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SOME REMARKS ON IDEALS OF THE ALGEBRA OF SMOOTH FUNCTION GERMS

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Resumo

Considere $\epsilon(n)$ o álgebra local de germens suaves de $\mathbb{R}^n$ a $\mathbb{R}$ e $m(n)$ seu ideal maximal. Se I é um ideal de $\epsilon(n)$ precisamos saber quando $I \cap m(n)^k$ é finitamente gerado. Neste trabalho mostramos que esta afirmação é sempre certa. Para a recíproca precisamos de condições mas fortes, suporemos que I seja ideal radical e k um número particular. Com estas condições obtemos também a equivalência de $I \cap m(n)^k$ e $I \cdot m(n)^k$ sejam finitamente gerados.

Some Remarks on Ideals of the Algebra of Smooth Function Germs

BRASIL TERRA LEME and LEÓN KUSHNER

Abstract

In this paper we study for the local algebra $\varepsilon(n)$ of germs of $\mathbb{R}^n$ to $\mathbb{R}$ with maximal ideal $m(n)$ and an ideal I, conditions for the ideals I, $I \cap m(n)^k$ and $I \cdot m(n)^k$ to be finitely generated.

Introduction

Consider $\varepsilon(n)$ the local algebra of smooth germs of $\mathbb{R}^n$ to $\mathbb{R}$ and $m(n)$ its maximal ideal. If I is an ideal of $\varepsilon(n)$ we need to know for the theory of finite relative determination when $I \cap m(n)^k$ is finitely generated. In this paper we prove that this assertion is always true. For the converse we need stronger conditions, I must be a radical ideal and for a suitable k. We also get with these conditions the equivalence of $I \cap m(n)^k$ be finitely generated and $I \cdot m(n)^k$ be finitely generated.

Main results

Lemma 1 *Let A be a commutative ring,I, J and K ideals of A such that $I \subseteq K$. Then we have the equality*

$$(I + J) \cap K = I + (J \cap K).$$

Proof: Let $x \in (I+J) \cap K$, then we have $x \in K$ and $x = y + z$ with $y \in I$ and $z \in J$. Since x and y are elements of K so is z and $x = y + z$ with $y \in I$ and $z \in J \cap K$, therefore $(I+J) \cap K \subseteq I + J \cap K$. The other inclusion is obvious. $\square$

Remark The same proof will work for V_1, V_2 and V_3 subspaces of V with $V_1 \subseteq V_3$, i.e. $(V_1 + V_2) \cap V_3 = V_1 + (V_2 \cap V_3)$.

Theorem 2 *Let I be a finitety generated ideal of $\varepsilon(n)$. Then $I \cap m(n)^k$ is also finitely generated for all k.*

Proof: Let $g_1, \ldots, g_s$ be generators of I and $x \in I$, then $x = \sum_{i=1}^{s} a_i g_i = \sum_{i=1}^{s} a_i^{(k)} g_i + \sum_{i=1}^{s} m_i^{(k)} g_i$ with $a_i^{(k)}$ polynomials of degree at most $k-1$ and $m_i^{(k)} \in m(n)^k$. If we denote by S the vector space generated by $\{b_i g_j\}$ where b_i are the monomials of degree at most $k-1$, we will get $I = S + Im(n)^k$ and hence $I \cap m(n)^k = S \cap m(n)^k + Im(n)^k$. Let $\{v_1, \ldots, v_\ell\}$ be a basis of the subspace $S \cap m(n)^k$. If $f \in I \cap m(n)^k$, there exist real numbers $a_1, \ldots, a_\ell$ such that $f - \sum_{i=1}^{\ell} a_i v_i \in Im(n)^k$, hence we get

$$f = \sum_{i=1}^{\ell} a_i v_i + \sum_{j=1}^{s} \sum_{|I|=k} \left(g_j x^I\right) h_{I \cdot j}$$

where $x^I = x_1^{i_1} \ldots x_n^{i_n}$ and $|I| = \sum_{k=1}^{n} i_k$. Therefore $\{v_i\}$ and $\{g_j x^I\}$, with $1 \leq i \leq \ell$, $1 \leq j \leq s$ and $|I| = k$ generate $I \cap m(n)^k$ as an ideal of $\varepsilon(n)$ $\square$

Consider now $\pi : \varepsilon(n) \longrightarrow \mathbb{R}[[x]]$ the canonical projection to the local algebra of formal series which assigns to each germ its taylor series around the origin. Let I an ideal of $\varepsilon(n)$, hence $\pi(I)$ is a finitely generated ideal, say $\pi(I) = \{s_1, \ldots, s_t\}$. Let $g_i \in I$ such that $\pi(g_i) = s_i$, hence we get $I + m(n)^\infty = \langle g_1, \ldots, g_t \rangle + m(n)^\infty$, and if we intersect each side of the equality with I we get

$$I = \langle g_1, \ldots, g_t \rangle + I \cap m(n)^\infty \qquad (*)$$

Definition 3 *Let I ideal of $\varepsilon(n)$ and $z(I)$ the germ of common zeroes of I, we let $\widehat{I}$ be the ideal of germs vanishing at $z(I)$ (we will suppose $\overline{0} \in z(I)$). We say that I is radical if $\widehat{I} = I$.*

Lemma 4 ([K]) *Let I be a radical ideal then $Im(n)^\infty = I \cap m(n)^\infty$ and we write equation $(*)$ as $I = \langle g_1, \ldots, g_t \rangle + I \cdot m(n)^\infty$.*

In particular comparing I with $I \cap m(n)^k$ we have $\{s_1, \ldots, s_r\}$ generators of $\pi(I \cap m(n)^k)$ and $\{s_1, \ldots, s_r, s_{r+1}, \ldots, s_t\}$ generators of $\pi(I)$, hence we will get

$$I \cap m(n)^k = \langle g_1, \ldots, g_r \rangle + I \cap m(n)^\infty \tag{**}$$

Definition 5 *Let I ideal of $\varepsilon(n)$ and $\pi(I)$ ideal in $\mathbb{R}[[x]]$, hence there exists a minimum k such that $\pi(I) \cap M^m = M^{m-k}\left(\pi(I) \cap M^k\right)$ for all $m \geq k$, (Here M is the maximal ideal of $\mathbb{R}[[x]]$). This k will be denoted by $A(I)$ and be called the Artin-Rees number of I.*

Lemma 6 ([K]) *Consider I a radical ideal and $k = A(I)$, then the following assertions are equivalent for $(**)$*

i) $I \cap m(n)^k$ *is a finitely generated ideal.*

ii) $I \cap m(n)^k = \langle g_1, \ldots, g_r \rangle$.

iii) $\langle g_1, \ldots, g_r \rangle \supseteq I \cap m(n)^\infty$.

Theorem 7 *Let I be a radical ideal, $k = A(I)$ and $I \cap m(n)^k$ finitely generated. Then I is finitely generated too.*

Proof: From Lemma 6 we get $\langle g_1, \ldots, g_r \rangle \supseteq I \cap m(n)^\infty$, hence $\langle g_1, \ldots, g_t \rangle \supseteq I \cap m(n)^\infty$ and from $(*)$, $I = \langle g_1, \ldots g_t \rangle$. □

Joining Theorems 2 and 7 we get the following

Theorem 8 *Let I be a radical ideal, $k = A(I)$. Then I is finitely generated if and only if $I \cap m(n)^k$ is finitely generated.*

Therefore with the conditions $I = \hat{I}$ and $k = A(I)$, we get that if $I \cap m(n)^k$ is finitely generated so is $I \cdot m(n)^k$.

With a similar argument as in Lemma 6, in fact, since $Im(n)^\infty = I\left(m(n)^k m(n)^\infty\right) = \left(Im(n)^k\right) m(n)^\infty$. Then $Im(n)^k = \langle h_1, \ldots, h_\ell\rangle + Im(n)^\infty$, and we get the following

Lemma 9 *Let I be a radical ideal of $\varepsilon(n)$ and $k = A(I)$. Then the following assertions are equivalent*

i) $Im(n)^k$ is finitely generated.

ii $Im(n)^k = \langle h_1, \ldots, h_\ell\rangle$.

iii) $\langle h_1, \ldots, h_\ell\rangle \supseteq Im(n)^\infty$.

Proposition 10 *Let I be a radical ideal and $k = A(I)$. Then $I \cap m(n)^k$ is finitely generated if and only if $Im(n)^k$ is finitely generated.*

Proof: Necessity was proven in Theorem 8. For the sufficency, let us suppose that $Im(n)^k = \langle h_1, \ldots, h_\ell\rangle$ and $I \cap m(n)^k = \langle h_1, \ldots, h_\ell, h_{\ell+1}, \ldots, h_m\rangle + I \cap m(n)^\infty$. Since $I \cap m(n)^\infty = Im(n)^\infty \subseteq Im(n)^k = \langle h_1, \ldots, h_\ell\rangle$, we get $I \cap m(n)^k = \langle h_1, \ldots, h_\ell, h_{\ell+1}, \ldots h_m\rangle$.
□

Theorems 8 and 10 give rise to our main theorem.

Theorem 11 *Let I be a radical ideal, $k = A(I)$. Consider the ideals I, $I \cap m(n)^k$ and $I \cdot m(n)^k$. If one of them is finitely generated so are the other two.*

Bibliography

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