_\epsilon(x_0)$, hence $P_i(x) \equiv 0 \ \ \forall i$ and $\forall x \in B_{\epsilon_j}(x_j)$, but P_i are polynomials, hence $P_i \equiv 0 \ \ \forall i$.

Remark The result is also true for analytic map using the same proof.

Definition 6 *Let $A \subseteq \mathbb{R}^n$ a semialgebraic set defined by inequalities $P_i(x) \leq 0$, $1 \leq i \leq s$. We say that A is good if when we decompose $A = \bigcup_{j=1}^{t} A_j$, $\{A_j\}_{j=1}^{t}$ disjoint semialgebraic subsets each of them semialgebraically homeomorphic to a product $(0,1)^{s_j}$, then given A_ℓ with $s_\ell < n$, there exists j such that $A_\ell \subseteq \overline{A_j}$ and $s_j = n$.*

The example 2) is good, while the examples 1) and 3) are not good. We remark that example 3 is homogeneous.

Proposition 7 *Let $P_1(x), \ldots, P_s(x)$ polynomials, A the semialgebraic set defined by $A = \{x \mid P_i(x) \leq 0,\ 1 \leq i \leq s\}$. Suppose that A is good, then $cl\,\{x \mid P_i(x) < 0,\ 1 \leq i \leq s\} = \{x \mid P_i(x) \leq 0,\ 1 \leq i \leq s\}$.*

Proof: Let x_0 such that $P_i(x_0) \leq 0\ \ \forall i$, if $P_i(x_0) < 0\ \ \forall i$, $x_0 \in \{x \mid P_i(x) < 0,\ 1 \leq i \leq s\}$. If not, there exists i_0 with $P_{i_0}(x_0) = 0$. Then there exists a sequence (x_j) converging to x with $P_i(x_j) < 0$ for $1 \leq i \leq s$ and for all j. The other inclusion is obvious. □

Corollary 8 *For S a good semialgebraic set, we get $S = cl(\text{int } S)$.*

Proposition 9 *Suppose S is a closed subset of $\mathbb{R}^n$ containing the origin and such that $S = cl(\text{int } S)$. If $\varepsilon(S,n) \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle \varepsilon(s,n)$ then f is S-stable.*

Proof: The same proof as proposition 1.13 of [P-L] works. We recall our remark that if $f \in \varepsilon(S,n)$ then $\frac{\partial^I f}{\partial x^I} \subset \varepsilon(S,n)$ for each I multi-index. Now let $g(x)$ germ such that $g(x) = f(x)\ \ \forall x \in S$. We define the trivial homotopy $F(x,t) = tg(x) + (1-t)f(x)$ and $N = C_{S\times\mathbb{R}}(\mathbb{R}^{n+1}) \left\langle \frac{\partial f}{\partial x_i} \right\rangle$, $K = C_{S\times\mathbb{R}}(\mathbb{R}^{n+1}) \left\langle \frac{\partial F}{\partial x_i} \right\rangle$. Since $g - f \in \varepsilon(S,n)$, we have that if $h(x,t) \in N$, $h(x,t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)h_i(x,t) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x,t)h_i(x,t) + t\sum_{i=1}^{n} \frac{\partial (f-g)}{\partial x_i} h_i(x,t)$, $h_i(x,t) \in C_{S\times\mathbb{R}^n}(\mathbb{R}^{n+1})$. But $\frac{\partial (f-g)}{\partial x_i} \in \varepsilon(S,n) \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle \varepsilon(S,n)$, therefore $N \subseteq K + \varepsilon(S,n)N$ and by Nakayama's lemma $N \subseteq K$, and hence we have $\varepsilon(S\times\mathbb{R})\left\langle \frac{\partial f}{\partial x_i} \right\rangle \subseteq \varepsilon(S\times\mathbb{R})\left\langle \frac{\partial F}{\partial x_i} \right\rangle$. Now, since $g - f \in \varepsilon(S,n)$, we get by hypothesis $\frac{\partial F}{\partial t} = (g-f) \in \varepsilon(S\times\mathbb{R})\left\langle \frac{\partial f}{\partial x_i} \right\rangle \subseteq \varepsilon(S\times\mathbb{R})\left\langle \frac{\partial F}{\partial x_i} \right\rangle$. We now proceed as in the usual proof. $\square$

Definition 10 *A germ f is S-infinitesimally stable if* $\varepsilon(S,n) \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle \varepsilon(S,n)$.

Remark The inclusion in fact is an equality.

Corollary 11 *If f is a finitely (infinitely) determined germ, then f is S-infinitesimally stable and therefore S-stable for $S = cl(\text{int } S)$.*

Proof: Since $\varepsilon(S,n) = m(n)^\infty \varepsilon(S,n)$ and $m(n)^k \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle$ for some $k \leq \infty$ we get that $\varepsilon(S,n) \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle \varepsilon(S,n)$. We now use proposition 9. $\square$

Examples We will suppose $S = cl(\text{int } S)$.

1. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, $I_k = \langle f^k \rangle$ such that $\varepsilon(S,n) \subseteq I_k$, $z(f) \cap \text{int } S = \emptyset$ and $f^k \in \left\langle f^{k-1} \frac{\partial f}{\partial x_i} \right\rangle$. If $h \in \varepsilon(S,n)$, $h = f^k g$ and then $g(x) = 0\ \forall x \in \text{int } S$, so $g \in \varepsilon(S,n)$ and $\varepsilon(S,n) \subseteq \left\langle f^{k-1} \frac{\partial f}{\partial x_i} \right\rangle \varepsilon(S,n)$. Therefore f^k is S-infinitesimally stable.

2. Let $f_1(x_1,\ldots,x_n) = \sum_{i=1}^{s} x_i^2$, $f_2(x_1,\ldots,x_n) = x_1$ and $S = \{(x_1,\ldots,x_n) \mid x_1 \leq 0\}$. We clearly have $z(f_1) = \{0\} \times \mathbb{R}^{n-s}$, $z(f_2) = \{(x_1,\ldots,x_n) \mid x_1 = 0\}$, $z(f_1) \cap \text{int } S = \emptyset$

and $z(f_2) \cap \operatorname{int} S = \phi$. In both cases for any k we get $\varepsilon(S,n) \subseteq \langle f_i^k \rangle$ (Prop 5-4, Chap V, [T]). Moreover, $\dfrac{\partial f_1}{\partial x_i} = 2x_i$ for $1 \le i \le s$ and $\dfrac{\partial f_1}{\partial x_i} = 0$ for $i > s$ hence $\sum_{i=1}^{n} \left(\dfrac{1}{2} f_1^{k-1} \dfrac{\partial f_1}{\partial x_i} \right) x_i = f_1^{k-1} \cdot f_1 = f_1^k$. We also have $\dfrac{\partial f_2}{\partial x_i} = \delta_{i_1}$ (kronecker's delta) and $\sum_{i=1}^{n} f_2^{k-1} \dfrac{\partial f_2}{\partial x_i} x_i = f_2^k$. By the previous example f_1^k and f_2^k are S-infinitesimally stable ($k \ge 2$) ([P-L]).

Lemma 12 *Let $f(\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a germ and $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ a germ of diffeomorphism. Then $g \in \left\langle \dfrac{\partial f}{\partial x_i}, \ldots, \dfrac{\partial f}{\partial x_n} \right\rangle$ if and only if $g \circ \phi \in \left\langle \dfrac{\partial (f \circ \phi)}{\partial x_i}, \ldots, \dfrac{\partial (f \circ \phi)}{\partial x_n} \right\rangle$.*

Proof: There exist $h_i, \ldots, h_n$ germs such that $g = \sum_{i=1}^{n} \dfrac{\partial f}{\partial x_i} \cdot h_i$, but $\dfrac{\partial f}{\partial x_i}(y) = \sum_{j=1}^{n} \dfrac{\partial (f \circ \phi)}{\partial x_j} (\psi(y)) \dfrac{\partial \psi_j}{\partial x_i}(y)$ where ψ is the inverse of ϕ. hence $g(y) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \dfrac{\partial (f \circ \phi)}{\partial x_j} (\psi(y)) \dfrac{\partial \psi_j}{\partial x_i}(y) \right) h_i$ and $g(\phi(x)) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \dfrac{\partial f \circ \phi}{\partial x_j}(x) \dfrac{\partial \psi_j}{\partial x_i}(\phi(x)) \right) (h_i \circ \phi)(x)$. Therefore $g \circ \phi \in \left\langle \dfrac{\partial (f \circ \phi)}{\partial x_i} \right\rangle$. For the converse use $h = f \circ \phi$ and ψ then $g \circ \phi \in \left\langle \dfrac{\partial h}{\partial x_i}, \ldots, \dfrac{\partial h}{\partial x_n} \right\rangle$ implies $g \circ \phi \circ \psi \in \left\langle \dfrac{\partial (h \circ \psi)}{\partial x_1}, \ldots, \dfrac{\partial (h \circ \psi)}{\partial x_n} \right\rangle$. □

Theorem 13 *Let $P(x,y)$ be a polynomial in two variables. We consider two cases 1) $P(x,y) = x^k y^\ell Q(x,y)$ and 2) $P(x,y) = x^k Q(x,y) + y^\ell$. where x nor y divide $Q(x,y)$. Then $P \in \left\langle \dfrac{\partial P}{\partial x}, \dfrac{\partial P}{\partial y} \right\rangle$ if and only if*

Case 1: *There exist $h_3(x,y)$ and $h_4(x,y)$ germs such that*

a) $Q(x,y) = \left(kQ(x,y) + x \dfrac{\partial Q}{\partial x} \right) h_3(x,y) + \left(\ell Q(x,y) + y \dfrac{\partial Q}{\partial y} \right) h_4(x,y)$ *for* $k > 0$, $\ell > 0$.

b) $Q(x,y) = \dfrac{\partial Q}{\partial x}(x,y) h_3(x,y) + \left(\ell Q(x,y) + y \dfrac{\partial Q}{\partial y} \right) h_4(x,y)$ *for* $k = 0$, $\ell > 0$.

c) $P(x,y) = \dfrac{\partial P}{\partial x}(x,y)h_3(x,y) + \dfrac{\partial P}{\partial y}(x,y)h_4(x,y)$ *for* $k = \ell = 0$.

Case 2: *There exists* $h_1(x,y)$ *and* $h_4(x,y)$ *germs such that*

a) $xQ(x,y) = \left(kQ(x,y) + x\dfrac{\partial Q}{\partial x}\right)h_1(x,y) + \dfrac{xy}{\ell}\dfrac{\partial Q}{\partial y} + x^k\dfrac{\partial Q}{\partial y}h_4(x,y) + \ell y^{\ell-1}h_4(x,y)$ *for* $k > 0$, $\ell > 0$.

b) $Q(x,y) = \dfrac{\partial Q}{\partial x}(x,y)h_1(x,y) + \dfrac{y}{\ell}\dfrac{\partial Q}{\partial y} + x\dfrac{\partial Q}{\partial y}h_3(x,y) + \ell xy^{\ell-1}h_3(x,y)$ *for* $k = 0$.

Proof:

Case 1: For $k > 0$ and $\ell > 0$

$$\begin{aligned}\frac{\partial P}{\partial x} &= kx^{k-1}y^{\ell}Q(x,y) + x^k y^{\ell}\frac{\partial Q}{\partial x}\\ \frac{\partial P}{\partial y} &= \ell x^k y^{\ell-1}Q(x,y) + x^k y^{\ell}\frac{\partial Q}{\partial y}\,.\end{aligned}$$

Now $P(x,y) = h_1(x,y)\dfrac{\partial P}{\partial x} + h_2(x,y)\dfrac{\partial P}{\partial y}$, then $xy\,Q(x,y) = \left(ky\,Q(x,y) + xy\,\dfrac{\partial Q}{\partial x}\right)h_1(x,y) + \left(\ell x\,Q(x,y) + xy\,\dfrac{\partial Q}{\partial y}\right)h_2(x,y)$. Since $h_1(0,y) \equiv 0$ and $h_2(x,0) \equiv 0$, we get $Q(x,y) = \left(k\,Q(x,y) + x\,\dfrac{\partial Q}{\partial x}\right)h_3(x,y) + \left(\ell\,Q(x,y) + y\,\dfrac{\partial Q}{\partial y}\right)h_4(x,y)$.

The proof is similar for b) and c).

Case 2: For $k > 0$ and $\ell > 0$

$$\begin{aligned}\frac{\partial P}{\partial x} &= kx^{k-1}Q(x,y) + x^k\frac{\partial Q}{\partial x}\\ \frac{\partial P}{\partial y} &= x^k\frac{\partial Q}{\partial y} + \ell\,y^{\ell-1}\,.\end{aligned}$$

Then our case is equivalent to

$$x^kQ(x,y) + y^{\ell} = \left(kx^{k-1}Q(x,y) + x^k\frac{\partial Q}{\partial x}\right)h_1(x,y) + \left(x^k\frac{\partial Q}{\partial y} + \ell\,y^{\ell-1}\right)h_2(x,y)\,.$$

Since $h_2(x,y) = \dfrac{y}{\ell} + xh_3(x,y)$, we get

$$x^k Q(x,y) = \left(kx^{k-1}Q(x,y) + x^k\frac{\partial Q}{\partial x}\right) h_1(x,y) + \frac{x^k y}{\ell}\frac{\partial Q}{\partial y} + x^{k+1}\frac{\partial Q}{\partial y} h_3(x,y) + \ell\, x\, y^{\ell-1} h_3(x,y)\,.$$

From here, $h_3(x,y) = x^{k-2}h_4(x,y)$ and

$$xQ(x,y) = \left(kQ(x,y) + x\frac{\partial Q}{\partial x}\right) h_1(x,y) + \frac{xy}{\ell}\frac{\partial Q}{\partial y} + x^k\frac{\partial Q}{\partial y} h_4(x,y) + \ell\, y^{\ell-1} h_4(x,y)\,.$$

For $k = 0$ and $\ell > 0$ it is similar. $\square$

The case for quasihomogeneous polynomials is easier.

Definition 14 *Let P be a polynomial in variables $x_1, \ldots, x_n$. We say that P is quasihomogeneous if there exists numbers $k_1, \ldots, k_n$ and ℓ such that $P(t^{k_1}x_1, \ldots, t^{k_n}x_n) = t^{\ell}P(x_1, \ldots, x_n)$.*

Observe that if $k_1 = \cdots = k_n = 1$ we have homogeneity of degree ℓ and that for P quasihomogeneous $\dfrac{\partial P}{\partial x_j}(t^{k_1}x_1, \ldots, t^{k_n}x_n) = t^{\ell-k_j}\dfrac{\partial P}{\partial x_j}(x_1, \ldots, x_n)$.

If we write $P = \Sigma a_I x^I$ for a quasihomogeneous polynomial we obtain for $I = (i_1, \ldots, i_n)$, $i_1k_1 + \cdots + i_nk_n = \ell$ $(a_I \neq 0)$.

Lemma 15 *If P is a quasihomogeneous polynomial then $P(x) = \displaystyle\sum_{j=1}^{n} c_j x_j \frac{\partial P}{\partial x_j}$, where $c_j = k_j/\ell$.*

Proof: If $P(x) = \sum_I a_I x^I$, we get $\sum c_j i_j = 1$ and $\dfrac{\partial P}{\partial x_j} = \displaystyle\sum_{\substack{I\\ i_j\geq 1}} a_I i_j x^{I-e_j}$ where e_j is the canonical vector in $\mathbb{R}^n$ with 1 in its j-coordinate and 0 in all others, then

$$P(x) = \sum_I a_I x^I = \sum_I a_I\left(\sum_{j=1}^{n} c_j i_j\right) x^I = \sum_{j=1}^{n} c_j x_j \sum_{\substack{I\\ i_j\geq 1}} a_I i_j x^{I-e_j} = \sum_{j=1}^{n} c_j x_j \frac{\partial P}{\partial x_j}\,.$$

$\square$

Theorem 16 *Let $P(x)$ be a quasihomogeneous polynomial and S a closed subset of $\mathbb{R}^n$ containing the origin and such that $S = cl(\text{int } S)$. Suppose that $\varepsilon(S,n) \subseteq \langle P \rangle$ and that $z(P) \cap \text{int } S = \phi$. Then f is S-infinitesimal stable. In the case $S = \{x \mid P(x) \leq 0\}$ we only need S to be a good semialgebraic subset of $\mathbb{R}^n$.*

Proof: By hypothesis we get $\varepsilon(S,n) \subset \langle P \rangle$ and $P \in \left\langle \frac{\partial P}{\partial x_i} \right\rangle$, this together with $z(P) \cap \text{int } S = \phi$ gives the result using example 1 after Corollary 11. For the second part it is obvious that $z(P) \cap \text{int } S = \phi$ and since S is a good semialgebraic set, we finish.

□

Corollary 17 *Let $f(x,y)$ be a simple germ and ϕ the germ of a diffeomorphism such that $f \circ \phi$ is in canonical form. Then if $S' = \{x \mid P(x) \leq 0\}$, f is S-infinitesimally stable, where $S = \phi(S')$.*

Proof: From Lemma 12 we have to show everything for P the canonical form of f. From the previous theorem we only reed the inclusion $\varepsilon(S,2) \subseteq \langle P \rangle$. Let $I = \langle P \rangle$, $J_1(I) = \left\langle P, \frac{\partial P}{\partial x}, \frac{\partial P}{\partial y} \right\rangle$, hence $z(J_1(I)) = \{(0,0)\}$ and if $I_1 = \left\langle x^2 \pm y^{k+1} \right\rangle$, $I_2 = \left\langle x^2 y + y^{k-1} \right\rangle$, $I_3 = \langle x^3 + y^4 \rangle$, $I_4 = \langle x^3 + xy^3 \rangle$, and $I_5 = \langle x^3 + y^5 \rangle$ we get the following inclusions

$$(x^2 + y^2)^{[\frac{k}{2}]+1} \in J_1(I_1),\ (x^2 + y^2)^{[\frac{k-1}{2}]+1} \in J_1(I_2),$$

$$(x^2 + y^2)^2 \in J_1(I_3),\ (x^2 + y^2)^4 \in J_1(I_4),\ (x^2 + y^2)^2 \in J_1(I_5).$$

Hence by Prop 5.4. Chap V. [T] the ideals $J_1(I_i)$, $1 \leq i \leq 5$, are Lojasiewicz and $\varepsilon(S',2) \subseteq I_i$, $i \leq i \leq 5$.

□

Remark: The above corollary is true by adding a nondegenerate quadratic form in $n-2$ variables and hence for the general A_k, D_4, E_6, E_7 and E_8 singularities.

Examples:

1. Let $P(x,y) = x^2 + y^3 + x^2y^3$, one sees that P is not quasihomogeneous but P is simple and directly
$$P(x,y) = \left(2x(1+y^3)\right)\left(\frac{x}{2}\right) + \left(3y^2(1+x^2)\right)\frac{y}{3(1+x^2)}\,.$$

2. Let $P(x,y) = x^2y + x^2y^2 + x^3y^3$, which is not quasihomogeneous, using Theorem 13 we need to solve
$$1 + y + xy = (2 + 2y + 3xy^2)h_3(x,y) + (1 + 2y + 3xy^2)h_4(x)$$
which has obviously many solutions, since the factors of h_3 and h_4 are invertible.

Definition 18 *a) Let S be a closed subset of $\mathbb{R}^n(\overline{0} \in S)$ and f a germ with $f(\overline{0}) = 0$. We say that f satisfies a Lojasiewicz inequality for S if given K germ of a compact set with $\overline{0} \in K$, there exist constants $c > 0$ and $\alpha \geq 0$ such that $|f(x)| \geq c\,d(x,X)^\alpha$ for all $x \in K$.*

b) Let I be a finitely generated ideal of $\varepsilon(n)$ and S the germ of its common zeroes. We say that I is a Lojasiewicz ideal if there exists a germ f in I satisfying a Lojasiewicz inequality for S.

Remark: If $\{f_1,\ldots,f_s\}$ is a set of generators of a Lojasiewicz ideal I, then $\sum_{i=1}^{s} f_i^2$, $\sum_{i=1}^{s} |f_i|$ and $\max\{f_1^2,\ldots,f_s^2\}$ also satisfy a Lojasiewicz inequality for S.

c) Let $f \in m(n)$ and S be a closed subset of $\mathbb{R}^n$, $\overline{0} \in S$, we say that f satisfies a Jacobi-Lojasiewicz condition for S if $|\nabla f(x)|$ does.

Definition 19 *Let (b_i) a sequence of positive real numbers converging to zero. We say that a sequence of real numbers (a_i) is flat along (b_i) if given $r > 0$ there exists a natural number $N = N(r)$ such that $|a_i| \leq b_i^r$ for $i \geq N$. Vectors, matrices, ∞-jets are flat along a sequence (b_i) if each entry is flat. A sequence is flat along a sequence (x_i) of nonzero vectors in $\mathbb{R}^n$ converging to the vector $\overline{0}$ if it is flat along the sequence $(|x_i|)$.*

Remark We can change $r > 0$ for $r = n$ natural since for $n > r$ $b_i^n \leq b_i^r$ $(0 \leq b_i \leq 1)$.

Lemma 20 *A germ g does not satisfy a Lojasiewicz inequality for a closed subset S if and only if there exists a sequence of vectors $x_i \in \mathbb{R}^n - S$ converging to the vector $\overline{0}$ such that $g(x_i)$ is flat along $d(x_i, S)$.*

Proof: Necessity: We have a germ of a compact set K_0 such that given constants $c_n > 0$ and $\alpha_n \geq 0$, there exists a point $x_n \in K_0 - S$ with $|g(x_n)| < c_n\, d(x_n, S)^{\alpha_n}$. We let $\alpha_n = n$ and $c_n = 1$, we can suppose $x_n \to \overline{0}$, hence given N, we have for $n \geq N$, $|g(x_n)| < d(x_n, S)^n \leq d(x_n, S)^N$.

Sufficiency: We have a sequence (x_i) converging to $\overline{0}$ such that $g(x_i)$ is flat along $d(x_i, S)$, hence for $r > 0$ there exists $N = N(r)$ such that $|g(x_i)| \leq d(x_i, S)^r$ for $i \geq N$. If we suppose that g is Lojasiewicz we would have for a germ of a compact set containing (x_i) for $i \geq M$ that $|g(x_i)| \geq c\, d(x_i, S)^\alpha$ for constants $c > 0$ and $\alpha \geq 0$. Hence joining both inequalities $c\, d(x_i, S)^\alpha \leq d(x_i, S)^r$ for $i \geq \max\{M, N\}$. If we choose $r > \alpha$ and let $i \to \infty$ we get $c = 0$. □

Remark For g germ not identically zero we can choose $g(x_i) \neq 0 \ \ \forall i$.

Definition 21 *Let S be a closed subset of $\mathbb{R}^n$, then $M(S,n)$ are the maps $\phi : \mathbb{R}^n - S \to \mathbb{R}$ such that if K is a germ of a compact set and k is a multi-index of natural numbers, there exists constants $c > 0$ and $\alpha > 0$ such that $|D^k\phi(x)| \leq c\, d(x,S)^{-\alpha}$ for all x in $K - S$.*

Theorem 22 *Let $\phi \in M(S,n)$ and $f \in \varepsilon(S,n)$. Then we can extend ϕf in a unique way to a germ in $\varepsilon(S,n)$ denoted also by ϕf.*

Proof: Let $x_0 \in S$ and $K_{x_0} \subseteq K_{x_0}$ compact sets containing x_0 such that, for $x \in K_{x_0}$ there exists $a_x \in S \cap K_{x_0}$ with $d(x,S) = d(x,a_x)$. Let $f \in \varepsilon(S,n)$, since f is flat along S, given m natural number, there exists c' (independent of f) such that $|D^k f(x)| \leq c' \, |f|_m^{K'_{x_0}} \, d(x,S)^{m-|k|} \ \forall x \in K_{x_0}$ and $\forall |k| \leq m$. Let $x \in K - S$ and m to find, then

$$
\begin{aligned}
|D^\ell(\varphi \cdot f)(x)| &\leq \sum_{s \leq \ell} \binom{\ell}{s} |D^s\phi(x)| \left|D^{\ell-s} f(x)\right| \\
&\leq \sum_{s \leq \ell} \binom{\ell}{s} c\, d(x,X)^{-\alpha} c' |f|_m^{K'_{x_0}} \, d(x,X)^{m-|\ell-s|} \\
&\leq c'' \, |f|_m^{K'_{x_0}} \sum_{s \leq \ell} \binom{\ell}{s} d(x,X)^{m-|\ell-s|-\alpha} \\
&\leq c''' \, |f|_m^{K'_{x_0}} \, d(x,X) \quad \text{for} \quad m \geq |\ell| + \alpha + 1
\end{aligned}
$$

Therefore $D^\ell(\varphi \cdot f)$ extends continuously as zero in S. Using Proposition 4.2, Chapter IV of [T], $\varphi \cdot f$ extends smoothly and it is flat on X. □

Theorem 23 *Let f be a germ and S closed with $S = cl(\text{int } S)$ and $\bar{0} \in S$. Suppose that f satisfies a Jacobi-Lojasiewicz condition at S. Then f is S-infinitesimally stable and therefore stable.*

Proof: Consider $g = |\nabla f|^2$, we shall show that $\varepsilon(S,n) \subseteq \varepsilon(S,n)\left\langle \frac{\partial f}{\partial x_1} \right\rangle$. Let K be a compact germ and g_1 representative of g, then for each k multi-index, there exists c_k constant such that $\left| D^k \left(\frac{1}{g_1} \right)(x) \right| \leq \frac{c_k}{|g_1(x)|^{|k|+1}}$ $\forall x \in K$. Since g_1 satisfies a Lojasiewicz inequality for S, there exists $c > 0$ and $\alpha \geq 0$ such that $|g_1(x)| \geq c\, d(x,S)^\alpha$ $\forall x \in K$ and therefore $\left| D^k \left(\frac{1}{g_1(x)} \right) \right| \leq \frac{c_k}{c^{|k|+1} d(x,S)^{\alpha(|k|+1)}}$ $\forall x \in K - S$, hence $\frac{1}{g_1} \in M(S,n)$. Now for $h \in \varepsilon(S,n)$ and $x \notin S$ we have $h(x) = \frac{h(x)}{g_1(x)} g_1(x)$, extend $\frac{h(x)}{g_1(x)}$ to a germ H in $\varepsilon(S,n)$ and we get $h = Hg_1 \in \varepsilon(S,n)\left\langle \frac{\partial f}{\partial x_i} \right\rangle$. Therefore we get $\varepsilon(S,n) \subseteq \varepsilon(S,n)\left\langle \frac{\partial f}{\partial x_i} \right\rangle$ and by Proposition 9, f is S-infinitesimally stable. □

Lemma 24 (Lemma 3.3, [W]) *Suppose there exist a sequence (w_i) in $\mathbb{R} \times J^k(n,1)$ ($k \leq \infty$), a sequence (x_i) in $\mathbb{R}^n - \{\bar{0}\}$ converging to the origin and a germ f such that $g_i = w_i - j^k f(x_i)$ is flat along (x_i). Then there exists a germ g such that $j^k g(x_{i_j}) = (w_{i_j})$ for (x_{i_j}) subsequence of (x_i).*

Definition 25 *Consider q_i map germs from $\mathbb{R}^n$ to $\mathbb{R} \times J^k(\mathbb{R}^n, \mathbb{R})$, S a closed set containing the origin and (x_i) a sequence in $\mathbb{R}^n - S$ converging to the origin. We say that (q_i) are flat along (x_i) if given $r > 0$ there exists $N = N(r)$ such that $|q_i(x_i)| \leq d(x_i, S)^r$ $\forall i \geq N$.*

Lemma 26 *Let f be a germ, (w_i) sequence in $\mathbb{R} \times J^k(\mathbb{R}^n, \mathbb{R})$, (x_i) in $\mathbb{R}^n - S$ converging* *to zero, where S is a closed subset of $\mathbb{R}^n$ containing the origin. Suppose that $(q_i(y)) = (w_i - j^k f(y))$ is a sequence flat along (x_i). Then there exists a g such that $(j^k g)(x_i) = w_i$ $\forall i$ and $j^\infty g(x) = j^\infty f(x)$ $\forall x \in S$.*

Proof: Following Lemma 3.3 [W], we can suppose $k = \infty$ by considering zero all terms of order bigger than k. We define Q a Taylor vector field given by $q_i(x_i)$ at x_i and the zero series at S. Moreover, given $r > 0$, there exists $N = N(r)$ such that $|q_i(x_i)| \leq d(x_i, S)^r$ for $i \geq N$. It is enough to show that $\left(R_y^m Q\right)^k(x) = o\left(|x-y|^{m-|k|}\right)$, where $\left(R_y^m Q\right)^k(x) = Q^k(x) - \sum_{|L| \leq m-|k|} Q^{k+L}(y)(x-y)^L/L!$ (Prop 1.5, Chapter IV of [T]).

(i) If x and y are in $\{x_i\} \cup \{\bar{0}\}$ same proof as Lemma 20.

(ii) If x and y in S is obvious.

(iii) If $x \in \{x_j\}$ and $y \in S$, $\left(R_y^m Q\right)^k(x_j) = Q^k(x_j)$ and $|Q^k(x_j)| \leq d(x_j, S)^r \leq d(x_j, y)^r$, put $r = m+1$.

(iv) If $x \in S$ and $y = x_j$, $\left(R_{x_j}^m Q\right)^k(x) = -\sum_{|L| \leq m-|k|} Q^{k+L}(x_j)\dfrac{(x-x_j)}{L!}$, same as iii).

Hence $\left(R_y^m Q\right)^k(x) = o\left(|x-y|^{m-|k|}\right)$ and by Whitney Extension Theorem (Thm 3.1, Chapter IV of [T]), there exists a smooth germ q such that $j^\infty q(x_i) = q_i$ and $j^\infty q(x) = 0$ $\forall x \in S$, let $g = f + q$, hence g has the desired properties. In fact if $S = cl(\text{int } S)$, we get $q \equiv 0$ on S and $g \equiv f$ on S.

Theorem 27 *Let f be a germ, S a closed subset of $\mathbb{R}^n$ with $S = cl(\text{int } S)$. Hence if f is S-stable, then $|\nabla f|$ satisfies a Jacobi-Lojasiewicz condition for S.*

Proof: We shall prove the theorem by contradiction. We have a sequence (x_i) converging to the origin such that for each natural number n there exists a natural number $N(n)$ such that $|\nabla f(x_i)| < d(x_i, S)^n \quad \forall\, i \geq N$. Choose (y_i) a sequence of regular values of f converging to zero and such that $|f(x_n) - y_n| \leq d(x_n, S)^n \quad \forall n$. Let m natural number. then if $n \geq m$ we get $|f(x_n) - y_n| \leq d(x_n, S)^n \leq d(x_n, S)^m$, therefore the sequences $(\nabla f(x_i))$ and $(f(x_i) - y_i)$ are flat along (x_i) and so is $(y_i, \overline{0}) - (f(x_i), \nabla f(x_i))$. If we denote $q_i(y) = (y_i, \overline{0}) - (f(y), \nabla f(y))$, by the previous lemma with $k = 1$, there exists a germ g such that $j^1 g(x_i) = (y_i, \overline{0})$ and $g - f \in \varepsilon(S, n)$. But since f is S-stable, f and g must have the same critical and regular values, which is not the case, since the points y_i are regular values for f, but critical for g. $\square$

As a consequence of Proposition 9 and theorems 23 and 27 we get the following:

Theorem 28 *The concepts of S-infinitesimally stability, S-stability and the Jacobi-Lojasiewicz condition at S are equivalent for* $S = cl(\text{int } S)$.

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BRASIL TERRA LEME

Departamento de Matemática
Universidade Federal de São Carlos
13565-905 - São Carlos, S.P. - BRASIL

LEON KUSHNER

Departamento de Matemática
Facultad de Ciencias
UNAM
México D.F. 04510 - MEXICO

and

Instituto de Ciências Matemáticas de São Carlos
Caixa Postal, 668
13560-970 - São Carlos - S.P. - BRASIL