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ABSTRACT PARABOLIC PROBLEMS WITH CRITICAL NONLINEARITIES AND APPLICATIONS TO NAVIER-STOKES AND HEAT EQUATIONS

JOSÉ M. ARRIETA
ALEXANDRE N. CARVALHO

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Abstract Parabolic Problems with Critical Nonlinearities and Applications to Navier-Stokes and Heat Equations.

José M. Arrieta[*]
and
Alexandre N. Carvalho[†]

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Abstract

We prove a local existence and uniqueness theorem for abstract parabolic problems of the type $\dot{x} = Ax + f(t,x)$ when the nonlinearity f satisfies certain critical conditions. We apply this abstract result to the Navier-Stokes and Heat equations.

1 Introduction

In this paper we consider problems of the type

$$\begin{aligned} \dot{x} &= Ax + f(t,x), \ \ t > t_0 \\ x(t_0) &= x_0, \end{aligned} \tag{1}$$

where the linear operator $A : D(A) \subset X^0 \to X^0$ is a sectorial operator in the Banach space X^0. We will denote by X^α, $\alpha \geq 0$ the fractional power spaces associated to the

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operator A (see [HE, AM]) and by e^{At} the analytic semigroup generated by A. Without loss of generality we can assume that it is uniformly bounded, that is

$$\begin{aligned} \|e^{At}x\|_{X^\alpha} &\leq M\|x\|_{X^\alpha}, \ \alpha \geq 0 \\ \|e^{At}x\|_{X^\alpha} &\leq Mt^{-\alpha}\|x\|_X, \ \alpha \geq 0. \end{aligned} \tag{2}$$

In order to initiate the discussion let us consider for a moment that the map f is time independent and $t_0 = 0$. Therefore the problem above reads

$$\begin{aligned} &\dot{x} = Ax + f(x), \ t > 0 \\ &x(0) = x_0. \end{aligned} \tag{3}$$

It is well known now that if the map $f : X^1 \to X^\alpha$, for certain $\alpha > 0$, and it is Lipschitz on bounded sets of X^1, that is $\|f(x) - f(y)\|_{X^\alpha} \leq C(R)\|x - y\|_{X^1}$ for $\|x\|_{X^1}, \|y\|_{X^1} \leq R$, then the problem (3) is locally well posed in X^1. For each $x_0 \in X^1$ one seeks fixed points of the map T in the space $K(\tau, \mu) = \{x(t) \in C([0, \tau], X^1);\ x(0) = x_0,\ \|x(t)\|_{L^\infty(0,\tau,X^1)} \leq \|x_0\|_{X^1} + \mu\}$, where T is given by

$$(Tx)(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(x(s))ds. \tag{4}$$

The simple computations,

$$\|(Tx)(t) - (Ty)(t)\|_{X^1} \leq \textstyle\int_0^t (t-s)^{-1+\alpha}\|f(x(s)) - f(y(s))\|_{X^\alpha}ds$$

$$\leq C \textstyle\int_0^t (t-s)^{-1+\alpha}\|x(s) - y(s)\|_{X^1}ds \leq C \int_0^t (t-s)^{-1+\alpha}ds \sup_{0\leq s\leq t}\{\|x(s) - y(s)\|_{X^1}\}$$

and

$$\|(Tx)(t)\|_{X^1} \leq \|e^{At}x_0\|_{X^1} + \textstyle\int_0^t (t-s)^{-1+\alpha}\|f(x(s))\|_{X^\alpha}ds$$

$$\leq \|e^{At}x_0\|_{X^1} + C\textstyle\int_0^t (t-s)^{-1+\alpha}ds + C\int_0^t (t-s)^{-1+\alpha}ds \sup_{0\leq s\leq t}\{\|x(s)\|_{X^1}\}$$

together with the fact that $\|e^{At}x_0\|_{X^1} \stackrel{t\to 0^+}{\longrightarrow} \|x_0\|_{X^1}$, $\int_0^t (t-s)^{-1+\alpha}ds = t^\alpha \int_0^1 (1-s)^{-1+\alpha}ds \to 0$ as $t \to 0^+$, suggest that for $\mu > 0$ fixed we can choose $\tau > 0$ small enough such that $T : K(\tau, \mu) \to K(\tau, \mu)$ and that T is an strict contraction in $K(\tau, \mu)$. Once this is accomplished, the Banach fixed point theorem takes care of the existence and uniqueness of solutions of the integral equation. With some extra effort one can show that the solution found is a solution of the differential equation (3).

In the analysis above, the convergence of the improper integral $\int_0^1 (1-s)^{-1+\alpha}ds$, which is equivalent to the fact that $\alpha > 0$, is essential and the whole argument breaks down when $\alpha = 0$. In other words, since $A : X^1 \to X^0$, the fact that $f : X^1 \to X^\alpha$ with $\alpha > 0$

means that the solutions of problem (3) can be obtained as perturbations of the solutions of the linear problem $\dot{x} = Ax$.

In this paper we address the question of local solvability of problem (1), (3) when $\alpha = 0$.

It is clear that if the only requirements on f is that $f : X^1 \to X^0$ being locally Lipschitz, it will be impossible to show that problem (3) is well posed. For example, taking $f(x) = -2Ax$, which satisfies $f : X^1 \to X^0$ and is globally Lipschitz, we will have $\dot{x} = Ax + f(x) = -Ax$, which is not locally well posed, in general (if $A = \Delta$ then $\dot{x} = -Ax$ is the backwards heat equation). Hence, some extra conditions should be imposed on f to guarantee the existence of solutions of the above problem.

In order to illustrate the main ideas and techniques contained in this paper let us consider the particular example given by the equation,

$$\begin{aligned} &u_t = \Delta u + u|u|^{\rho-1}, \quad \Omega \\ &u = 0 \quad \partial\Omega \\ &u = (0) = u_0 \end{aligned} \tag{5}$$

where Ω is a bounded and smooth domain in $\mathbb{R}^3$ and $\rho > 1$.

It is well known that the operator Δ can be regarded as an unbounded operator in $X^0 = H^{-1}(\Omega)$ with domain $X^1 = H_0^1(\Omega)$. Moreover, the fractional power spaces are given by X^α which satisfy the following embedding properties,

$$\begin{aligned} &X^\alpha \subset H^{2\alpha-1}(\Omega), \qquad \alpha > 1/2 \\ &X^{\frac{1}{2}} = L^2(\Omega), \\ &X^\alpha \supset H^{2\alpha-1}(\Omega), \qquad \alpha < 1/2 \end{aligned} \tag{6}$$

(see [HE, AM]).

If $f(u) = u|u|^{\rho-1}$, then with some Sobolev embeddings and with (6), we can show that for $1 < \rho \le 3$, we have $f : X^1 \equiv H_0^1(\Omega) \to X^{\frac{1}{2}} \equiv L^2$; for $4 < \rho \le 5$ we have $f : X^1 \equiv H_0^1(\Omega) \to H^{\frac{3-\rho}{2}} \subset X^{\frac{5-\rho}{4}}$. Hence, for $1 < \rho < 5$, $f : X^1 \to X^\alpha$ for some $\alpha > 0$. For $\rho = 5$, $f : X^1 \to X^0$ and we are in the critical case $\alpha = 0$. But observe that for $\rho = 5$, again with some some Sobolev embeddings and (6), we get that if $\epsilon > 0$ is small then $f : X^{1+\epsilon} \to X^{5\epsilon}$, while the linear operator $A : X^{1+\epsilon} \to X^\epsilon$. This means that, although A and f can be regarded as of the same order in X^1, if we consider an slightly better space, $X^{1+\epsilon}$, then the map f regularizes more than A. Moreover, it can be seen that f satisfies,

$$\|f(u) - f(v)\|_{X^{5\epsilon}} \le c\|u - v\|_{X^{1+\epsilon}}(\|u\|^4_{X^{1+\epsilon}} + \|v\|^4_{X^{1+\epsilon}} + 1) \qquad \forall u, v \in X^{1+\epsilon}$$

$$\|f(u)\|_{X^{5\epsilon}} \le c\|u\|^5_{X^{1+\epsilon}}$$

In particular, this means that we can solve problem (5) with initial data in $X^{1+\epsilon}$.

Moreover, if we consider now a sequence of initial data $u_n \in X^{1+\epsilon}$ with $u_n \to u_0 \in X^1$ in X^1, with the following computation,

$$t^\epsilon \|u_n(t)\|_{X^{1+\epsilon}} \leq t^\epsilon \|e^{At}u_n\|_{X^{1+\epsilon}} + t^\epsilon \int_0^t (t-s)^{-1+4\epsilon} \|u_n(s)^5\|_{X^{5\epsilon}} ds$$

$$\leq t^\epsilon \|e^{At}u_n\|_{X^{1+\epsilon}} + t^\epsilon \int_0^t (t-s)^{-1+4\epsilon} s^{-5\epsilon} ds \sup_{0<s<t}\{s^\epsilon \|u_n(s)\|_{X^{1+\epsilon}}\}^5$$

and the fact that $t^\epsilon \|e^{At}u_n\|_{X^{1+\epsilon}} \stackrel{t\to 0^+}{\longrightarrow} 0$ uniformly on compacts of X^1 (see Lemma2, below), it is not difficult to see that if $\mu > 0$ is small enough, we can get a uniform time $\tau_1 > 0$, independent of n such that $t^\epsilon \|u_n(t)\|_{X^{1+\epsilon}} \leq \mu$ for all $t \in (0, \tau_1]$ and all n.

But also, for $0 < t \leq \tau_1$, we have

$$t^\epsilon \|u_n(t) - u_m(t)\|_{X^{1+\epsilon}} \leq t^\epsilon \|e^{At}(u_n - u_m)\|_{X^{1+\epsilon}}$$

$$+t^\epsilon \int_0^t (t-s)^{-1+4\epsilon} \|u_n(s) - u_m(s)\|_{X^{1+\epsilon}} (1 + \|u_n(s)\|^4_{X^{1+\epsilon}} + \|u_m(s)\|^4_{X^{1+\epsilon}}) ds$$

$$\leq M\|u_n - v_n\|_{X^1} + t^\epsilon \int_0^t (t-s)^{-1+4\epsilon} \|u_n(s) - u_m(s)\|_{X^{1+\epsilon}} ds$$

$$+\mu^4 t^\epsilon \int_0^t (t-s)^{-1+4\epsilon} s^{-4\epsilon} \|u_n(s) - u_m(s)\|_{X^{1+\epsilon}} ds$$

which implies that for some $0 < t \leq \tau_0 \leq \tau_1$ we get

$$t^\epsilon \|u_n(t) - u_m(t)\|_{X^{1+\epsilon}} \leq C\|u_n - u_m\|_{X^1}.$$

This will allow us to go to the limit when $n \to \infty$ and obtain solutions in the space $C([0\tau_0], X^1) \cap C((0, \tau_0], X^{1+\epsilon})$ with initial conditions in X^1.

2 Abstract Results

Motivated by the discussion in the introduction, we define the concept of ϵ-regular solution and ϵ-regular map:

Definition 1 *We say that $x : [t_0, \tau] \to X^1$ is an $\epsilon-$regular mild solution ($\epsilon-$solution for short) to (1) if $x \in C([t_0, \tau], X^1) \cap C((t_0, \tau], X^{1+\epsilon})$, and $x(t)$ satisfies*

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-s)} f(s, x(s)) ds. \tag{7}$$

Definition 2 *For $\epsilon \geq 0$, we will say that a map g, is an $\epsilon-$regular map relatively to the pair (X^1, X^0), if there exist $\rho > 1$, $\gamma(\epsilon)$ with $\rho\epsilon \leq \gamma(\epsilon) < 1$ and a constant c, such that $g : X^{1+\epsilon} \to X^{\gamma(\epsilon)}$ and*

$$\|g(x) - g(y)\|_{X^{\gamma(\epsilon)}} \leq c\|x - y\|_{X^{1+\epsilon}}(\|x\|^{\rho-1}_{X^{1+\epsilon}} + \|y\|^{\rho-1}_{X^{1+\epsilon}} + 1) \qquad \forall x, y \in X^{1+\epsilon} \tag{8}$$

Let us consider the following class of nonlinearities: with ϵ, ρ, $\gamma(\epsilon)$, c, positive constants, and $\nu(t)$ with $0 \leq \nu(t) \leq \delta$ $\lim_{t\to 0^+} \nu(t) = 0$, define $\mathcal{F} := \mathcal{F}(\epsilon, \rho, \gamma(\epsilon), c, \nu(\cdot))$ as the family of functions f such that for $t > 0$ $f(t, \cdot)$ is an ϵ-regular map relative to the pair (X^1, X^0), satisfying:

$$\|f(t, x) - f(t, y)\|_{X^{\gamma(\epsilon)}} \leq c\|x - y\|_{X^{1+\epsilon}}(\|x\|^{\rho-1}_{X^{1+\epsilon}} + \|y\|^{\rho-1}_{X^{1+\epsilon}} + \nu(t)t^{-\gamma(\epsilon)+\epsilon}) \tag{9}$$

$$\|f(t, x)\|_{X^{\gamma(\epsilon)}} \leq c(\|x\|^{\rho}_{X^{1+\epsilon}} + \nu(t)t^{-\gamma(\epsilon)}). \tag{10}$$

for all $x, y \in X^{1+\epsilon}$.

Without loss of generality we can assume that the function $\nu(t)$ is non-decreasing.

In most cases in the argument below we will fix the parameters ϵ, ρ, $\gamma(\epsilon)$ and c, and we will denote the class $\mathcal{F}$ defined above by $\mathcal{F}(\nu(\cdot))$.

Let M be such that the following holds,

$$t^{1+\alpha-\beta}\|e^{At}x\|_{X^{1+\alpha}} \leq M\|x\|_{X^\beta}, \ 0 \leq \beta \leq 1 + \alpha \leq 2.$$

With these definitions we have the following

Theorem 1 *Let $f \in \mathcal{F}(\epsilon, \rho, \gamma(\epsilon), c, \nu(\cdot))$. If $y_0 \in X^1$ there exist $r > 0$, and $\tau_0 > 0$ with the property that for any $x_0 \in B_{X^1}(y_0, r)$ there exists a continuous function $x(\cdot, x_0) : [0, \tau_0] \to X^1$, with $x(0) = x_0$, which is the unique $\epsilon-$regular mild solution starting at x_0 of the problem*

$$\begin{aligned} &\dot{x} = Ax + f(t, x), \ t > 0 \\ &x(0) = x_0. \end{aligned} \tag{11}$$

This solution satisfies

$$x \in C((0, \tau_0], X^{1+\theta}), \quad 0 \leq \theta < \gamma(\epsilon)$$

$$t^\theta\|x(t, x_0)\|_{X^{1+\theta}} \to 0, \qquad \text{as } t \to 0, \quad 0 < \theta < \gamma(\epsilon)$$

Moreover, if $x_0, z_0 \in B_{X^1}(y_0, r)$ the following holds

$$t^\theta\|x(t, x_0) - x(t, z_0)\|_{X^{1+\theta}} \leq C\|x_0 - z_0\|_{X^1}, \ \forall t \in [0, \tau_0], \qquad 0 \leq \theta \leq \theta_0 < \gamma(\epsilon)$$

Also, if $\gamma(\epsilon) > \rho\epsilon$, then r can be chosen arbitrarily large. That is, the time of existence is uniform on bounded sets of X^1.

If $t \to f(t, x)$, as a map from $(0, \infty)$ to $X^{\gamma(\epsilon)}$, is locally Hölder continuous, uniformly on bounded sets of $x \in X^{1+\gamma(\epsilon)}$, then $x \in C^1((0, \tau_0], X^{\gamma(\epsilon)}) \cap C((0, \tau_0], X^{1+\gamma(\epsilon)})$, and $x(\cdot, x_0)$ is an strict solution of (11).

The constants above depend on the following: $\tau_0 = \tau_0(y_0, A, \nu(\cdot), \epsilon, \rho, \gamma(\epsilon), c, M)$, $r = r(y_0, \epsilon, \rho, \gamma(\epsilon), c, M)$, $C = C(\theta_0, \epsilon, \rho, \gamma(\epsilon), M)$.

Before we prove this theorem we will need some lemmas.

Lemma 1 *The operators* $t^\alpha e^{-At} : X^1 \to X^{1+\alpha}$, $t > 0$, *are bounded linear operators satisfying* $\|t^\alpha e^{-At}\|_{L(X^1,X^{1+\alpha})} \leq M$, *with* M *independent of* t. *Moreover given a compact subset* J *of* X^1 *we have*

$$\lim_{t\to 0^+} \sup_{x\in J} \|t^\alpha e^{-At}x\|_{X^{1+\alpha}} = 0.$$

Proof: The fact that $\|t^\alpha e^{-At}\|_{L(X^1,X^{1+\alpha})} \leq M$ is standard and can be found, for example, in [HE]. For the remaining part we proceed as follows.

Given $\eta > 0$ there are $x_1, \cdots, x_n$ in J such that $J \subset \cup_{i=1}^n B_{X^1}(x_i, \eta)$. Since $X^{1+\alpha}$ is dense in X^1, there are $y_1, \cdots, y_n$ in $X^{1+\alpha}$ such that $J \subset \cup_{i=1}^n B_{X^1}(y_i, 2\eta)$. Then, we have

$$\|t^\alpha e^{-At}x\|_{X^{1+\alpha}} \leq \|t^\alpha e^{-At}(x-y_i)\|_{X^{1+\alpha}} + \|t^\alpha e^{-At}y_i\|_{X^{1+\alpha}}$$

for i such that $x \in B_{X^1}(y_i, 2\eta)$. Therefore

$$\begin{aligned}\|t^\alpha e^{-At}x\|_{X^{1+\alpha}} &\leq M\|(x-y_i)\|_{X^1} + t^\alpha M\|y_i\|_{X^{1+\alpha}} \\ &\leq 2M\eta + t^\alpha M \max_{1\leq i\leq n} \|y_i\|_{X^{1+\alpha}}.\end{aligned}$$

Thus

$$\limsup_{t\to 0^+} \sup_{x\in J} \|t^\alpha e^{-At}x\|_{X^{1+\alpha}} \leq 2M\eta,$$

since $\eta > 0$ is arbitrary, we have proved the lemma.

Let us recall the definition of the beta function $\mathbf{B}(\cdot,\cdot) : (0,\infty) \times (0,\infty) \to (0,\infty)$, which is

$$\mathbf{B}(a,b) = \int_0^1 (1-x)^{a-1}x^{b-1}\,dx.$$

Define

$$\mathbf{B}_\epsilon^\theta = \max_{0\leq\xi\leq\theta}\{\mathbf{B}(\gamma(\epsilon), 1-\gamma(\epsilon)), \mathbf{B}(\gamma(\epsilon), 1-\rho\epsilon), \mathbf{B}(\gamma(\epsilon)-\xi, 1-\gamma(\epsilon)), \mathbf{B}(\gamma(\epsilon)-\xi, 1-\rho\epsilon)\}.$$

Lemma 2 *Let* $f \in \mathcal{F}(\nu(\cdot))$. *If* $x \in C((0,\tau], X^{1+\epsilon})$, *then for all* $0 \leq \theta < \gamma(\epsilon)$, *we have*

$$t^\theta \|\int_0^t e^{A(t-s)} f(s,x(s))ds\|_{X^{1+\theta}} \leq cM\mathbf{B}_\epsilon^\theta(\nu(t) + t^{\gamma(\epsilon)-\rho\epsilon}[\lambda(t)]^\rho) \quad 0 < t \leq \tau,$$

where $\lambda(t) := \sup_{s\in(0,t]}\{s^\epsilon \|x(s)\|_{X^{1+\epsilon}}\}$.

Proof. It is not difficult to see that the following holds,

$$t^{\theta} \quad \| \int_0^t e^{A(t-s)} f(s,x(s))ds\|_{X^{1+\theta}} \leq Mt^{\theta} \int_0^t (t-s)^{-1+\gamma(\epsilon)-\theta} \|f(s,x(s))\|_{X^{\gamma(\epsilon)}} ds$$

$$\leq cMt^{\theta} \int_0^t (t-s)^{-1+\gamma(\epsilon)-\theta} (\nu(s)t^{-\gamma(\epsilon)} + \|x(s)\|^{\rho}_{X^{1+\epsilon}})ds$$

$$\leq cMt^{\theta}\delta \int_0^t (t-s)^{-1+\gamma(\epsilon)-\theta} s^{-\gamma(\epsilon)} ds + cMt^{\theta} \int_0^t (t-s)^{-1+\gamma(\epsilon)-\theta} s^{-\rho\epsilon} [s^{\epsilon} \|x(s)\|_{X^{1+\epsilon}}]^{\rho} ds$$

$$\leq cM\nu(t) \int_0^1 (1-s)^{-1+\gamma(\epsilon)-\theta} s^{-\gamma(\epsilon)} ds + cMt^{\gamma(\epsilon)-\rho\epsilon} [\lambda(t)]^{\rho} \int_0^1 (1-s)^{-1+\gamma(\epsilon)-\theta} s^{-\rho\epsilon} ds$$

$$= cM\mathbf{B}^{\theta}_{\epsilon} [\nu(t) + t^{\gamma(\epsilon)-\rho\epsilon}[\lambda(t)]^{\rho}],$$

from where the lemma follows.

Lemma 3 *Let $f \in \mathcal{F}(\nu(\cdot))$ and $x, y \in C((0,\tau], X^{1+\epsilon})$ such that $t^{\epsilon}\|x(t)\|_{X^{1+\epsilon}} \leq \mu$, $t^{\epsilon}\|y(t)\|_{X^{1+\epsilon}} \leq \mu$, for some $\mu > 0$. Then, for all $0 \leq \theta < \gamma(\epsilon)$, we have*

$$t^{\theta}\| \int_0^t e^{A(t-s)} [f(s,x(s)) - f(s,y(s))]ds\|_{X^{1+\theta}} \leq \Gamma_{\theta}(t) \sup_{s\in[0,\tau]} s^{\epsilon}\|x(s) - y(s)\|_{X^1}$$

where

$$\Gamma_{\theta}(t) = cM\mathbf{B}^{\theta}_{\epsilon} \left[\nu(t) + t^{\gamma(\epsilon)-\rho\epsilon} 2\mu^{\rho-1}\right].$$

Proof: Using the $\epsilon-$ regularity property of f we have that

$$t^{\theta}\| \int_0^t e^{A(t-s)} [f(s,x(s)) - f(s,y(s))]ds\|_{X^{1+\theta}}$$

$$\leq t^{\theta} \int_0^t cM(t-s)^{-1+\gamma(\epsilon)-\theta} \|x(s) - y(s)\|_{X^{1+\epsilon}} (\nu(t)s^{-\gamma(\epsilon)+\epsilon} + \|x(s)\|^{\rho-1}_{X^{1+\epsilon}} + \|y(s)\|^{\rho-1}_{X^{1+\epsilon}})ds$$

$$\leq cMt^{\theta}\nu(t) \int_0^t (t-s)^{-1+\gamma(\epsilon)-\theta} s^{-\gamma(\epsilon)} s^{\epsilon}\|x(s) - y(s)\|_{X^{1+\epsilon}} ds +$$

$$cMt^{\theta} \int_0^t (t-s)^{-1+\gamma(\epsilon)-\theta} s^{-\rho\epsilon} \left[(s^{\epsilon}\|x(s)\|_{X^{1+\epsilon}})^{\rho-1} + (s^{\epsilon}\|y(s)\|_{X^{1+\epsilon}})^{\rho-1}\right] s^{\epsilon}\|x(s) - y(s)\|_{X^{1+\epsilon}} ds$$

$$= \Gamma_{\theta}(t) \sup_{t\in[0,\tau]} \{s^{\epsilon}\|x(s) - y(s)\|_{X^{1+\epsilon}}\}$$

Proof of Theorem 1: We will divide the proof in two parts, existence and uniqueness.

Existence. Define μ by

$$cM\mathbf{B}^{\epsilon}_{\epsilon}\mu^{\rho-1} = \frac{1}{8}$$

Let us choose now $r = r(\mu, M) > 0$ such that

$$r = \frac{\mu}{4M} = \frac{1}{4M(8cMB_\epsilon^\epsilon)^{\frac{1}{\rho-1}}} \tag{12}$$

Also, for y_0 fixed, choose $\tau_0 = \tau_0(y_0, A, \mu, \nu(\cdot), \epsilon, \rho, \gamma(\epsilon), c, M) \in (0, 1]$ such that $\nu(t) < \delta$ for $t \in (0, \tau_0]$ and

$$\begin{array}{l} \|t^\epsilon e^{-At} y_0\|_{X^{1+\epsilon}} \leq \frac{\mu}{2}, \quad 0 \leq t \leq \tau_0, \\ cM\delta \mathbf{B}_\epsilon^\epsilon = \min\{\frac{\mu}{4}, \frac{1}{4}\} \end{array} \tag{13}$$

Notice that these choices imply that $\Gamma_\epsilon(t) \leq \frac{1}{2}$ for $t \in (0, 1)$.

Since we will be looking for solutions which regularize immediately we search for solutions in

$$K(\tau_0) = \{x \in C((0, \tau_0], X^{1+\epsilon}) : \sup_{t \in (0, \tau_0]} t^\epsilon \|x(t)\|_{X^{1+\epsilon}} \leq \mu\}.$$

with the norm

$$\|x\|_{K(\tau_0)} = \sup_{t \in [0, \tau_0]} t^\epsilon \|x(t)\|_{X^{1+\epsilon}}.$$

Consider $x_0 \in X^1$ with $\|x_0 - y_0\|_{X^1} < r$, $f \in \mathcal{F}(\delta)$ and on $K(\tau_0)$ define the map

$$(Tx)(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} f(s, x(s)) ds.$$

We will show that, for all $x_0 \in B_{X^1}(y_0, r)$, T takes $K(\tau_0)$ into itself and that T is a strict contraction in $K(\tau_0, x_0)$.

Let us first prove that T is a well defined map and that $T(K(\tau_0)) \subset K(\tau_0)$. We start showing the following,

$$\text{if } x \in K(\tau_0) \quad \text{then} \quad Tx \in C((0, \tau_0], X^{1+\theta}), \qquad \forall \theta \in [0, \gamma(\epsilon)). \tag{14}$$

Fix $t_2 \in (0, \tau_0]$ and let $\tau_0 \geq t_1 > t_2$; then, for $0 \leq \theta < \gamma(\epsilon)$, we have:

$$\begin{aligned} \|(Tx)(t_1) - (Tx)(t_2)\|_{X^{1+\theta}} &\leq \|(e^{-At_1} - e^{-At_2}) x_0\|_{X^{1+\theta}} + \| \textstyle\int_{t_2}^{t_1} e^{A(t_1-s)} f(s, x(s)) ds\|_{X^{1+\theta}} \\ &+ \|[I - e^{-A(t_1-t_2)}] \textstyle\int_0^{t_2} e^{A(t_2-s)} f(s, x(s)) ds\|_{X^{1+\theta}}. \end{aligned}$$

In the above, the first and third term trivially go to zero when $t_1 \to t_2$. Let us consider the second term. For that we have

$$\| \textstyle\int_{t_2}^{t_1} e^{A(t_1-s)} f(s, x(s)) ds\|_{X^{1+\theta}} \leq c \textstyle\int_{t_2}^{t_1} M(t_1 - s)^{-1+\gamma(\epsilon)-\theta} (\delta s^{-\gamma(\epsilon)} + \|x(s)\|_{X^{1+\epsilon}}^\rho) ds$$

$$\leq cM\delta \textstyle\int_{t_2}^{t_1} (t_1 - s)^{-1+\gamma(\epsilon)-\theta} s^{-\gamma(\epsilon)} ds + cM \textstyle\int_{t_2}^{t_1} (t_1 - s)^{-1+\gamma(\epsilon)-\theta} s^{-\rho\epsilon} (s^\epsilon \|x(s)\|_{X^{1+\epsilon}})^\rho ds$$

$$\leq cM\delta t_1^{-\theta} \textstyle\int_{\frac{t_2}{t_1}}^1 (1 - s)^{-1+\gamma(\epsilon)-\theta} s^{-\gamma(\epsilon)} ds + cM\mu^\rho t_1^{\gamma(\epsilon)-\theta-\rho\epsilon} \textstyle\int_{\frac{t_2}{t_1}}^1 (1 - s)^{-1+\gamma(\epsilon)-\theta} s^{-\rho\epsilon} ds.$$

Which goes to zero as $t_1 \to t_2^+$. The case $t_1 < t_2$ is similar.

Let us now show that $t^\epsilon \|x(t)\|_{X^{1+\epsilon}} \leq \mu$, for all $t \in (0, \tau_0]$. In fact

$$t^\epsilon \quad \|(Tx)(t)\|_{X^{1+\epsilon}} \leq \|t^\epsilon e^{-At} x_0\|_{X^{1+\epsilon}} + cMt^\epsilon \int_0^t (t-s)^{-1+\gamma(\epsilon)-\epsilon} (\delta s^{-\gamma(\epsilon)} + \|x\|^\rho_{X^{1+\epsilon}}) ds$$

$$\leq \|t^\epsilon e^{-At} x_0\|_{X^{1+\epsilon}} + cMt^\epsilon \delta \int_0^t (t-s)^{-1+\gamma(\epsilon)-\epsilon} s^{-\gamma(\epsilon)} ds$$

$$+ cMt^\epsilon \int_0^t (t-s)^{-1+\gamma(\epsilon)-\epsilon} s^{-\rho\epsilon} (s^\epsilon \|x\|_{X^{1+\epsilon}})^\rho ds$$

$$\leq \|t^\epsilon e^{-At} x_0\|_{X^{1+\epsilon}} + cM\delta \mathbf{B}^\epsilon_\epsilon + cM\mathbf{B}^\epsilon_\epsilon \mu^\rho$$

$$\leq Mr + \|t^\epsilon e^{-At} y_0\|_{X^{1+\epsilon}} + cM\mathbf{B}^\epsilon_\epsilon \delta + cM\mathbf{B}^\epsilon_\epsilon \mu^\rho \leq \mu.$$

This shows that T takes $K(\tau_0)$ into itself.

The next step is to prove that the map T is a contraction from $K(\tau_0)$ into itself.

It follows from Lemma 3, by taking $\theta = \epsilon$, that T is a strict contraction in $K(\tau_0)$ and that

$$\|T(x) - T(y)\|_{K(\tau_0)} \leq \frac{1}{2} \|x - y\|_{K(\tau_0)}.$$

By the Banach contraction principle we have that T has a unique fixed point in $K(\tau_0)$. We will denote this fixed point by $X(t, x_0)$ which is defined for $\|x_0 - y_0\|_{X^1} < r$, $0 \leq t \leq \tau_0$. Note that, from (14) $X(\cdot, x_0) \in C((0, \tau_0], X^{1+\theta})$, for all $0 \leq \theta < \gamma(\epsilon)$.

Let us prove that $t^\theta \|X(t, x_0)\|_{X^{1+\theta}} \to 0$ as $t \to 0$ for all $0 < \theta < \gamma(\epsilon)$.

From Lemma 2

$$t^\theta \|X(t, x_0)\|_{X^{1+\theta}} \leq t^\theta \|e^{At} x_0\|_{X^{1+\theta}} + t^\theta \int_0^t \|e^{A(t-s)} f(s, X(s, x_0))\|_{X^{1+\theta}} ds$$

$$\leq t^\theta \|e^{At} x_0\|_{X^{1+\theta}} + cM\mathbf{B}^\theta_\epsilon \nu(t) + cM\mathbf{B}^\theta_\epsilon \mu^{\rho-1} \sup_{0<s\leq t} \{t^\epsilon \|X(t, x_0)\|_{X^{1+\epsilon}}\}$$

Therefore if $\theta = \epsilon$ we have

$$t^\epsilon \|X(t, x_0)\|_{X^{1+\epsilon}} \leq t^\epsilon \|e^{At} x_0\|_{X^{1+\epsilon}} + cM\mathbf{B}^\epsilon_\epsilon \nu(t) + \frac{1}{8} \sup_{0<s\leq t} \{t^\epsilon \|X(t, x_0)\|_{X^{1+\epsilon}}\}$$

from where we obtain

$$\sup_{0<s\leq t} \{t^\epsilon \|X(t, x_0)\|_{X^{1+\epsilon}} \leq \frac{8}{7} (t^\epsilon \|e^{At} x_0\|_{X^{1+\epsilon}} + cM\mathbf{B}^\epsilon_\epsilon \nu(t)) \to 0 \quad \text{as } t \to 0$$

If $0 < \theta < \gamma(\epsilon)$, from the above expressions we also obtain $t^\theta \|X(t, x_0)\|_{X^{1+\theta}} \to 0$ as $t \to 0$.

Let us prove now that

$$\lim_{t \to 0^+} \|X(t, x_0) - x_0\|_{X^1} = 0$$

In fact, from Lemma 2

$$\|X(t,x_0)-x_0\|_{X^1} \le \|e^{At}x_0 - x_0\|_{X^1} + \int_0^t \|e^{A(t-s)} f(s,X(s,x_0))\|_{X^1} ds$$

$$\le \|e^{At}x_0 - x_0\|_{X^1} + cM\mathbf{B}_\epsilon(\nu(t) + [\sup_{0<s\le t}\{t^\epsilon \|X(t,x_0)\|_{X^{1+\epsilon}}\}]^\rho) \xrightarrow{t\to 0^+} 0,$$

With all these we have that $X(t,x_0)$ is an $\epsilon-$regular solution starting at x_0 and it is the unique ϵ-regular solution starting at x_0, in the set $K(\tau_0)$. We will hereafter call it the $K-$solution starting at x_0.

Moreover, if $x_0,\ z_0 \in B_{X^1}(y_0,r)$, taking into account the estimates of Lemma 3, and our choice of τ_0, we have

$$\begin{aligned} & t^\theta \|X(t,x_0)-X(t,z_0)\|_{X^{1+\theta}} \\ & \le t^\theta \|e^{At}(x_0-z_0)\|_{X^{1+\theta}} + t^\theta \int_0^t \|e^{A(t-s)}[f(s,X(s,x_0)) - f(s,X(s,z_0))]\|_{X^{1+\theta}} ds \\ & \le M\|x_0-z_0\|_{X^1} + \Gamma_\theta(t)\sup_{s\in[0,\bar{\tau}_0]} s^\epsilon \|X(s,x_0)-X(s,z_0)\|_{X^{1+\epsilon}}. \end{aligned} \tag{15}$$

For $\theta=\epsilon$ we get

$$t^\epsilon \|X(t,x_0)-X(t,z_0)\|_{X^{1+\epsilon}} \le M\|x_0-z_0\|_{X^1} + \frac{1}{2}\sup_{s\in[0,\bar{\tau}_0]} s^\epsilon \|X(s,x_0)-X(s,z_0)\|_{X^{1+\epsilon}}$$

which implies

$$t^\epsilon \|X(t,x_0)-X(t,z_0)\|_{X^{1+\epsilon}} \le 2M\|x_0-z_0\|_{X^1},$$

for $0\le\theta\le\theta_0<\gamma(\epsilon)$ we have from (15) that

$$t^\theta \|X(t,x_0)-X(t,z_0)\|_{X^{1+\theta}} \le M\|x_0-z_0\|_{X^1} + \Gamma_\theta(t)2M\|x_0-z_0\|_{X^1} \le C(\theta_0)\|x_0-z_0\|_{X^1}.$$

where $C(\theta_0)=M(1+2\sup\{\Gamma_\theta(t);\ t\in[0,\tau_0],\ 0\le\theta\le\theta_0\})$

This concludes the existence part of the theorem.

Uniqueness. Notice that from the existence part we have that for any $x_0\in B_{X^1}(y_0,r)$ and for any $f\in\mathcal{F}(\nu(\cdot))$ there exists a unique K-solution defined in $[0,\tau_0]$, of the problem

$$\begin{aligned} \dot{x} &= Ax + f(t,x) \\ x(0) &= x_0 \end{aligned} \tag{16}$$

To stress the dependence of the K-solution on f we will denote it by $X_f(t,x_0)$.

Consider the following

Lemma 4 *If $\phi(t)$ is an ϵ-regular solution in $[0,t_0]$ of (16) and $t^\epsilon\|\phi(t)\|_{X^{1+\epsilon}}\to 0$ as $t\to 0$, then $\phi(t)=X_f(t,x_0)$ for all $0\le t\le \min\{\tau_0,t_0\}$*

Proof. It is clear that $\phi \in K(\bar{\tau})$ for certain $\bar{\tau} \leq \tau_0$ small. Since we also have $X_f(\cdot, x_0) \in K(\bar{\tau})$ and both ϕ and $X_f(\cdot, x_0)$ are solutions of the integral equation we get $X_f(t, x_0) = \phi(t)$ for all $0 \leq t \leq \bar{\tau}$. With an standard continuation argument it is easy to see that we must have $X_f(t, x_0) = \phi(t)$ for all $0 \leq t \leq \min\{t_0, \tau_0\}$. This concludes the proof of the lemma.

Lemma 5 *If $f \in \mathcal{F}(\nu(\cdot))$ then $f_a \in \mathcal{F}(\nu_a(\cdot))$ for all $a \geq 0$, where $f_a(t, x) \equiv f(t + a, x)$ and $\nu_a(t) = \nu(t + a)(t/t + a)^{\gamma(\epsilon)} \leq \nu(t + a)$. Moreover, there exists $a_0 > 0$, small, such that for all $a \in [0, a_0]$ the time of existence $\tau_0(a)$ given by (13) can be chosen independent of a.*

Proof. The first part of the lemma is trivial.

For the second one we just need to observe that if $\nu(t) < \delta$ for $t \in [0, \tau_0]$ then, for a small, we will also have $\nu(t + a) < \delta$ for $t \in [0, \tau_0]$.

Following similar ideas as in [BC], we can prove now the uniqueness of ϵ-regular solutions.

Let $\phi(t)$, $0 \leq t \leq t_0$, be an $\epsilon-$regular solution starting in $x_0 \in B_{X^1}(y_0, r)$. From the fact that $\phi \in C([0, t_0], X^1)$ we have that there exists a $0 < a_0 \leq t_0$ such that $\phi(a) \in B_{X^1}(y_0, r)$. Notice that $\phi_a(\cdot) \equiv \phi(a + \cdot) \in C([0, t_0 - a], X^{1+\epsilon})$, and therefore $t^\epsilon \|\phi_a(t)\|_{X^{1+\epsilon}} \to 0$ as $t \to 0$. Moreover, ϕ_a is an ϵ-regular mild solution of

$$\begin{aligned} \dot{x} &= Ax + f_a(t, x) \\ x(0) &= \phi(a) \end{aligned} \tag{17}$$

From Lemma 5 and the results of the existence part we have that there exists a unique K-solution of problem (17), $X_{f_a}(t, \phi(a))$, defined in $[0, \tau_0]$. Moreover, from Lemma 4, we get $X_{f_a}(t, \phi(a)) = \phi_a(t)$ for all $0 \leq t \leq \min\{\tau_0, t_0 - a\}$, for all $0 < a \leq a_0$. In particular this implies that without loss of generality we can assume that $t_0 \geq \tau_0$, since if this in not the case we can define $\tilde{\phi}(t) = \phi(t)$ for $0 \leq t \leq t_0$ and $\tilde{\phi}(t) = X_{f_{a_0}}(t - a_0, \phi(a_0))$ for $t_0 \leq t \leq \tau_0$, and from the results above $\tilde{\phi}$ is also an ϵ-regular solution starting at x_0.

In view of the definition of K-solution, the only thing we need to show is that $t^\epsilon \|\phi(t)\|_{X^{1+\epsilon}} \leq \mu$ for all $0 < t \leq \tau_0$. But, for $0 < a < a_0$,

$$t^\epsilon \|\phi(t)\|_{X^{1+\epsilon}} \leq t^\epsilon \|\phi(t) - \phi(t + a)\|_{X^{1+\epsilon}} + t^\epsilon \|X_{f_a}(t, \phi(a))\|_{X^{1+\epsilon}} \leq t^\epsilon \|\phi(t) - \phi(t + a)\|_{X^{1+\epsilon}} + \mu$$

For $0 < t \leq \tau_0$ fixed, letting $a \to 0$ we have that $t^\epsilon \|\phi(t) - \phi(t + a)\|_{X^{1+\epsilon}} \to 0$, which implites that $t^\epsilon \|\phi(t)\|_{X^{1+\epsilon}} \leq \mu$ for all $t \in (0, \tau_0]$. This concludes the uniqueness part of the theorem.

For the case where $\gamma(\epsilon) > \rho\epsilon$, we proceed as follows. Let us define $y(t) = x(a\,t)$, for some $a < 1$. The equation for y is $\dot{y} = \tilde{A}y + \tilde{f}(t, y)$ where $\tilde{f}(t, x) = af(at, x)$, $\tilde{A} = aA$. Moreover, notice that $x(t)$ is a solution of the original equation in $(0, \tau_0]$, if and only if

$y(t)$, $\tilde{t} \in [0, a\tau_0]$ is a solution of the new equation. For this new equation, applying the existence part of the theorem, we will have a positive number $\tilde{r}$ such that the conclusions of the theorem are valid. Notice that from (12), we will have that

$$\tilde{r} = \frac{1}{4\tilde{M}(8\tilde{c}\tilde{M}B_\epsilon^\epsilon)^{\frac{1}{\rho-1}}}$$

Where $\tilde{r}, \tilde{c}$ and $\tilde{M}$ are constants related to the new equations. Let us relate $\tilde{r}, \tilde{c}, \tilde{M}$ with r, c and M. Note that

$$t^{\alpha-\beta}\|(aA)^\alpha e^{aAt}\|_{X^0} = a^\beta (at)^\alpha \|A^\alpha e^{Aat}\|_{X^0} \leq Ma^\beta \|A^\beta x\|_{X^0} = M\|(aA)^\beta x\|_{X^0},$$

which implies that $\tilde{M} = M$. To see that $\tilde{c} = a^{\gamma(\epsilon)-\rho\epsilon+1-\rho}c$ observe that

$$\begin{aligned}
\|(aA)^{\gamma(\epsilon)}\tilde{f}(t,y)\|_{X^0} &= a^{\gamma(\epsilon)+1}\|A^{\gamma(\epsilon)}f(at,y)\|_{X^0} \\
&\leq a^{\gamma(\epsilon)+1-\rho(1+\epsilon)}c\left(\nu(at)a^{\rho(1+\epsilon)}(at)^{-\gamma(\epsilon)} + \|(aA)^{1+\epsilon}y\|_{X^0}^\rho\right) \\
&\leq a^{\gamma(\epsilon)+1-\rho(1+\epsilon)}c\left(\nu(at)(at)^{-\gamma(\epsilon)} + \|(aA)^{1+\epsilon}y\|_{X^0}^\rho\right) \\
&\leq a^{\gamma(\epsilon)+1-\rho(1+\epsilon)}c\left(\tilde{\nu}(t)t^{-\gamma(\epsilon)} + \|(aA)^{1+\epsilon}y\|_{X^0}^\rho\right)
\end{aligned}$$

where $\tilde{\nu}(t) = \nu(at)a^{-\gamma(\epsilon)}$. The computations with the Lipschitz properties of f are similar. From this we have that

$$\tilde{r} = \frac{1}{4\tilde{M}(8\tilde{c}\tilde{M}B_\epsilon^\epsilon)^{\frac{1}{\rho-1}}} = ra^{\frac{\rho\epsilon-\gamma(\epsilon)}{\rho-1}+1}.$$

This implies that if $\|aA(y_0 - x_0)\| < \tilde{r}$ there exists a $\tilde{\tau}_0$ such that the solution of $\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{f}(t,\tilde{x})$ starting in x_0 is defined in $[0, \tilde{\tau}_0]$. Therefore, if $\|A(y_0 - x_0)\| < ra^{\frac{\rho\epsilon-\gamma(\epsilon)}{\rho-1}}$ then $\dot{x} = Ax + f(t,x)$ has the solution $x(t,x_0) = \tilde{x}(\frac{t}{a}, x_0)$ defined in $[0, a\tilde{\tau}_0]$. Since a can be chosen arbitrarily small and $\gamma(\epsilon) > \rho\epsilon$ we have that solutions have a common interval of existence on bounded subsets of X^1.

Finally, if $t \to f(t,x)$ is locally Hölder continuous for $t > 0$, uniformly on bounded sets of $X^{1+\epsilon}$, using standard regularity arguments, see for example [HE], we obtain the regularity stated in the theorem.

This concludes the proof of the theorem.

From Theorem 1 we have the following

Corollary 1 *If f is as in Theorem 1 and if K is a precompact set in X^1, then there exists a $\tau_0 = \tau(K)$ such that the $\epsilon-$regular solution starting at x_0 exists for time τ_0 for any $x_0 \in K$*

Proof. From Theorem 1, for any $y_0 \in \bar{K} \equiv Cl(K)$ there exists a $r(y_0)$ and a $\tau(y_0)$ such that for any $x_0 \in X^1$ with $\|x_0 - y_0\|_{X^1} < r(y_0)$ the unique $\epsilon-$regular solution exists in $[0, \tau(y_0)]$. From compactness of $\bar{K}$ we can choose $y_1, \cdots, y_n \in \bar{K}$ such that $K \subset \cup B_{X^1}(y_i, r(y_i))$. Choosing $\tau_0 = \min\{\tau(y_i) : 1 \leq i \leq n\}$ we prove the corollary.

With this corollary it is not difficult to prove a result on the maximal time of existence of $\epsilon-$regular solutions,

Proposition 1 *If f is as in Theorem 1 and $x(t, x_0)$ is an $\epsilon-$regular solution starting at x_0 with a maximal time of existence τ_m, then either $\tau_m = \infty$ or $\lim_{t \to \tau_m^-} \|x(t, x_0)\|_{X^{1+\epsilon}} = \infty$.*

Proof. Assume $\tau_m < \infty$ and also that $\liminf_{t \to \tau_m^-} \|x(t, x_0)\|_{X^{1+\epsilon}} = M < \infty$. This means that there exists a sequence of $t_n \to \tau_m^-$ such that $\|x(t_n, x_0)\|_{X^{1+\epsilon}} \leq 2M$ for all n. This implies that the sequence $\{x(t_n, x_0)\}_{n \in N}$ is a bounded sequence in $X^{1+\epsilon}$ and therefore a precompact sequence in X^1. From the previous corollary and the uniqueness of $\epsilon-$regular solution we obtain the result of the proposition.

In many applications the map f is independent of time. In this respect we have the following results:

Corollary 2 *Assume that f is independent of time and that it is an $\epsilon-$regular map, for some $\epsilon > 0$. Then, if $y_0 \in X^1$, there exist $r = r(y_0) > 0$ and $\tau_0 = \tau_0(y_0) > 0$ such that for $x_0 \in X^1$ with $\|x_0 - y_0\|_{X^1} < r$ there exists a continuous function $x : [0, \tau_0] \to X^1$, with $x(0) = x_0$, which is the unique $\epsilon-$regular mild solution to (3) starting at x_0. This solution satisfies*

$$t^\theta \|x(t, x_0)\|_{X^{1+\theta}} \to 0, \qquad \text{as } t \to 0, \quad 0 < \theta \leq \epsilon$$

Moreover, if $x_0, z_0 \in B_{X^1}(y_0, r)$ the following holds

$$t^\theta \|x(t, x_0) - x(t, z_0)\|_{X^{1+\theta}} \leq 2M \|x_0 - z_0\|_{X^1}, \ \forall t \in [0, \tau_0], \qquad 0 \leq \theta \leq \epsilon.$$

Also, if $\gamma(\epsilon) > \rho\epsilon$, then τ_0 can be chosen uniform on bounded sets of X^1. Furthermore, $x \in C^1((0, \tau_0], X^{\gamma(\epsilon)}) \cap C((0, \tau_0], X^{1+\gamma(\epsilon)})$, that is, $x(\cdot, x_0)$ is an strict solution of (3).

In the autonomous case, it is often the case were the map f is an $\epsilon-$regular map for a range of values of the parameter ϵ. In this direction we have the following,

Corollary 3 *If f is an $\epsilon-$regular map for all $\epsilon \in (\epsilon_0, \epsilon_1]$ then if we denote by $x_\epsilon(t, x_0)$ the unique $\epsilon-$regular solution starting at x_0, for $\epsilon \in (\epsilon_0, \epsilon_1]$, then $x_\epsilon = x_{\epsilon_1}$ and $x \in C((0, \tau], X^{1+\gamma(\epsilon_1)})$.*

If f a is time independent map which is $\epsilon-$regular map in an interval I. We classify the map in the following way:

- If $I = [0, \epsilon_1]$ for some $\epsilon_1 > 0$ and $\gamma(0) > 0$. We say that f is a *subcritical* map relative to (X^1, X^0).

- If $I = [0, \epsilon_1]$ for some $\epsilon_1 > 0$ and $\gamma(0) = 0$, and f is not subcritical; then, we say that f is a *critical* map relative to (X^1, X^0).

- If $I = (0, \epsilon_1]$ for some $\epsilon_1 > 0$ and f is not subcritical or critical; then, we say that f is a *double-critical* map relative to (X^1, X^0).

- If $I = [\epsilon_0, \epsilon_1]$ for some $\epsilon_1 > \epsilon_0 > 0$ with $\gamma(\epsilon_0) > \rho\epsilon_0$ and f is not subcritical, critical or double critical; then, we say that f is an *pseudo-subcritical* map relative to (X^1, X^0).

- If $I = [\epsilon_0, \epsilon_1]$ for some $\epsilon_1 > \epsilon_0 > 0$ and f is not subcritical, critical, double critical or pseudo-subcritical; then, we say that f is an *ultra-critical* map relative to (X^1, X^0).

Note that if f is subcritical then $f : X^1 \to X^{\gamma(0)}$, $\gamma(0) > 0$, which is the usual definition of subcritical map. When f is a critical map it takes X^1 into X^0 but there is no positive constant α such that f takes X^1 into X^α. When f is double-critical (this name first appears in [BC]) it is not defined as a map from X^1 into X^0 but it is $\epsilon-$regular for arbitrarily small positive values of ϵ. When f is pseudo-subcritical or ultra-critical is not a well defined map in $X^{1+\epsilon}$ for small values of $\epsilon > 0$, and it is only an $\epsilon-$regular map when $\epsilon \geq \epsilon_0 > 0$, for some ϵ_0. The main difference between pseudo-subcritical and ultra-critical maps is that for the first one the time of existence of the solution can be chosen uniformly on bounded sets of X^1, while for the last one this is still an unknown property.

3 Applications:

It is clear from the results in the previous section that for a given problem $\dot{x} = Ax + f(t, x)$, where A is a sectorial operator with fractional powers X^α, $\alpha \in \mathbb{R}$, we need to identify these fractional powers spaces and to study the $\epsilon-$regularity properties of the map f. This last part is usually a consequence of standard techniques involving Sobolev embeddings. In this way the local existence of solutions for this problem is reduced to a good knowledge of the linear operator A. In the following examples we show how this technique simplifies considerably the study of local existence in parabolic equations and Navier-Stokes equations with critically growing nonlinearities as seen in [KF, FK, BC, W1, W2].

In this section we will constantly use certain well known Sobolev embeddings that we summarize as:

$$
\begin{aligned}
&W^{l_1,p_1}(\Omega) \subset W^{l_2,p_2}(\Omega), \quad if\ \tfrac{l_1}{N} - \tfrac{1}{p_1} \geq \tfrac{l_2}{N} - \tfrac{1}{p_2},\ 1 < p_1 \leq p_2 < \infty \\
&W^{l,p}(\Omega) \subset C^{\eta}(\bar{\Omega}),\ if\ l - \tfrac{N}{p} > \eta > 0
\end{aligned}
\tag{18}
$$

with continuous embeddings.

3.1 Navier-Stokes Equations:

Consider the N-dimensional Navier-Stokes system; that is, if $\Omega \subset \mathbb{R}^N$, is a bounded smooth domain,

$$
\begin{aligned}
&u_t = \Delta u - \nabla p + g - (u \cdot \nabla)u, \quad x \in \Omega \\
&\mathrm{div} u = 0, x \in \Omega \\
&u = 0, \quad x \in \partial\Omega
\end{aligned}
\tag{19}
$$

where $u \in \mathbb{R}^N$ is the velocity field, p is the pressure and g is the external force.

It is an standard procedure to set this problem in an abstract context by using the orthogonal projection $P : L^2(\Omega, \mathbb{R}^N) \to H$ where H is the closure of $\{u \in C^2(\Omega, \mathbb{R}^N) : \mathrm{div} u = 0, u \cdot n = 0\}$ in $L^2(\Omega, \mathbb{R}^N)$. In this way the problem becomes

$$
\dot{u} = Au + f(u) + h(t) \tag{20}
$$

where $A : D(A) \subset H \to H$, $D(A) = H^2(\Omega, \mathbb{R}^N) \cap V$, $V = \{u \in H_0^1(\Omega, \mathbb{R}^N) : \mathrm{div} u = 0\}$, $A = P\Delta$ and $f(u) = -P(u \cdot \nabla)u$ and $h = Pg$. The operator A is self-adjoint and positive, and its fractional power spaces that we denote by E^{α}, $\alpha > 0$.

In their very nice papers [KF, FK], Fujita and Kato search for the largest fractional power space in which a local-existence and uniqueness theorem for the problem (20) can be proved, for $N = 3$. They arrive at the following result: If $u_0 \in D(A^{\frac{1}{4}})$ and $\|h(t)\|_H = o(t^{-\frac{3}{4}})$, then there is a $T > 0$ and a curve $u(t)$ such that

- $u : [0, T] \to H$ is continuous and $u(0) = u_0$;
- $A^{\frac{1}{2}}u(t)$ is continuous in H for $t \in (0, T]$ and $\lim_{t\to 0^+} t^{\frac{1}{4}}\|A^{\frac{1}{2}}u(t)\|_H = 0$;
- $A^{\frac{3}{4}}u(t)$ is continuous in H for $t \in (0, T]$ and $\lim_{t\to 0^+} t^{\frac{1}{2}}\|A^{\frac{3}{4}}u(t)\|_H = 0$;
- $u(t)$ satisfies the integral equation 7 for $t \in [0, T]$;

- If h is locally Hölder continuous in $(0,T]$, then $u : (0,T] \to H$ is continuously differentiable and satisfies 20.
- The solution found is unique in the class of functions satisfying the first four properties above.

We will show that this result can be easily obtained (in a even more general form) from the results in the previous section. Moreover we will obtain uniqueness of local-solutions for this problem in a larger class of functions.

The operator A has an associated scale of Banach Spaces E^α, $\alpha \in \mathbb{R}$, satisfying $E^0 = H$ and $E^\alpha \hookrightarrow W^{2\alpha,2}$, $\alpha \geq 0$. From this and the continuity of the projection $P : L^p(\Omega, \mathbb{R}^N) \to L^p_\sigma$, $1 < p < \infty$ we obtain the following embeddings

$$\left.\begin{aligned} E^\alpha &\hookrightarrow L^q_\sigma(\Omega, \mathbb{R}^N)\ q \leq \tfrac{2N}{N-4\alpha} \\ E^{-\alpha} &\hookleftarrow L^{q'}_\sigma(\Omega, \mathbb{R}^N)\ q' \geq \tfrac{2N}{N+4\alpha} \end{aligned}\right\} \alpha > 0$$

where L^q_σ is the closure of $\{u \in C^2(\Omega, \mathbb{R}^N) : \operatorname{div} u = 0, u \cdot n = 0\}$ in $L^q(\Omega, \mathbb{R}^N)$. The realization of A in E^α (denoted by A_α) is an isometry from $E^{\alpha+1}$ into E^α and $A_\alpha : D(A_\alpha) = E^{\alpha+1} \subset E^\alpha \to E^\alpha$ is a sectorial operator. Furthermore, $D(A^\beta_\alpha) = E^{\alpha+\beta}$.

Denote by $X^\alpha := E^{\alpha-\frac{3}{4}}$, $\alpha \in \mathbb{R}$ and by $A : X^1 \subset X^0 \to X^0$ the operator $A_{-\frac{3}{4}}$.

Lemma 6 *For $N = 3$, the nonlinearity f in (20) is an $\epsilon-$regular map relatively to (X^1, X^0), for $0 < \epsilon \leq \frac{3}{8}$. In this case $\rho = 2$ and $\gamma(\epsilon) = 2\epsilon$.*

Proof: If $0 \leq \epsilon \leq \frac{3}{8}$, then

$$L_\sigma^{\frac{6}{6-8\epsilon}} \hookrightarrow X^{2\epsilon}. \tag{21}$$

From this we have that

$$\begin{aligned} \|P(u\cdot\nabla)u\|_{X^{2\epsilon}} &\leq c\|P(u\cdot\nabla)u\|_{L_\sigma^{\frac{6}{6-8\epsilon}}} \leq c\|(u\cdot\nabla)u\|_{L^{\frac{6}{6-8\epsilon}}} \\ &\leq c\|u\|_{L^{\frac{6}{6-8\epsilon}r}}\|\nabla u\|_{L^{\frac{6}{6-8\epsilon}r'}} \end{aligned}$$

choosing $r = \frac{3-4\epsilon}{1-2\epsilon}$ and $r' = \frac{3-4\epsilon}{2-2\epsilon}$ we have that

$$\|P(u\cdot\nabla)u\|_{X^{2\epsilon}} \leq c\|u\|_{L^{\frac{6}{2-4\epsilon}}}\|\nabla u\|_{L^{\frac{3}{2-2\epsilon}}} \leq c\|u\|^2_{X^{1+\epsilon}}.$$

Similarly we obtain that

$$\|P(u\cdot\nabla)u - P(v\cdot\nabla)v\|_{X^{2\epsilon}} \leq c\|u-v\|_{X^{1+\epsilon}}(1 + \|u\|_{X^{1+\epsilon}} + \|v\|_{X^{1+\epsilon}}).$$

This concludes the proof of the lemma.

The fact that (21) does not hold for $\epsilon = 0$ indicates that the map $f(u) = P(u \cdot \nabla)u$ is a double critical map.

It is clear from the previous lemma that the following holds.

Lemma 7 *If the forcing h goes from $(0,T]$ to $X^{\gamma(\epsilon)}$, for some $\epsilon \in (0, \frac{3}{8}]$, and satisfies $\|h(t)\|_{X^{\gamma(\epsilon)}} \leq \nu(t)t^{-2\epsilon}$, where $0 \leq \nu(t) \leq \delta$, $\nu(t) \to 0$ as $t \to 0^+$, then $f(u) + h(t) \in \mathcal{F}(\delta)$.*

From the Theorem 1 and the above lemmas we have that

Theorem 2 *If there exists an $\epsilon \in (0, \frac{3}{8}]$ such that $h : (0,T] \to X^{2\epsilon}$ and $t^{2\epsilon}\|h(t)\|_{X^{\gamma(\epsilon)}} = o(1)$; then, for any $u_0 \in X^1$, the problem (20) has a unique $\epsilon-$regular solution starting in u_0. Moreover this solution satisfies $t^\theta\|u(t,u_0)\|_{X^{1+\theta}} \to 0$, $\forall\ 0 < \theta \leq \epsilon$.*

Corollary 4 (Fujita & Kato) *If $h : (0,T] \to X^{\frac{3}{4}} = H$ and satisfies $t^{\frac{3}{4}}\|h(t)\|_H = o(1)$; then, for any $u_0 \in X^1 = D(A^{\frac{1}{4}})$, the problem (20) has a unique $\frac{3}{8}-$regular solution starting in u_0. Moreover this solution satisfies $t^\theta\|u(t,u_0)\|_{X^{1+\theta}} \to 0$, $\forall\ 0 < \theta \leq \frac{3}{8}$. In particular for $\theta = \frac{1}{4}$ and $\theta = \frac{1}{2}$ we have that*

$$\lim_{t\to 0^+} t^{\frac{1}{4}}\|A^{\frac{1}{2}}u(t)\|_H = 0$$

$$\lim_{t\to 0^+} t^{\frac{1}{2}}\|A^{\frac{3}{4}}u(t)\|_H = 0$$

3.2 Heat Equations - L^q Theory

There is a sequence of very interesting papers [W1, W2, BC] that study in the spaces $L^q(\Omega)$ the model equation

$$\begin{aligned} &u_t = \Delta u + u|u|^{\rho-1},\ x \in \Omega \\ &u = 0,\ x \in \partial\Omega \end{aligned} \tag{22}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain. The aim is to establish for each value of q the largest value of ρ for which one may have existence and (maybe) uniqueness of solutions for (22).

In this section we show that most of the results in [W1, W2, BC] can be easily obtained from the results in Section 2. The basic results obtained in [W1, W2, BC] are the following

Theorem 3 (Brezis & Cazenáve) *Assume that $q > N\frac{(p-1)}{2}$ and $q \geq 1$ (resp. $q = N\frac{(p-1)}{2}$) and $q > 1$), $N \geq 1$. Given any $u_0 \in L^q(\Omega)$ there exist a time $T = T(u_0) > 0$ and a unique function $u \in C([0,T], L^q(\Omega))$ with $u(0) = u_0$ which is a classical solution of (22) on $(0,T) \times \bar{\Omega}$ in the sense that u is C^1 in $t \in (0,t)$ and C^2 in $x \in \bar{\Omega}$. Moreover, we have:*

- $\|u(t) - v(t)\|_{L^q(\Omega)} + t^{\frac{N}{2q}}\|u(t) - v(t)\|_{L^\infty(\Omega)} \leq C\|u_0 - v_0\|_{L^q(\Omega)}$ *for all $t \in (0,T]$ where $T = \min\{T(u_0), T(v_0)\}$ and C can be estimated in terms of $\|u_0\|_{L^q(\Omega)}$ and $\|v_0\|_{L^q(\Omega)}$*

- $\lim_{t\to 0^+} t^{\frac{N}{2q}}\|u(t)\|_{L^\infty(\Omega)} = 0.$

Furthermore the time T can be chosen uniformly in compact subsets of $L^q(\Omega)$.

The operator $L = \Delta$ (or in general a second order elliptic differential operator), can be seen as an unbounded operator in $L^p(\Omega, \mathbb{R}^n)$ for $1 < p < \infty$ with domain contained in $W^{2,p}(\Omega, \mathbb{R}^n)$. Following [AM], L has an associated scale of Banach Spaces E_p^α, $\alpha \in \mathbb{R}$, satisfying $E^0 = L^p(\Omega, \mathbb{R}^n)$ and

$$\left.\begin{array}{l} E_p^\alpha \subset W^{2\alpha,p}(\Omega, \mathbb{R}^n) \\ \\ E_p^{-\alpha} \supset W^{-2\alpha,p}(\Omega, \mathbb{R}^n) = (W^{2\alpha,p'}(\Omega, \mathbb{R}^n))' \end{array}\right\} \alpha > 0$$

with continuous embeddings, in such a way that the realization of L in E_p^α (denoted by L_α) is an isometry from $E_p^{\alpha+1}$ into E^α and such that $L_\alpha : D(L_\alpha) = E_p^{\alpha+1} \subset E_p^\alpha \to E_p^\alpha$ is a sectorial operator. Furthermore, $D(L_\alpha^\beta) = E_p^{\alpha+\beta}$.

Denote by $X_p^\alpha := E_p^{\alpha-1}$, $\alpha \in \mathbb{R}$ and by $A_p : X_p^1 \subset X_p^0 \to X_p^0$ the operator L_{-1}. Moreover the fractional power spaces associated to A_p satisfy,

$$\begin{aligned} &X_p^\alpha \subset W^{2\alpha-2,p}(\Omega, \mathbb{R}^n), \qquad \alpha > 1 \\ &X_p^1 = L^p(\Omega, \mathbb{R}^n), \\ &X_p^\alpha \supset W^{2\alpha-2,p}(\Omega, \mathbb{R}^n), \qquad \alpha < 1 \end{aligned} \tag{23}$$

with continuous embeddings.

If we consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(u) = u|u|^{p-1}$, or in general f satisfying $|f(u) - f(v)| \leq c|u - v|(|u|^{p-1} + |v|^{p-1} + 1)$, we have the following,

Lemma 8 (Critical Nonlinearities) *If $1 < q < \infty$ and if $q = \frac{N(p-1)}{2}$, then*

- *If $q > \frac{N}{N-2}$, then f is an $\epsilon-$regular map relative to (X_q^1, X_q^0) for $0 = \epsilon_0(q) \leq \epsilon < \frac{N}{N+2q}$. Therefore f is a critical map.*

- *If $q = \frac{N}{N-2}$, then f is an $\epsilon-$regular map relative to (X_q^1, X_q^0) for $0 = \epsilon_0(q) < \epsilon < \frac{N}{N+2q}$. Therefore, f is a double-critical map.*
- *If $1 < q < \frac{N}{N-2}$, then f is an $\epsilon-$regular map relative to (X_q^1, X_q^0) for $0 < \epsilon_0(q) < \epsilon < \frac{N}{N+2q}$ for $\epsilon_0(q) = \frac{N}{N+2q}\left(1 - \frac{N}{2}\left(1 - \frac{1}{q}\right)\right) > 0$. Therefore, f is an ultra-critical map.*

Proof. Just use Sobolev embeddings and (23).

Now it is clear that applying Theorem 1, for each $u_0 \in L^q(\Omega)$ we have the existence of a unique $\epsilon-$regular solution of the above problem, starting at u_0, for any $\epsilon \in (\epsilon_0(q), \frac{N}{N+2q})$. Moreover, for any $0 < \theta < \gamma(\frac{N}{N+2q}) = 1$, this solution satisfies

$$t^{\theta}\|u(t)\|_{X^{1+\theta}} \to 0, \quad \text{as} \quad t \to 0^+$$

$$t^{\theta}\|u(t,u_0) - u(t,v_0)\|_{X^{1+\theta}} \le C\|u_0 - v_0\|_{L^q}, \; 0 < t < \tau(u_0, v_0).$$

In particular, since $X_q^1 = L^q(\Omega)$, we get that $X^{1+\theta} \hookrightarrow W^{2\theta,q} \subset L^p$ for $p = \frac{Nq}{N-2\theta q}$; then, taking θ close enough to 1, and taking into account that $\frac{N+2q}{N} < \frac{Nq}{N-2q}$, we have that $p_0 > \max\{\frac{N+2q}{N}, q\}$, such that

$$\begin{aligned} &t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p_0})}\|u(t)\|_{L^{p_0}} \to 0, \qquad \text{as} \qquad t \to 0^+ \\ &t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p_0})}\|u(t,u_0) - u(t,v_0)\|_{L^{p_0}} \le C\|u_0 - v_0\|_{L^q}, \; 0 < t < \tau(u_0, v_0). \end{aligned} \tag{24}$$

Now it is possible to apply a bootstrap argument to show that in fact

$$\begin{aligned} &t^{\frac{N}{2q}}\|u(t)\|_{L^\infty} \to 0 \text{ as } t \to 0^+ \\ &t^{\frac{N}{2q}}\|u(t,u_0) - u(t,v_0)\|_{L^\infty} \le C\|u_0 - v_0\|_{L^q}, \; 0 < t < \tau(u_0, v_0). \end{aligned} \tag{25}$$

For this let us show the following lemma:

Lemma 9 *There exists a $\eta > 0$ small enough such that if $p \ge p_0$, then*

$$\frac{N}{2}\left(\frac{\rho}{p} - \frac{1}{p+\eta}\right) < 1, \; \forall p \ge p_0.$$

Proof: Note that

$$\frac{N}{2}\left(\frac{\rho}{p} - \frac{1}{p+\eta}\right) = \frac{q}{p} + \frac{N}{2}\left(\frac{1}{p} - \frac{1}{p+\eta}\right) \le \frac{q}{p_0} + \frac{N}{2}\left(\frac{1}{p} - \frac{1}{p+\eta}\right).$$

Choose η small so that $\frac{N}{2}\left(\frac{1}{p} - \frac{1}{p+\eta}\right) < 1 - \frac{q}{p_0}$, for all $p \ge p_0$. This proves the result.

The next is to use an induction argument as follows:

Lemma 10 *Let η and p_0 be as above and define the sequence p_n by $p_{n+1} = p_n + \eta$. If*

$$t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p_n})}\|u(t)\|_{L^{p_n}(\Omega)} \to 0 \ \textit{as} \ t \to 0^+$$

then

$$t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|u(t)\|_{L^{p}(\Omega)} \to 0 \ \textit{as} \ t \to 0^+ \ \forall p \in [p_n, p_{n+1}].$$

Proof: Using the expression

$$u(t) = e^{-\Delta\frac{t}{2}}u(\frac{t}{2}) + \int_{\frac{t}{2}}^{t} e^{-\Delta(t-s)}f(u(s))ds$$

we have,

$$t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|u\|_{L^p} \leq t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|e^{-\Delta\frac{t}{2}}u(\tfrac{t}{2})\|_{L^p} + t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \int_{\frac{t}{2}}^{t}(t-s)^{-\frac{N}{2}(\frac{\rho}{p_n}-\frac{1}{p})}\|f(u)\|_{L^{\frac{p_n}{\rho}}}$$

$$\leq t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|e^{-\Delta\frac{t}{2}}u(\tfrac{t}{2})\|_{L^p} + ct^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \int_{\frac{t}{2}}^{t}(t-s)^{-\frac{N}{2}(\frac{\rho}{p_n}-\frac{1}{p})}(1+\|u\|_{L^{p_n}})$$

$$\leq t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\tfrac{t}{2}^{\frac{N}{2}(\frac{1}{p}-\frac{1}{p_n})}\|u(\tfrac{t}{2})\|_{L^{p_n}} + ct^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \int_{\frac{t}{2}}^{t}(t-s)^{-\frac{N}{2}(\frac{\rho}{p_n}-\frac{1}{p})}ds$$

$$+ct^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})} \int_{\frac{t}{2}}^{t}(t-s)^{-\frac{N}{2}(\frac{\rho}{p_n}-\frac{1}{p})}s^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p_n})\rho}ds \sup_{0\leq s\leq t}\left(s^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p_n})}\|u(s)\|_{L^{p_n}(\Omega)}\right)^{\rho}$$

$$\leq c\sup_{0<s<t}\left(s^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p_n})}\|u(s)\|_{L^{p_n}}\right) + ct^{\frac{N\rho}{2}(\frac{1}{q}-\frac{1}{p_n})} \int_{\frac{1}{2}}^{1}(1-s)^{-\frac{N}{2}(\frac{\rho}{p_n}-\frac{1}{p})}ds$$

$$+c\int_{\frac{1}{2}}^{1}(1-s)^{-\frac{N}{2}(\frac{\rho}{p_n}-\frac{1}{p})}s^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p_n})\rho}ds \sup_{0\leq s\leq t}\left(s^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p_n})}\|u(s)\|_{L^{p_n}(\Omega)}\right)^{\rho}.$$

Notice that from the previous lemma we have that $\frac{N}{2}\left(\frac{\rho}{p_n}-\frac{1}{p}\right) < 1$ and therefore all the integrals are well defined. From the hypothesis, we have that $\sup\left(s^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p_n})}\|u(s)\|_{L^{p_n}}\right) \to 0$ as $t \to 0^+$. Also, since $p_n > q$ we have that $t^{\frac{N\rho}{2}(\frac{1}{q}-\frac{1}{p_n})} \to 0$ as $t \to 0$.

This concludes the proof.

Note that from (24) and the previous lemma we get that

$$t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|u\|_{L^p} \to 0, \ \forall p \geq p_0 \tag{26}$$

In order to obtain the estimates for the L^∞ norm, we choose $p > \frac{N\rho}{2}$, and with a similar argument as above we can show that:

$$t^{\frac{N}{2q}}\|u\|_{L^\infty} \leq t^{\frac{N}{2q}}\|e^{-\Delta\frac{t}{2}}u(\tfrac{t}{2})\|_{L^\infty} + t^{\frac{N}{2q}}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{N\rho}{2p}}\|f(u)\|_{L^{p/\rho}}$$

$$\leq t^{\frac{N}{2q}}(t/2)^{-\frac{N}{2p}}\|u(\tfrac{t}{2})\|_{L^p} + t^{\frac{N}{2q}}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{N\rho}{2p}}c(1+\|u\|_{L^p}^{\rho}ds$$

$$\leq t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|u(t)\|_{L^p} + ct^{\frac{N}{2q}}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{N\rho}{2p}}ds$$

$$+ct^{\frac{N}{2q}}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{N\rho}{2p}}s^{-\rho\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}ds\,\sup\nolimits_{0<s<t}[s^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|u(s)\|_{L^p}]^{\rho} \to 0 \text{ as } t\to 0,$$

because of similar reasons as the argument above.

This shows the first statement of (25).

For the second one the analysis is similar. Starting with the second statement of (24) and with a similar bootstrap argument as above we can obtain the desired estimate.

Now it is not difficult to obtain certain estimates of the behavior of the C^α norm of the solutions as $t \to 0^+$.

We have the following

Proposition 2 *If $q = \frac{N(\rho-1)}{2} > 1$ and f satisfies $|f(u)-f(v)| \leq c|u-v|(|u|^{\rho-1}+|v|^{\rho-1}+1)$ then, for any $\eta < 1$, we have*

$$\begin{aligned} &t^{1+\frac{N}{2q}}\|u(t)\|_{C^{1,\eta}} \to 0 \text{ as } t \to 0,\\ &t^{1+\frac{N}{2q}}\|u(t)-v(t)\|_{C^{1,\eta}} \leq C\|u_0-v_0\|_{L^q}\end{aligned} \tag{27}$$

Proof. Let us start with the following result

Lemma 11 *With the hypothesis above, if p is large enough, we have*

$$f : X_p^1 = L^p \to L^{\frac{p}{\rho}} \hookrightarrow W^{-2\alpha,p} \hookrightarrow X_p^\alpha,$$

with $\alpha = \frac{N}{2p}(\rho-1)$, and satisfies

$$\|f(u)-f(v)\|_{X_p^\alpha} \leq c\|u-v\|_{X_p^1}(\|u\|_{X_p^1}^{\rho-1} + \|v\|_{X_p^1}^{\rho-1} + 1).$$

That is, f is a subcritical map relative to the pair (X_p^1, X_p^0).

Proof. Sobolev embeddings again.

Proof of the proposition. We estimate the norms $W^{2\beta,p}$ for p large enough and β close to 1. Therefore, let $\beta < 1$ and choose p large enough such that $\alpha = \frac{N}{2p}(\rho - 1) < 1 - \beta$. Then,

$$t^{\beta+\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|u(t)\|_{X_p^{1+\beta}} \leq t^{\beta+\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|e^{-\Delta\frac{t}{2}}u(\tfrac{t}{2})\|_{X_p^{1+\beta}}$$

$$+t^{\beta+\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\int_{\frac{t}{2}}^{t} e^{-\Delta(t-s)}f(u(s))ds\|_{X_p^{1+\beta}}$$

$$\leq t^{\beta+\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\left(\tfrac{t}{2}\right)^{-\beta}\|u(\tfrac{t}{2})\|_{X_p^1} + t^{\beta+\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\int_{\frac{t}{2}}^{t}(t-s)^{-(\beta+\alpha)}c(1+\|u(s)\|_{X_p^1}^{\rho})ds$$

$$\leq ct^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\|u(\tfrac{t}{2})\|_{X_p^1} + ct^{\beta+\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\int_{\frac{t}{2}}^{t}(t-s)^{-(\beta+\alpha)}ds$$

$$+ct^{\beta+\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}\int_{\frac{t}{2}}^{t}(t-s)^{-(\beta+\alpha)}s^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})\rho}ds\,\sup_{0<s<t}\left(s^{\frac{N}{2}(\frac{1}{q}-\frac{1}{p})}1\|u(s)\|_{X_p^1}\right)^{\rho} \overset{t\to 0^+}{\longrightarrow} 0$$

because of similar reasons as the arguments above.

The proof of the first statement of (27), follows from the above result and the fact that for any $\eta < 1$, we can choose a $\beta < 1$ and a p large enough, with the property that $X_p^{1+\beta} \hookrightarrow W^{2\beta,p} \hookrightarrow C^{1,\eta}$.

For the second part of statement (27) the argument is similar. This concludes the proof of the proposition.

So far we have treated the case where $q = \frac{N}{2}(\rho-1)$, that is, critical cases. In principle, the treatment of the cases $q > \frac{N}{2}(\rho - 1)$ is simpler. It can be seen that

- If $\frac{N}{N-2} \leq q$ and $\frac{N}{2}(\rho - 1) < q$, then f is a subcritical map relative to (X_q^1, X_q^0)
- If $1 < q < \frac{N}{N-2}$ and $\rho \leq q$ then f is a subcritical map relative to (X_q^1, X_q^0)
- If $1 < q < \frac{N}{N-2}$ and $\frac{N}{2}(\rho - 1) < q < \rho$ then f is a pseudo-subcritical map relative to (X_q^1, X_q^0).

In order to visualize this classification we design the figures 1 2 and 3.

3.3 Heat Equations - $W^{1,q}$ Theory

Once we have the results from Section 2, the $W^{1,q}$-Theory for Heat Equations is not much different from the L^q theory. Again, what we need is a good understanding of the fractional power spaces of the linear operator (already done in the L^q setting) and to study the $\epsilon-$regularity properties of the nonlinearities involved.

Roughly speaking, we will see that in this case, for $1 < q < N$ the critical growth exponents are $\rho = \frac{N+q}{N-q}$, for $q > N$ there is no critical exponent due to the embedding of $W^{1,q}(\Omega) \hookrightarrow C(\bar{\Omega})$ and for $q = N$ the critical growth is larger than exponential and is established by Trudinger's Inequality (see, [Tr, Mo]).

Let E^α, $\alpha \in \mathbb{R}$ be as in the previous section. Denote by $X_p^\alpha := E_p^{\alpha-\frac{1}{2}}$, $\alpha \in \mathbb{R}$ and by $A_p : X_p^1 \subset X_p^0 \to X_p^0$ the operator $L_{-\frac{1}{2}}$. Moreover the fractional power spaces associated to A_p satisfy,

$$
\begin{aligned}
&X_p^\alpha \subset W^{2\alpha-1,p}(\Omega, \mathbb{R}^n), \qquad \alpha > 1/2\\
&X_p^{\frac{1}{2}} = L^p(\Omega, \mathbb{R}^n), \\
&X_p^\alpha \supset W^{2\alpha-1,p}(\Omega, \mathbb{R}^n), \qquad \alpha < 1/2
\end{aligned}
\tag{28}
$$

with continuous embeddings.

Assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^1 map. In the $W^{1,q}$ theory we will need the following two growth conditions

$$|f(u) - f(v)| \le c|u-v|(1 + |u|^{\rho-1} + |v|^{\rho-1}), \quad u \in \mathbb{R} \tag{29}$$

$$\lim_{|u|\to\infty} \frac{|f'(u)|}{e^{\eta|u|^{\frac{N}{N-1}}}} = 0, \ \forall \eta > 0, \quad u \in \mathbb{R} \tag{30}$$

The $\epsilon-$regularity properties of the map f are given by the following lemma.

Proposition 3 *The nonlinearity f can be classified as follows:*

- *If $q > N$ then any $f \in C^1$ is a subcritical map relatively to (X_q^1, X_q^0).*
- *If $q = N$ then any $f \in C^1$ which satisfies (30) is a subcritical map relatively to (X_q^1, X_q^0).*
- *If $\frac{N}{N-1} < q < N$ and $f \in C^1$ satisfying (29)*
  1. *If $\rho = \frac{N+q}{N-q}$, f is an $\epsilon-$regular map relative (X_q^1, X_q^0) for $0 \le \epsilon < \frac{1}{2\rho}$. Therefore f is a critical map.*
  2. *If $\rho < \frac{N+q}{N-q}$, f is a subcritical map relatively to (X_q^1, X_q^0).*
- *If $\frac{N}{N-1} = q$ and $f \in C^1$ satisfying (29)*
  1. *If $\rho = \frac{N+q}{N-q} = \frac{N}{N-2}$, f is an $\epsilon-$regular map relative (X_q^1, X_q^0) for $0 < \epsilon < \frac{1}{2\rho}$. Therefore f is a double-critical map.*
  2. *If $\rho < \frac{N}{N-2}$, f is a subcritical map relatively to (X_q^1, X_q^0).*

- *If $1 < q < \frac{N}{N-1}$ and $f \in C^1$ satisfying (29)*

   1. *If $\rho = \frac{N+q}{N-q}$, f is an $\epsilon-$regular map relative (X_q^1, X_q^0) for $0 < \epsilon_0 = \frac{N-q}{2q} - \frac{N}{2\rho} < \epsilon < \epsilon_1 = \frac{N-q}{2q} - \frac{N}{2q\rho}$, with $\gamma(\epsilon) = \rho\epsilon$. Therefore f is an ultra-critical map.*
   2. *If $\frac{Nq}{N-q} < \rho < \frac{N+q}{N-q}$, f is an $\epsilon-$regular map relative (X_q^1, X_q^0) for $0 < \epsilon_0 = \frac{N-q}{2q} - \frac{N}{2\rho} < \epsilon < \epsilon_1 = \frac{N-q}{2q} - \frac{N}{2q\rho}$, with $\gamma(\epsilon) > \rho\epsilon$. Therefore f is a pseudo-subcritical map.*
   3. *If $1 < \rho \leq \frac{Nq}{N-q}$, f is a subcritical map relatively to (X_q^1, X_q^0).*

In order to visualize this classification we include the figures 4, 5 and 6, which are similar to the figures from the L^q theory.

Proof: All the proofs follow from Sobolev embeddings (18) and the embeddings (28) except for the case $q = N$ for which the proof is based in the following lemma due to N. Trudinger (see, [Tr, Mo]).

Lemma 12 *Given $p \geq 1$ and $\sigma \leq \frac{1}{p} N \omega_{N-1}^{\frac{1}{N-1}}$ there exists a positive constant K such that, if $u \in W^{1,N}(\Omega, \mathbb{R}^n)$, $\|u\|_{W^{1,N}} \leq 1$, then*

$$\|e^{\sigma\, |u(\cdot)|^{\frac{N}{N-1}}}\|_{L^p} \leq K.$$

Where ω_{N-1} is the $(N-1)-$dimensional surface of the unit sphere.

Let us prove that the function $f : X_N^1 \to X_N^{\frac{1}{2}}$ is Lispschitz continuous in bounded subsets of X^1.

Let $r > 0$ and u, v functions in $W^{1,N}$ such that $\|u\|_{W^{1,N}} \leq r$, $\|v\|_{W^{1,N}} \leq r$. Let $\eta < \frac{\sigma}{2N\, r^{\frac{N}{N-1}}}$. Then, from (30), there exists $c_\eta > 0$ such that

$$|f(u) - f(v)|^N \leq c_\eta (e^{N\eta|u|^{\frac{N}{N-1}}} + e^{N\eta|v|^{\frac{N}{N-1}}})|u - v|^N,$$

and

$$\|f(\phi) - f(\psi)\|_{L^N(\Omega)}^N \leq c_\eta \int_\Omega [e^{N\eta|\phi(x)|^{\frac{N}{N-1}}} + e^{N\eta|\psi(x)|^{\frac{N}{N-1}}}]\, |\phi(x) - \psi(x)|^N dx$$

$$\leq c_\eta \left(\int_\Omega [e^{N\eta|\phi(x)|^{\frac{N}{N-1}}} + e^{N\eta|\psi(x)|^{\frac{N}{N-1}}}]^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |\phi(x) - \psi(x)|^{2N}\, dx\right)^{\frac{1}{2}}$$

$$\leq c_\eta\, \|\phi - \psi\|_{L^{2N}(\Omega)}^N \left(\int_\Omega [e^{N\eta|\phi(x)|^{\frac{N}{N-1}}} + e^{N\eta|\psi(x)|^{\frac{N}{N-1}}}]^2\, dx\right)^{\frac{1}{2}}$$

$$\leq \bar{c}_\eta\, \|\phi - \psi\|_{W^{1,N}}^N \left(\int_\Omega [e^{2N\eta|\phi(x)|^{\frac{N}{N-1}}} + e^{2N\eta|\psi(x)|^{\frac{N}{N-1}}}]\, dx\right)^{\frac{1}{2}}.$$

The result now follows from the fact $X_N^1 \subset W^{1,N}$ with continuous embedding, $X_N^{\frac{1}{2}} = L^N$ and from the fact that

$$\|e^{2N\eta|\phi(\cdot)|^{\frac{N}{N-1}}}\|_{L^1} \leq K$$

which comes from Lemma 12.

4 A comment on uniqueness

In Section 2 we have been able to establish the existence and uniqueness of ϵ-regular solutions for the problem $\dot{x} = Ax + f(x)$, where f is an $\epsilon-$regular map. This ϵ-regular solutions are characterized by an immediate regularization properties for $t > 0$. Therefore, uniqueness is established in the class of functions $C((0,\tau], X^{1+\epsilon})$. It is a natural and interesting question to ask whether uniqueness can be obtained in the larger space $C([0,\tau], X^1)$. This is equivalent to establish uniqueness of mild solutions (not just ϵ-regular mild solutions).

In this respect there are several results in the literature that can give some insight to this problem in the abstract setting.

In [NS], Ni and Sacks were able to give a non-uniqueness result in $C([0,\tau], L^q(\Omega))$ for the heat equation (22) where $q = \frac{N}{2}(\rho - 1)$ and $q = \rho$ (see also [BC]). This is exactly the case where $q = \rho = \frac{N}{N-2}$ which in our classification of nonlinearities corresponds to the *double critical* case. In view of this result, it seems very difficult to give a better uniqueness result than the one obtained in Theorem 1 for the double critical case. In particular for the example on the Navier-Stokes equations discussed above it also seems unlikely to obtain uniqueness of solution in the space $C([0,\tau], X^1)$ where in this case $X^1 = D(A^{\frac{1}{4}})$.

In [BC] (Theorem 4), Brezis and Cazenáve were able to give a uniqueness result in the space $C([0,\tau], X^1)$ for the problem (22) when $q = \frac{N}{2}(\rho-1)$ and $q > \rho$. In our classification of nonlinearities this corresponds to the *critical case*.

We tried, unsuccessfully, to prove an abstract uniqueness result in $C([0,\tau], X^1)$, for the *critical case*, but it seems it should be plausible to obtain this uniqueness result. The main difference between the *critical* case and the *double critical case* is that in the first one the map f is ϵ-regular even for $\epsilon = 0$. This means that f transforms X^1 into X^0. In the double critical case the ϵ-regularity properties of f start for $\epsilon > 0$ and therefore f does not transform X^1 into X^0. Any proof of a uniqueness result for the critical case should exploit this fact.

Also, in view of the above argument it seems also unlikely to obtain a uniqueness result for the *ultracritical* case.

Needless to mention that to prove or disprove any of the uniqueness results mentioned above will be extremely important for the whole understanding of the subject.

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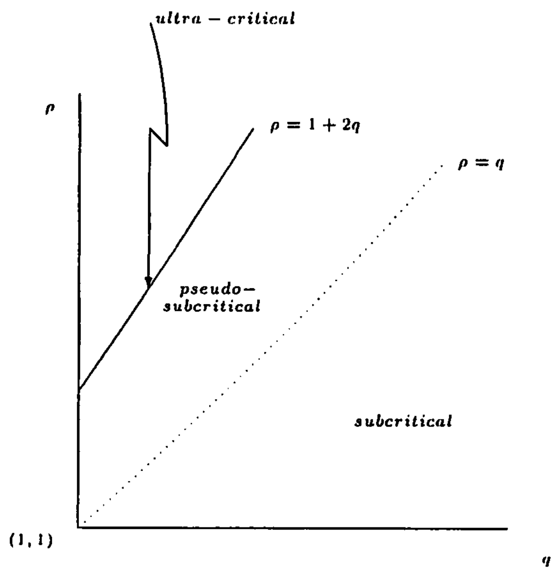

Figure 1: $N = 1$.

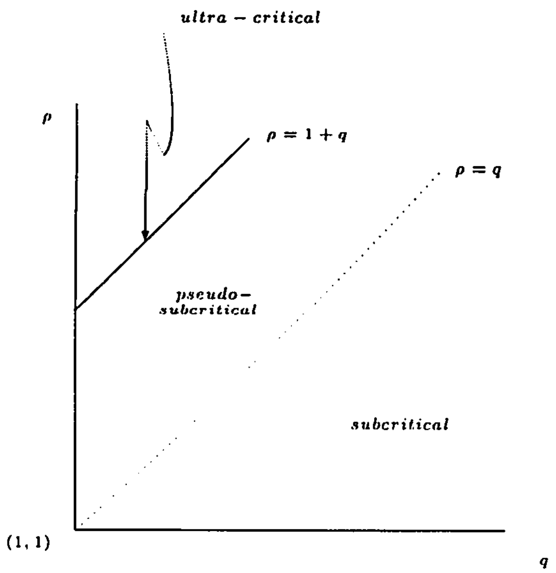

Figure 2: $N = 2$

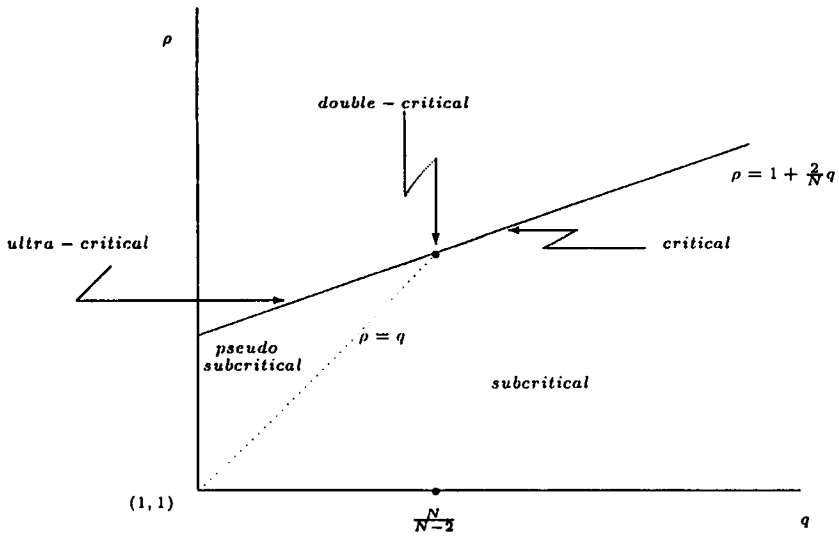

Figure 3: $N \geq 3$

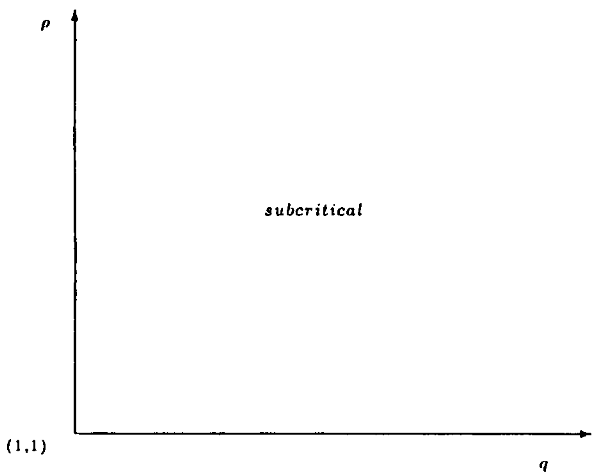

Figure 4: $N = 1$

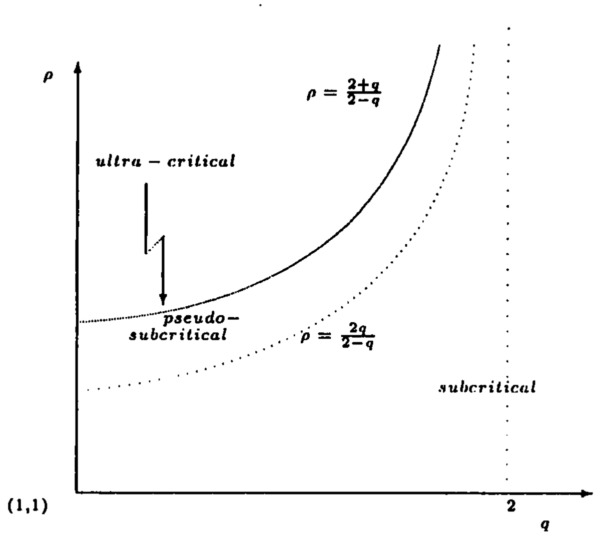

Figure 5: $N = 2$

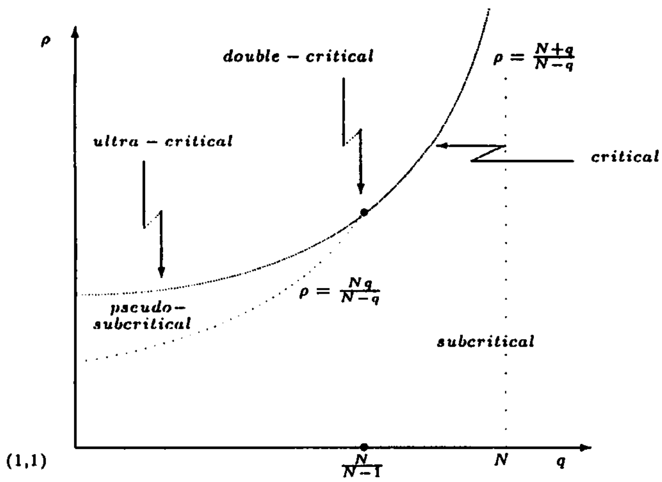

Figure 6: $N \geq 3$

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