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ON THE MULTIPLICITY OF IMPLICIT DIFFERENTIAL EQUATIONS

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RESUMO

Neste artigo apresentamos um método para calcular a multiplicidade local de um germe singular de uma equação diferencial implícita (IDE)

$$F(x, y, dy/dx) = 0$$

onde F é uma função analítica em (x, y). A multiplicidade é o número máximo de pontos singulares da IDI que aparecem quando a equação F é perturbada.

A motivação inicial é o problema concreto sobre as equações diferenciais binárias (BDE) que são equações do tipo

$$a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2 = 0.$$

Distinguimos dois tipos de BDEs. O primeiro consiste em BDEs cujos coeficients a, b, c não são todos nulos na origem, e o segundo quando os coeficientes se anulam na origem. No primeiro caso aplicamos os resultados gerais obtidos sobre IDEs. No entanto, no segundo caso estudamos IDEs cujas codimensões como IDEs são infinitas mas cujas codimensões como BDE são finitas. Relacionamos tambem os numeros de Milnor da singularidade do discriminant Δ de uma BDE e da superfície M associada a esta equação.

On the Multiplicity of Implicit Differential Equations

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1 Introduction

In this paper we consider the problem of the computation of the local multiplicity of a germ of a singularity of an implicit differential equation (IDE)

$$F(x,y,dy/dx)=0$$

where F is an analytic function in (x,y). By this we mean the maximum number of singular points of the IDE which emerge when perturbing the equation F. (As usual we count complex solutions also.) We shall use results from a paper of Montaldi and van Straten [17], where they define the local multiplicity of a 1-form on a possibly singular curve. One particular case we shall consider essentially reduces to the classical theorem of Plucker determining the class of a singular plane curve.

The original motivation for this work however lay in some concrete problems concerning binary differential equations (BDE's) that is IDE's of the form

$$a(x,y)dy^2+2b(x,y)dxdy+c(x,y)dx^2=0$$

where a, b, c are smooth real functions in (x,y), and we consider this case separately. The corresponding discriminant function $\delta=b^2-ac$ plays a key role. The BDE defines pairs of directions at points (x,y) in the plane where $\delta>0$. These directions coincide on the discriminant Δ given by $\delta=0$. (The BDE has no solution at points where $\delta<0$.) Such implicit differential equations have been studied by several authors (eg [3], [4], [6], [7], [8], [9], [11], [12], [13], [14], [15], [20]). They occur in a number of branches of mathematics, and in particular when studying the differential geometry of surfaces (see [7] for examples).

If $\mathbb{R}P$ denotes the real projective line then one way to proceed in the study of BDE's is to consider in $\mathbb{R}^2\times\mathbb{R}P$ the set M of points $(x,y,[\alpha:\beta])$ where $\delta(x,y)\geq 0$ and the direction $[\alpha:\beta]$ is a solution of the BDE at (x,y). One can lift the bivalued field defined by the BDE to a single valued field on M. Generically M is smooth, and there is a natural involution on M that interchanges points with the same image under the projection to $\mathbb{R}^2$. The set of fixed points of this involution is the lift of the discriminant. By studying this single field

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together with the involution, Davydov produced a topological classification of the integral curves of generic BDE's with smooth discriminants [11].

When Δ is singular, and in the case where a, b, c vanish at the origin, we showed in [6] that the surface M is smooth if and only if Δ has a singularity of Morse type. In that case the bivalued field in the plane can be lifted to a single field on M in a natural way and the above involution can be extended to the exceptional fibre. The approach in [4] can then be extended to produce topological normal forms of BDEs whose discriminant Δ has a Morse singularity [6]. The problem remains of studying BDEs whose discriminants have degenerate singularities. This situation can occur generically, for example, when studying principal directions on an immersed surface at a cross-cap point [22], and for all but the simplest BDE's. The difficulty here is that the surface M is no longer smooth.

We distinguish two types of BDEs. The first consists of cases where the coefficients a, b, c do not all vanish at the origin, and the second where they do. In the first case we can apply the general results we have concerning IDE's. In the second however we are studying objects which are of 'finite codimension' in some sense when considered as BDE's, but of 'infinite codimension' as IDE's. We consequently need a new definition of multiplicity in these cases. We look at the singularity type of the surface M and its relation with that of the discriminant Δ. We relate the multiplicity of the BDE with the Milnor numbers of the singularities of the surface M and that of the discriminant. In the final section we consider, in some detail, BDE's for which that discriminant is a simple plane curve singularity.

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2 Multiplicities of Implicit Differential Equations

Consider an implicit differential equation of the first order

$$F(x, y, dy/dx) = 0$$

for some analytic function $F(x, y, p)$. As usual we consider the surface $M = \{(x, y, p) \in \mathbf{R}^3 : F(x, y, p) = 0\}$, and the natural projection $\pi : M \to \mathbf{R}^2$, $(x, y, p) \mapsto (x, y)$. Generically M is a smooth surface (that is 0 is a regular value of F). The set of critical points of the projection is called the *criminant* of F and is given by the equations $F = F_p = 0$. The set of critical values of the projection is called the discriminant of F, and is obtained by eliminating p from the equations $F = F_p = 0$. Even in the case when M is singular we retain the same definitions (for example that the criminant is the locus of common zeros of F and F_p).

In the smooth case we can lift the multi-valued direction field to a single-valued direction field on the surface in a natural way. Indeed this direction field is determined by the vector field $\xi = F_p \partial/\partial x + pF_p \partial/\partial y - (F_x + pF_y)\partial/\partial p$.

Of course this lift makes sense in the singular case too. We are particularly interested in the zeros of this vector field. We shall refer to such points as singular points of the implicit differential equation (or IDE for short). Note that by definition these singular points lie on the criminant of F. An alternative way of viewing the singular points is as follows:

Proposition 2.1 *The direction field spans a subspace of the tangent space at each smooth point to the surface M given by the vanishing of the 1-form $dy - pdx$. Moreover the singular points correspond to zeros of the induced 1-form on M. Conversely if the given 1-form has a zero then either we have a zero of the lifted line field or $F_{pp} = 0$. The latter case corresponds to a non-fold point of the projection from $F = 0$ to the (x, y)-plane. The 1-form has a zero on the surface if and only if it has a zero on the criminant.*

Proof Note that the vector field above is clearly annihilated by the given 1-form. Moreover the 1-form has a zero on the criminant if and only if the three 1-forms dF, dF_p and $dy - pdx$ are linearly dependent. A short computation shows that either $F = F_p = F_{pp} = 0$ or $F = F_p = F_x + pF_y = 0$, as asserted. The last assertion is also a straightforward calculation.

Definition 2.2 *In what follows the 1-form $dy - pdx$ will be called the* canonical or contact 1-form *on* $\mathbf{R}^3$.

The initial aim of this paper is to solve the following problem: given an IDE $F(x, y, dy/dx) = 0$ and a singular point or zero (x_0, y_0, p_0) of the lifted field as above, find the multiplicity of the zero. In other words determine the maximum number of singular points which the zero can split up into under deformation. We see from the above that we should also include the points where $F = F_p = F_{pp} = 0$. For a generic surface these are the points of the surface where the projection has a cusp singularity. This motivates the following definition.

Definition 2.3 *A* singular point or zero *of the IDE given by $F(x, y, p) = 0$ is a zero of the canonical 1-form $dy - pdx$ on the criminant $F = F_p = 0$. The multiplicity of a singular point is the maximum number of zeros it can split up into under deformation of the equation $F = 0$ (including complex zeros).*

The general problem of computing the multiplicity of the zero of a 1-form on a curve has been considered by Montaldi and van Straten in [17]. Their results hold for meromorphic 1-forms and singular curves. As usual we shall complexify (we assumed that the function F above is analytic). The key point to be made concerning the results in [17] is that the definition of multiplicity given there (or more precisely sums of multiplicities) is invariant under deformation [17], Theorem 1.7. In what follows we shall assume that the criminant is an isolated complete intersection singularity (as defined by the pair (F, F_p)). See [16]. As we shall show later this only fails for a set of defining equations F of infinite codimension. Perturbing F ensures that the resulting family of discriminant curves is flat. Note also that the multiplicity is not defined if the 1-form vanishes identically on the curve.

Beyond the invariance under deformation we shall only need the following properties of the multiplicity in the sequel, also obtained from [17]:

Proposition 2.4 *In what follows let $C, 0$ denote the germ of a curve in $\mathbb{C}^n$ and let α denote the germ of a 1-form at $0 \in \mathbb{C}^n$. Then the multiplicity of α at 0 on C is denoted by $\rho(\alpha)$.*

(a) If C is smooth (we may assume $n = 1$ and $C = \mathbb{C}$) and we write $\alpha(t) = a(t)dt$ then $\rho(\alpha)$ is simply the order of $a(t)$ at 0, denoted $\nu(a(t))$.

(b) If $n : \mathbb{C}, 0 \to \mathbb{C}^n, 0$ is the normalisation map for C then the multiplicity of the pull-back $n^*(\alpha)$ is given by $\rho(n^*(\alpha)) = \rho(\alpha) - 2\delta$ where δ is the δ-invariant of the curve C.

Our first task is to describe what the stable singularities of the IDE's are. More precisely we shall describe necessary and sufficient conditions for singularities of the canonical 1-form $dy - pdx$ to have multiplicity 1 on the criminant.

Proposition 2.5 *(a) If the pull-back of the 1-form $dy - pdx$ has multiplicity 1 at a point z_0 on the criminant then this curve is smooth at z_0.*

(b) If the point on the criminant is a cusp of the projection then the multiplicity of the canonical form there is generically 1, and indeed is 1 precisely when the limiting tangent direction to the cusp is not in the same direction as that determined by the corresponding value of p at the cusp point. If the singular point of the discriminant is more degenerate then the multiplicity is at least 2. Moreover at a cusp point we have a singularity of multiplicity 1 if and only if the lifted vector field ξ is non-zero there.

(c) If the point on the criminant corresponds to a zero of the vector field ξ then its multiplicity is generically 1, and is 1 for a well-folded saddle, node or focus, as in [11]. Indeed it is 1 provided that the lifted field does not have an eigenvalue equal to zero at the singular point in question, or equivalently the corresponding point on the surface is a regular point of the map $(F, F_p, -(F_x + pF_y))$.

Proof (a) If this curve was not smooth at z_0 then its δ invariant would be at least 1. It follows that $\rho(dy - pdx)$ would be at least 2, since the pull-back $n^*(\alpha)$ is analytic, not meromorphic.

(b) If we have a cusp of the projection, then since the criminant is smooth we can parametrise it in the form $t \mapsto (x(t)+x_0, y(t)+y_0, t+p_0)$, with $t = 0$ corresponding to the cusp, $x'(0) = y'(0) = 0$, and $(x''(0), x'''(0))$, $(y''(0), y'''(0))$ independent vectors. Now the canonical form pulls back to $(y'(t) - (t+p_0)x'(t))dt$. We deduce that we have a zero of multiplicity 1 precisely when $y''(0) \neq p_0x''(0)$. But equality holds here if and only if the limiting tangent to the cusp in the (x, y)-plane has slope p_0, as required. If the criminant is singular we know that the multiplicity of the singular point is greater than 1. If the criminant is smooth then it is not hard to show that the 2-jet of the projection is equivalent to one of the form (x, y^2) or (x, xy). In the first case the discriminant is smooth. In the second it is not difficult to see that unless we have a cusp the discriminant has multiplicity greater than 2.

For the last part we may take $p_0 = 0$. Since we are at a cusp the orders of $x(t)$, $y(t)$ are at least 2 and for one of them the order is 2. Moreover $F_xF_{yp} - F_yF_{xp} \neq 0$ at $t = 0$. Now differentiate the identity $F(x(t), y(t), t) \equiv 0$ using $F_p(x(t), y(t), t) \equiv 0$. It is not difficult to establish that $y(t)$ has order > 2 if and only if $F_x(0,0,0) = 0$, but this is exactly the condition that ξ is zero on $F = 0$.

(c) Suppose now that the point on the criminant corresponds to a zero of the vector field and we can reduce F locally to $p^2 - y - \lambda x^2$. So the criminant is given by $p^2 - y - \lambda x^2 = p = 0$ which can be parametrised in the form $(t, -\lambda t^2, 0)$. The canonical form pulls back to $(-2\lambda t)dt$ and we have a zero of multiplicity 1 at $t = 0$ provided that λ is non-zero. Now a result of Davydov shows that this reduction is valid generically. In fact even if we only assume this reduction is possible to degree 3 (which is very easy, see [7]) the result still holds, as a straightforward computation shows. Note that in this case the linear part of the lifted field is $-p\partial/\partial x + (\lambda x - p)\partial/\partial p$ and $\lambda = 0$ if and only if this vector field on the surface $F = 0$ has a zero eigenvalue. Similarly it is easy to check that this holds if and only if $(0,0,0)$ is not a regular value of the map $(F, F_p, -(F_x + pF_y))$.

Proposition 2.6 *Let $F : \mathbb{C}^3, 0 \to \mathbb{C}, 0$ be a smooth map determining an IDE $F(x, y, p) = 0$. Then we can find a smooth family of mappings $\tilde{F} : \mathbb{C}^3 \times \mathbb{C}, (0,0) \to \mathbb{C}, 0$ with the property that for all t in a punctured neighbourhood of $0 \in \mathbb{C}$ the IDE given by $F_t = F(-,-,-,t) = 0$ only has singularities of multiplicity 1 near $0 \in \mathbb{C}^3$.*

Proof This is a straightforward transversality argument based on the criteria given in the proposition above. So for example it is well-known that we can deform F so that the criminant is smooth, and the only singularities of the projection are folds and cusps. Indeed we can do this by an arbitrary small rotation of the set $F = 0$ in (x, y, p)-space; see [19], or [5], p.170. At the cusp points we just need to check that the limiting tangent is not in the direction determined by the corresponding value of the p variable. This can be achieved by a translation of the surface in the p-direction, which will change the value of p but leave the criminant and discriminant unchanged. Similarly to obtain a discrete set of zeros of our lifted field, each of multiplicity 1, we only require that $(0,0,0)$ is a regular value of $(F, F_p, F_x + pF_y)$. This is easily arranged also.

In what follows we write $\mathcal{O}(x, y, p)$ for the ring of function germs $\mathbb{C}^3, 0 \to \mathbb{C}$. If I is an ideal in this ring then $\dim \mathcal{O}(x, y, p)/I$ denotes the dimension of the quotient as a complex vector space. With this notation we can state the following deductions from the results above.

Proposition 2.7 *(a) The multiplicity of a singular point ($(x, y, p) = (0,0,0)$) of the IDE $F = 0$ at a fold point of the projection (so $F_{pp} \neq 0$) corresponding to a zero of the vector field is given by $\dim \mathcal{O}(x, y, p)/\mathcal{O}(x, y, p)\langle F, F_p, F_x + pF_y\rangle$.*

(b) The multiplicity of a non-fold singularity of the projection $(x, y, p) \to (x, y)$ is given by $\dim \mathcal{O}(x, y, p)/\mathcal{O}(x, y, p)\langle F, F_p, F_{pp}\rangle$ provided that the vector field is non-zero on the lift.

(c) If we have a non-fold singular point of the projection where the vector field ξ vanishes, then the multiplicity is the sum of the numbers occuring in (a) and (b).

Proof If we deform the surface the given singular point will split up into a number of nearby ones each of multiplicity 1. In case (a) (resp. (b)) these new singular points will correspond to zeros of the associated vector field (resp. cusp points of the generic projection). These sets are in turn given by the three indicated equations, and the result follows in these cases, as then does (c).

Theorem 2.8 *(a) The multiplicity of the IDE $F = 0$ is finite if and only if the integers* $\dim \mathcal{O}(x,y,p)/\mathcal{O}(x,y,p)\langle F, F_p, F_x + pF_y\rangle$ *and* $\dim \mathcal{O}(x,y,p)/\mathcal{O}(x,y,p)\langle F, F_p, F_{pp}\rangle$ *are finite.*

(b) The set of IDE's of finite multiplicity is of infinite codimension in the set of all such equations F.

Proof (a) We have seen that if the multiplicity is finite so are the two given integers. Conversely suppose the multiplicity is not finite, and the criminant C has a component, parametrised as $(x(t), y(t), p(t))$, on which the 1-form $dy - pdx$ is identically zero. The pull-back of the 1-form is $y'(t) - p(t)x'(t)$ and differentiating the identity $F(x(t), y(t), p(t)) \equiv 0$, while using the fact that F_p vanishes along the curve shows that $F_x x' + F_y y' \equiv 0$. So $(F_x + pF_y)x' \equiv 0$ and consequently either $(F, F_p, F_x + pF_y)$ is not finite or x' and hence y' vanishes identically. This means that the p-axis is a component of the criminant, and then (F, F_p, F_{pp}) is not finite.

(b) Here we use the usual arguments found, for example, in [2], pp 115-119. We need to prove that given any k-jet $\bar{F}(x,y,p)$ we can find an l-jet $F(x,y,p)$ for some $l > k$ with $j^k F = \bar{F}$ and F having finite multiplicity. Note that following on from (a) this also establishes that the criminant is an ICIS.

By the above we need to show that (F, F_p, F_{pp}) and $(F, F_p, F_x + pF_y)$ are finite. We shall treat the second case; the first is similar but easier. Let H denote the vector space spanned by the monomials in x, y and p of degree $k+1$ and $k+2$, and consider the map $G : \mathbb{C}^3 \times H, (0,0) \to \mathbb{C}^3, 0$ defined by

$$G(x,y,p,Q) = (\bar{F} + Q, (\bar{F} + Q)_p, (\bar{F} + Q)_x + p(\bar{F} + Q)_y)(x,y,p).$$

We shall prove that 0 is a regular value of the restriction of G to $(U \setminus \{0\} \times H$, where U is some neighbourhood of $0 \in \mathbb{C}^3$. An elementary transversality result then tells us that the same is true for $G(-,Q) : U \setminus \{0\} \to \mathbb{C}^3$ for almost all $Q \in H$. So $G(-,Q)^{-1}(0)$ consists of discrete points in a punctured neighbourhood of $0 \in \mathbb{C}^3$. Consequently in some such neighbourhood it is empty, and the result will follow.

To prove that 0 is a regular value we look at the image of the tangent vectors determined by various monomials in H. The choices x^{k+1}, x^{k+2}, $x^k p$ yield the vectors $(x^{k+1}, 0, (k+1)x^k)$, $(x^{k+2}, 0, (k+2)x^{k+1})$ and $(x^k p, x^k, kx^{k-1}p)$ which show that we have a submersion off $x = 0$. Similarly p^{k+1}, p^{k+2}, xp^k show that we have a submersion off $p = 0$. Finally xy^k, y^{k+1}, $y^k p$ yield the required vectors when $x = p = 0$ and $y \neq 0$, and we are done.

Remark 2.9 (a) It is easy to see that if the criminant fails to be an ICIS then one of the integers in 2.8(a) is infinite. For then we can find a curve $(x(t), y(t), p(t))$ on $F = F_p = 0$ along which $dF \wedge dF_p$ vanishes. This implies that either F_{pp} or both F_x and F_y vanish along the curve. So (F, F_p, F_{pp}) or $(F, F_p, F_x + pF_y)$ vanishes on the curve and the corresponding maps are not finite.

(b) When the criminant is an ICIS then one can also compute the multiplicity ρ directly as $\dim \mathcal{O}(x,y,p)/\mathcal{O}(x,y,p)\langle F, F_p, \gamma\rangle$ where $\gamma = ((dy - pdx) \wedge (F, F_p)^*\omega_2)/\omega_3$ and ω_j is the volume form on $\mathbb{C}^j$. (See [17].) So γ is given by the determinant of the matrix

$$\begin{pmatrix} F_x & F_y & F_p \\ F_{px} & F_{py} & F_{pp} \\ -p & 1 & 0 \end{pmatrix}.$$

The vector space above then reduces to $\dim \mathcal{O}(x,y,p)/\mathcal{O}(x,y,p)\langle F, F_p, F_{pp}(F_x + pF_y)\rangle$, as we would expect. Our discussion separates out the two types of zero and distinguishes them geometrically.

Example 2.10 (a) Consider the surface given by $F(x, y, p) = p^3 + xp + (y + sx) = 0$, so that the criminant is given by $F = 0$ and $F_p = 3p^2 + x = 0$. Clearly the criminant is parametrised by $(-3t^2, 3st^2 + 2t^3, t)$ so the canonical form lifts to $6(2t^2 + st)dt$. When $s = 0$ this has multiplicity 2, although the projection is a simple cusp singularity. When $s \neq 0$ we have two zeros of the induced form, one corresponds to the cusp (at $t = 0$), and the other to a zero of the vector field ξ (the point corresponding to $t = -s/2$).

(b) Consider the surface given by $F(x, y, p) = p^2 + x^2 = 0$. The criminant here is given by $p = p^2 + x^2 = 0$, which is of course the y-axis, but (F, F_p) do not define it as an ICIS. Indeed $\dim \mathcal{O}(x,y,p)/\mathcal{O}(x,y,p)\langle F, F_p, F_x + pF_y\rangle$ is infinite. The same is true for $F(x, y, p) = p^3 + x = 0$, and here $\dim \mathcal{O}(x,y,p)/\mathcal{O}(x,y,p)\langle F, F_p, F_{pp}\rangle$ is also infinite. Note that the solution curves in this case are of the form $(x, y) = (t^3, (3/4)t^4 + c)$. The geometric reason for infinite codimension is that the integral curves are not ordinary cusps at generic points of the discrminant.

3 Binary Differential Equations and Plucker's Formulae

We now consider binary differential equations (or BDE), that is those IDE's which can be written in the form

$$a(x, y)dy^2 + 2b(x, y)dydx + c(x, y)dx^2 = 0.$$

The above analysis applies except that it is not possible for the projection to exhibit a cusp singularity, since the projection to the (x, y)-plane has 0, 1, 2 or infinitely many points in a fibre; the latter occuring when a, b, c have a common zero. Note that the criminant is given by $ap^2 + 2bp + c = ap + c = 0$ and the discriminant by $\delta = b^2 - ac = 0$. We seek an expression for the multiplicity in terms of the discriminant.

In this section we assume that a, b, c do not all vanish at $(0,0)$. This case will yield to the techniques from Section 2. Without loss of generality, we can assume that $a(0,0) \neq 0$.(If $a(0,0) = c(0,0) = 0$ but $b(0,0) \neq 0$ then change coordinates by $X = x + y$, $Y = x - y$.) By dividing by a we could assume that the coefficients of the BDE are of the form $(1, b, c)$. (We want a formula of maximum utility so do not make this simplification yet.)

Proposition 3.1 *Assume that $(0,0,p)$ is a point on the criminant and a does not vanish at $(0,0)$. Then the projection has a fold singularity. We have a zero for the lifted field ξ if and only if in addition $a_y p^3 + (a_x + 2b_y)p^2 + (2b_x + c_y)p + c_x = 0$. The multiplicity of this zero is then given by $m(\delta, a\delta_x - b\delta_y)$.*

Proof If there were to be a cusp (or worse) singularity then $F = F_p = F_{pp} = 0$, and this shows that a, b, c have a common zero. The condition for a zero of the lifted field is easily obtained.

For the multiplicity (from above) assume initially that the root is $p = 0$. We need $\dim \mathcal{O}(x,y,p)/\mathcal{O}(x,y,p)\langle F, F_p, F_x + pF_y\rangle$. But $F_p = 2(ap + b)$, so this quotient space has the same dimension as $\mathcal{O}(x,y)/\mathcal{O}(x,y)\langle ac - b^2, a^3c_x + aa_xb^2 - a^2bc_y - a_yb^3 + 2ab^2b_y - 2a^2bb_x\rangle$, using $p = -b/a$ and $a \neq 0$. Using the relation $ac = b^2$ in the second expression yields the result. Now if the root is at the point $p = p_0$ change coordinates by $Y = y - p_0x$, to reduce to the case $P = dY = 0$, and obtain a new BDE $AP^2 + 2BP + C = 0$, and discriminant Δ. One now easily checks that $m(\delta, a\delta_x - b\delta_y) = m(\Delta, A\Delta_x - B\Delta_Y)$ and the result follows. (Compare below where we consider the case when $a = 0$.)

Above we showed that we could assume that the coefficients of the BDE are of the form $(1, b, c)$. The following result shows that we may further suppose that $b = 0$.

Proposition 3.2 *A BDE with a non-zero 1-jet can be transformed by a change of coordinates to one of the form*

$$dy^2 + f(x,y)dx^2.$$

Proof Suppose given a BDE $a(x,y)dy^2 + 2b(x,y)dydx + c(x,y)dx^2 = 0$, with $a(0,0) \neq 0$. We define $x = X$ and $y = \phi(X,Y)$ for some, as yet unknown, function ϕ. So $dx = dX$ and $dy = \phi_X dX + \phi_Y dY$ and our BDE becomes

$$a(X,\phi(X,Y))(\phi_X dX + \phi_Y dY)^2 + 2b(X,\phi(X,Y))dX(\phi_X dX + \phi_Y dY)$$
$$+c(X,\phi(X,Y))(dX)^2 = 0.$$

We want the $dXdY$ term to vanish, so we seek a solution to the PDE

$$2a(X,\phi(X,Y))\phi_X\phi_Y + 2b(X,\phi(X,Y))\phi_Y = 0.$$

So it is enough to solve

$$\phi_X = -b(X,\phi)/a(X,\phi) = H(X,\phi)$$

say. If we fix Y this is an ODE for the function $\phi(-,Y)$. If we ask that $\phi(0,Y)=Y$ then it has a unique smooth solution, which depends smoothly on the initial value Y. Note that since $\partial\phi(0,0)/\partial X=1$ we have a genuine change of coordinates. Since the dY^2 coefficient is given by $a(X,\phi(X,Y))(\phi_Y)^2$, this is non-zero at $(0,0)$, so we can divide through to obtain the required normal form.

Remark 3.3 *The same argument shows that for a general IDE $F(x,y,p)=0$ with $F=F'=\ldots=F^{(r-1)}=0$ and $F^{(r)}\neq 0$ at $(0,0,0)$ we can change coordinates so that it has the form*

$$p^r+a_{r-2}(x,y)p^{r-2}+\ldots+a_0(x,y)=0.$$

The surface $F=0$ associated to the BDE has a singular point at $(0,0,0)$ if $(0,0)$ is a singular point of f, since the function $F=p^2+f(x,y)$. This singularity is of the same type as that of f (more accurately is a suspension of that singularity). Note also that in this case we can actually identify the criminant and discriminant, and the contact 1-form reduces to dy. The multiplicity is, of course, not defined if the pull-back of our 1-form vanishes on a component of the criminant, i.e. if $y'(t)$ is identically zero. This occurs only if $y=0$ is a component of $f=0$.

Proposition 3.4 *In this case (provided y=0 is not a component of $f=0$) the multiplicity of the BDE is given by*

$$m=m(f,f_x).$$

Alternatively, when none of the components of the curve $f=0$ have the y-axis as limiting tangent, it is one less than the sum of the Milnor numbers of f and of f restricted to $y=0$ at the origin; that is $m=\mu(f)+\mu(f(x,0))-1$.

Proof Here we have $a=1$ and $b=0$ so the first formula for m follows directly from Proposition 3.1.

For the second part note that the criminant and discriminant in this case essentially coincide. Suppose that we have r irreducible branches and parametrise the criminant as $(x_j(t),y_j(t),0)$. Working with the original definition above we see that the multiplicity of the zero of the induced form is the sum $\sum\nu(y_j'(t))+2\delta$ where δ is the δ-invariant of the discriminant. We now use the fact that $\mu(f)=2\delta-r+1$ to see that the above can be rewritten as $\mu+\sum a(j)-1$, where $a(j)=\nu(y_j(t))$. Now we suppose that we factor f as a product of r irreducible analytic functions f_j. Since the curve given by the equation $f_j=0$ is irreducible it is semi-quasihomogeneous with quasihomogeneous part $\alpha_j x^{c(j)}+\beta_j y^{d(j)}$ with $\alpha_j\beta_j\neq 0$, $c(j)$, $d(j)$ coprime. Now we know that this branch of the curve can be parametrised as $(\phi_j(t),t^{a(j)})$ for some $\phi_j(t)$. It is now easy to see that $c(j)=a(j)$ and we deduce that $f(x,0)$ has leading term some non-zero multiple of x^N where $N=\sum a(j)$. The result now follows.

Comments 3.5 The situation here is very similar to that encountered when establishing Plucker's formula for the class of a singular curve. Recall that one proceeds as follows. Given a plane projective algebraic curve $f(x,y,z) = 0$ we select a point Q not on the curve (say $(1:0:0)$). We then consider the number of tangents to the curve passing through the given point. If the curve is non-singular and the point p is in general position then there are a total of $d(d-1)$ such tangencies. If the curve is singular however this number decreases for each singularity. Indeed we can define an integer for each singular point, and a multiplicity for each tangency, so that their sum, over the singular points and points of tangency, is $d(d-1)$. This may be approached in the same way as above. The given point Q in the plane determines a projection to the projective line, and this in turn determines a differential 1-form (up to non-zero constants) on the complement of p. We now consider the zeros of this 1-form on our curve. Deforming the curve to a non-singular one of degree d in general position we find that the sum of the multiplicities of the zeros is $d(d-1)$. Some of these multiplicities come from tangencies at smooth points. The multiplicity in this case is simply one less than the order of contact, where tangency is 2 point contact. Others come from singular points. In this case, if the point $p = (1:0:0)$ is used, and we are considering a singular point at $(0:0:1)$ then writing the curve in affine coordinates $f(x,y,1) = 0$ we are again considering the differential form dy, and the relevant multiplicity can be computed as above. Indeed the affine version of Plucker's results can be obtained as a special case of the above by considering the BDE $dy^2 + f(x,y,1)dx^2$.

4 Binary Differential Equations with Vanishing Coefficients

In this section we study BDE's all of whose coefficients vanish at the origin. We start by noting that although generic BDE's are quite different in nature to generic IDE's, we can still deform them (within the set of BDE's) to one with non-degenerate singularities. Indeed this is established in the proof of Proposition 2.6; the procedure described there does not change the largest power of p that occurs.

Proposition 4.1 *Let $F = a(x,y)p^2 + 2b(x,y)pq + c(x,y)q^2 : \mathbb{C}^2 \times \mathbb{C}^2, 0 \to \mathbb{C}, 0$ be a smooth map determining an IDE $F(x,y,[p;q]) = 0$. Then we can find a smooth family of mappings $\bar{F} : \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}, (0,0) \to \mathbb{C}, 0$, with $\bar{F} = a(x,y,t)p^2 + 2b(x,y,t)pq + c(x,y,t)q^2$, $\bar{F}_0 = \bar{F}(-,-,0) = F$, and with the property that for all t in a punctured neighbourhood of $0 \in \mathbb{C}$ the IDE given by $\bar{F}_t = \bar{F}(-,-,-,t) = 0$ has smooth criminant, a discriminant with smooth branches, and only singularities of multiplicity 1 near $0 \times \mathbb{C}P \in \mathbb{C}^2 \times \mathbb{C}P$.*

Definition 4.2 *The multiplicity of the BDE above is defined to be the number of non-degenerate singular points of the perturbed equations $\bar{F}^t = 0$, where this is finite.*

Of course one might expect this to be the sum of the local multiplicities (as defined in the first section) of the singular points on the exceptional fibre

$0 \times \mathbb{C}P$. Unfortunately this is not well defined when a, b and c all vanish at the origin, since the canonical form $dy - pdx$ vanishes on the p-axis. This is nicely illustrated by the following example.

Example 4.3 *Consider the case of a BDE of Morse type $xp^2 + 2yp + x = 0$. As we shall see as a BDE this has multiplicity 3 at the origin. However if we consider it as an IDE we can deform by considering $tp^n + xp^2 + 2yp + x = 0$. The equations $F = F_p = F_{pp} = 0$ have a zero at the origin of multiplicity n. So using our definition of multiplicity from §2 the multiplicity of the above is infinite. Of course the reason is that the collapse of the p-axis under projection, unavoidable for many BDE's, is infinitely degenerate within the family ofIDE's. The exceptional fibre harbours infinitely many possible cusp points of the projection.*

As in the previous section we wish to compute the multiplicity in terms of the discriminant. To do this we use a geometrical characterisation of the singular points, namely that they correspond to points of the discriminant where the unique direction defined by the equation is tangent to the discriminant, or equivalently when the tangent direction to the discriminant is a solution of the equation. So the zeros are solutions of the system

$$\begin{aligned} \delta &= 0 \\ a(\delta_x)^2 - 2b\delta_x\delta_y + c(\delta_y)^2 &= 0. \end{aligned}$$

We remark here that on $\delta = 0$ the second equation is a perfect square, so the number of solutions of this system is counted twice (we make this precise below). This leads one to the conclusion that the multiplicity is given by

$$m = \frac{1}{2}\ dim\ \mathcal{O}_2/(\delta, a(\delta_x)^2 - 2b\delta_x\delta_y + c(\delta_y)^2).$$

To establish this we first show that the integer m is invariant under smooth changes of coordinates.

Proposition 4.4 *The multiplicity m is invariant under a smooth change of coordinates in the plane.*

Proof: Let $h : \mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ be a germ of local change of coordinates in the source. Write $h = (\phi, \psi)$ and denote the differential of ϕ by (α, β) and that of ψ by (γ, ξ). We have

$$\begin{array}{lcl} x = \phi(X,Y) & \quad\text{and}\quad & dx = \alpha dX + \beta dY \\ y = \psi(X,Y) & & dy = \gamma dX + \xi dY \end{array}.$$

The new BDE has coefficients (A, B, C) with

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \xi^2 & 2\beta\xi & \beta^2 \\ \gamma\xi & \alpha\xi + \beta\gamma & \alpha\beta \\ \gamma^2 & 2\alpha\gamma & \alpha^2 \end{pmatrix} \begin{pmatrix} a \circ h \\ b \circ h \\ c \circ h \end{pmatrix}.$$

We remark here that the matrix giving (A, B, C) in terms of (a, b, c) is invertible. It is not difficult to check that

$$B^2 - AC = (\det dh)^2.(b^2 - ac) \circ h.$$

Since h is a diffeomorphism $\det dh \neq 0$.

Let $\delta^* = B^2 - AC$ be the discriminant function of the BDE in the new system of coordinates so that $\delta^* = (\det dh)^2.\delta \circ h$. Using this equality and the expressions of A, B, C in terms of a, b, c, a calculation shows that the ideals $(\delta^*, A(\delta^*_X)^2 - 2B\delta^*_X\delta^*_Y + D(\delta^*_Y)^2)$ and $(\delta, a(\delta_x)^2 - 2b\delta_x\delta_y + c(\delta_y)^2)$ have the same codimension.

We can now establish the result we require.

Proposition 4.5 *The multiplicity of the BDE $a(x,y)p^2 + b(x,y)pq + c(x,y)q^2$ is given by the integer m above when this is finite.*

Proof We deform F to a generic BDE say $\bar{F}^t$ as above. The criminant is then smooth (and the discriminant has no cusps). There are only finitely many ordinary singular points, where locally the equation can be reduced to the form $F = p^2 - y - \lambda x^2$. One can check that in this case $\delta = -y - \lambda x^2$, and $m = \frac{1}{2}\, dim\, \mathcal{O}_2/(\delta, a(\delta_x)^2 - 2b\delta_x\delta_y + c(\delta_y)^2) = \frac{1}{2}\, dim\, \mathcal{O}_2/(y + \lambda x^2, \lambda(4\lambda - 1)x^2 - y) = 1$. The result now follows.

In the rest of the paper, we shall denote the multiplicity of the BDE by m, the dimension of the local algebra of a map-germ $(f, g) : \mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ by $m(f, g)$, and the Milnor number of a singularity of f by $\mu(f)$ $(\mu(f) = m(f_x, f_y))$. The following lemma simplifies the computation of the multiplicity. Compare with Proposition 3.1.

Lemma 4.6 $m = m(\delta, a\delta_x - b\delta_y) - m(a, b) = m(\delta, b\delta_x - c\delta_y) - m(b, c)$.

Proof Using $m(f, gh) = m(f, g) + m(f, h)$ and $m(f, g + fh) = m(f, g)$ (see for example [18] Lemma 3.2), we get

$$\begin{aligned}
2m(b^2 - ac, a\delta_x - b\delta_y) &= m(b^2 - ac, (a\delta_x - b\delta_y)^2) \\
&= m(b^2 - ac, a^2(\delta_x)^2 - 2ab\delta_x\delta_y + b^2(\delta_y)^2) \\
&= m(b^2 - ac, a(a(\delta_x)^2 - 2b\delta_x\delta_y + c(\delta_y)^2)) \\
&= m(b^2 - ac, a) + m(b^2 - ac, a(\delta_x)^2 - 2b\delta_x\delta_y + c(\delta_y)^2)) \\
&= 2m(a, b) + m(b^2 - ac, a(\delta_x)^2 - 2b\delta_x\delta_y + c(\delta_y)^2))
\end{aligned}$$

and the result follows for the first equality. The second equality in the Lemma follows in the same way.

Before doing a series of calculations we wish to relate this definition of multiplicity to that previously discussed. We are interested in those points where either the criminant is not smooth, or the lifted field vanishes. Of course since we are only interested in germs of BDE's we need only work in a neighbourhood of $0 \times \mathbb{C}P$, so need to find such points on the p-axis. (Of course we also need to work in the other chart on CP, but for purposes of exposition will ignore the the point $(p : q) = (1 : 0)$.) We have seen that there is a cubic in p determining zeros of the lifted field.

Proposition 4.7 *(a) At generic points of the p-axis F, F_p generate the ideal of germs vanishing on the axis if and only if the map $(a, b, c) : \mathbb{C}^2, 0 \to \mathbb{C}^3, 0$ has rank 2 at the origin.*

(b) In this case the singularities of the criminant occur at points $(0, 0, p)$ where $p^2(a_x b_y - a_y b_x) + p(a_x c_y - a_y c_x) + (b_x c_y - b_y c_x) = 0$, and $F = 0$ has at most one singularity on the p-axis.

(c) If (a, b, c) has rank 1 at the origin the surface $F = 0$ has 0, 1 or 2 singular points on the p-axis.

(d) If (a, b, c) has rank 0 at the origin, then the surface $F = 0$ is singular along the p-axis.

Proof Clearly the criminant is given (away from $(0, [1 : 0])$) by $ap+b = bp+c = 0$. The map with these components fails to be a submersion at points of the p-axis where $p^2(a_x b_y - a_y b_x) + p(a_x c_y - a_y c_x) + (b_x c_y - b_y c_x) = 0$. Here all derivatives are evaluated at the origin in $\mathbb{C}^2$ (recall that $a(0, 0) = b(0, 0) = 0$). This polynomial vanishes identically if and only if (a, b, c) is singular at $0 \in \mathbb{C}^2$. For the second part note that the conditions $F = F_p = 0$ hold along the p-axis. The conditions $F_x = F_y = 0$ (with $x = y = 0$) determine a pair of quadratic equations in p. If (a, b, c) has rank 2 these equations can have at most one common root (the precise condition is a given by the vanishing of a resultant). If it has rank 1 the two quadratics are multiples of each other, and this establishes (b) and (c). The last part is clear.

These singular points correspond to (generally two) other branches of the criminant meeting the p-axis. We need to understand the contribution to the multiplicity m from these singular points of the criminant, from the singular points of the lifted field, and from the singular points of the surface. Note that the simplest case, when the discriminant curve has a Morse singularity, has been dealt with in [6], so we shall assume that our BDE is more degenerate.

Proposition 4.8 *Suppose that our BDE $F = 0$ is not Morse, but that (a, b, c) has rank 2 at the origin, so that $F = F_p = 0$ at generic points determines the exceptional fibre.*

(a) A point $(0, 0, p)$ is a singular point of the criminant, and a singular point of the lifted field if and only if it is a singular point of $F = 0$.

(b) At a smooth point $(0, 0, p)$ of $F = 0$ where the lifted field vanishes (so p is a root of the cubic ϕ given in Proposition 3.1) the multiplicity of the zero of the lifted field is the multiplicity of p as a root of $\phi = 0$.

Proof (a) Without loss of generality we may suppose that the point on the p-axis is $(0, 0, 0)$, so $p = 0$. The conditions that the lifted field is zero becomes $b = c_x = 0$ (at $(0, 0)$ of course). On the other hand we have a singular point of the criminant if and only if $b_x c_y - b_y c_x = 0$. So if both occur $b_x c_y = 0$. On the other hand since we have a non-Morse BDE these conditions imply that $a_x c_y = 0$ (see [6]). So either $a_x = b_x = c_x = 0$, or $c_x = c_y = 0$. In the first case (a, b, c) has rank ≤ 1, in the second it is singular at the point in question.

(b) We need to consider the local degree of $(F, F_p, F_x + pF_y)$ at the origin, or equivalently $(ap + b, bp + c, F_x + pF_y)$. But by hypothesis the matrix of partial derivatives of the first two components of this mapping with respect to x and y at $(0,0)$ is invertible. So we can change coordinates so that the first two components are X and Y respectively. Now the map $(x, y, p) \mapsto (ap+b, bp+c, p) = (X, Y, p)$ preserves the p-axis. Hence so does the inverse map. It is now easy to see that after changing coordinates and setting $X = Y = 0$ the third component of the above map is unchanged, i.e. is $\phi(p)$, and the result follows.

The hypothesis above does not hold for all of the examples considered in the last section. The following result however is quite general and useful, since many of the simplest isolated singularities which occur are weighted homogeneous.

Proposition 4.9 *At an isolated singular point $(0,0,p)$ of the surface $F = 0$ which is right equivalent to a weighted homogeneous singularity, the contribution to the multiplicity of the BDE is at least that of the Milnor number of the singularity.*

Proof The germs $F_x + pF_y$, F_p lie in the Jacobian ideal of F, as does F itself since it is equivalent to a weighted homogeneous function. It follows that if $(F, F_p, F_x + pF_y)$ is a finite germ its multiplicity is at most that of (F_x, F_y, F_p), whence the result.

We shall also need to understand the singularities of the surface $F = 0$. The following two results will prove useful.

Lemma 4.10 *Let $F = a(x,y)p^2 + 2b(x,y)p + c(x,y) = 0$ be a surface with an isolated singularity at $0 = (0,0,0)$. Then by a change of coordinates of the form $(x,y,p) \mapsto (R(x,y), S(x,y), p-T(x,y))$ we can reduce F to $xp^2+2b(x)p+c(x,y)$, or $a(x,y)p^2 + 2xp + c(x,y)$, where $a, c \in \mathcal{M}^2(x,y)$. In the latter case the singularity is of type A_k. (If x does not divide $c(x,y)$ then k is one less than the order of $c(0,y)$.)*

Proof Since 0 is a singular point we have $b = c_x = c_y = 0$ at $(0,0)$. Clearly the germ of the p-axis will consist of singular points of $F = 0$ unless one of a_x, a_y, b_x, $b_y \neq 0$ at $(0,0)$. If say $a_x(0,0) \neq 0$ then we may suppose, after a change of (x,y)-coordinates, that $a(x,y) = x$. Now if $b(x,y) = B(y) + x\beta(x,y)$ the change of coordinates $p \mapsto p - \beta(x,y)$ will reduce to the first indicated normal form. The case $a_y(0,0) \neq 0$ is similar.

If $a_x(0,0) = a_y(0,0) = 0$, and say $b_x(0,0) \neq 0$, we can reduce b to x say, and the final result follows from the splitting lemma.

Note that the changes of coordinates employed here are much more general than those available below to simplify the normal forms of the BDE's.

Proposition 4.11 *Let F be as above, with $a_x(0,0)$ or $a_y(0,0) \neq 0$, and $F = 0$ having single (and isolated) singularity on the p-axis at $(0,0,p)$. Then the Milnor number of the singularity there, $\mu(F)$ is $\mu(\delta) - 1$, where $\delta = b^2 - ac$ is the discriminant of F, $\mu(\delta)$ its Milnor number at the origin.*

Proof First we translate so the singularity is at $(0,0,0)$. (This does not change the discriminant.) By the previous result we can change coordinates to reduce F to the first normal form F_1 above, so $\mu(F) = \mu(F_1)$. But because of the form of the change of coordinates the discriminants of F and F_1 are diffeomorphic. We can now appeal to a result of Wall, [21], which shows that $\mu(F_1, 0) = \mu(b^2 - xc)$.

Note that for the second form considered above $ap^2 + 2xp + c = 0$ there is no relation between the Milnor numbers of the singularity at the origin and that of the discriminant $x^2 - ac$. For example taking $a = y^k$, $b = y^l$ the discriminant has an A_{k+l-1}, and the surface an A_{k-1}.

5 Calculations for BDE's with Simple Discriminant

In this section we shall compute the multiplicity of many germs of BDE's whose discriminants are simple plane curve singularities (that is of type A, D or E). To do this we shall use two techniques which require some comment.

First we shall need to compute the codimension of various ideals $I = (f, g)$ by using their associated Newton polyhedra. When this is non degenerate, there is a formula in [18] that gives the codimension in terms of the area below the Newton polyhedra. In the case where one face (or more) of the Newton polyhedra is degenerate we use the crossword technique from [1]. So we add to the principal part of I (which consists of the parts in f and g that contribute to the Newton polyhedra) the lowest order monomials (with respect to the degree of the degenerate face) in f and g that are above the degenerate face. We use this new principal part to search for the level above which all chains are infinite. The codimension is then obtained by picking up the right number of monomials below this level.

As a second aid to our calculations we will formally reduce the BDE to a normal form. For single vector fields the theory of formal reduction to a normal form was developed by Poincaré. His aim was to transform a non-resonant vector field to its linear part at a singular point by a formal diffeomorphism. Poincaré's method consists of annihilating successively all terms of degree k ($k \geq 2$) in the equation. Although this process is not always convergent it provides a powerful device to study differential equations as the first terms of the series give significant information on the behaviour of the solutions. They are, for instance, sufficient to draw the phase portrait.

For BDEs we proceed similarly at the jet level. It is not possible in this case to annihilate all terms of higher degree, but we can reduce the coefficients of the BDE to simpler forms which make the computations of the multiplicity more accessible. Note that in a number of instances below we arrive at an initial normal form involving a parameter. Further reduction is then possible for almost all values of the parameter; the countable exceptional values are not made explicit, and we do not make further reductions in these cases. In particular we do not obtain normal forms for all BDE's with the given initial jet.

We start by classifying the linear part of the BDE. This is done in [6] for the case where the discriminant δ has a Morse singularity. Then the surface

$F = 0$ (also denoted by M) is smooth. If $j^1(a) = a_1x + a_2y$, $j^1(b) = b_1x + b_2y$ and $j^1(c) = c_1x + c_2y$, then M is smooth along the exceptional fibre if and only if

$$\begin{aligned} F_x(0,0,p) &= a_1p^2 + 2b_1p + c_1 \\ F_y(0,0,p) &= a_2p^2 + 2b_2p + c_2 \end{aligned}$$

do not vanish simultaneously (F is the equation for M). Since the Morse condition is equivalent to M being smooth, it follows that when δ has a singularity worse than Morse there exists a p_1 such that $F_x(0,0,p_1) = F_y(0,0,p_1) = 0$. Consider now the linear change of coordinates

$$\begin{aligned} x &= \alpha X + \beta Y \\ y &= \gamma X + \xi Y \end{aligned}$$

in the source so that the linear part of the BDE becomes

$$L = (A_1X + A_2Y)dX^2 + 2(B_1X + B_2Y)dXdY + (C_1X + C_2Y)dY^2$$

with

$$\begin{aligned} A_1 &= \xi^2(a_1\alpha + a_2\gamma) + 2\xi\beta(b_1\alpha + b_2\gamma) + \beta^2(c_1\alpha + c_2\gamma) \\ A_2 &= a_2\xi^3 + (2b_2 + a_1)\xi^2\beta + (2b_1 + c_2)\xi\beta^2 + c_1\beta^3 \end{aligned}$$

$$\begin{aligned} B_1 &= \gamma\xi(a_1\alpha + a_2\gamma) + (\alpha\xi + \gamma\beta)(b_1\alpha + b_2\gamma) + \alpha\beta(c_1\alpha + c_2\gamma) \\ B_2 &= \gamma\xi(a_1\beta + a_2\xi) + (\alpha\xi + \gamma\beta)(b_1\beta + b_2\xi) + \alpha\beta(c_1\beta + c_2\xi) \end{aligned}$$

$$\begin{aligned} C_1 &= a_2\gamma^3 + (2b_2 + a_1)\gamma^2\alpha + (2b_1 + c_2)\gamma\alpha^2 + c_1\alpha^3 \\ C_2 &= \gamma^2(a_1\beta + a_2\xi) + 2\gamma\alpha(b_1\beta + b_2\xi) + \alpha^2(c_1\beta + c_2\xi). \end{aligned}$$

One can write $A_2 = \beta^3\phi(\xi/\beta)$ and $C_1 = \alpha^3\phi(\gamma/\alpha)$ where

$$\begin{aligned} \phi(p) = &\; F_x(0,0,p) + pF_y(0,0,p) \\ = &\; a_2p^3 + (2b_2 + a_1)p^2 + (2b_1 + c_2)p + c_1. \end{aligned}$$

Let $\gamma = p_1\alpha$ where p_1 is a common root of $F_x(0,0,p)$ and $F_y(0,0,p)$, so that $C_1 = 0$ above. But in this case

$$C_2 = \alpha^2(F_x(0,0,p_1)\beta + F_y(0,0,p_1)\xi) = 0.$$

So, when the discriminant has a singularity worse than Morse, we can set $C_1 = C_2 = 0$, that is the initial 1-jet could be taken of the form $(a_1x+a_2y, b_1x+b_2y, 0)$. Making the same linear change of coordinates as above with $\gamma = 0$, we then obtain a new 1-jet with $C_1 = C_2 = 0$ and

$$\begin{aligned} A_1 &= \alpha\xi(a_1\xi + 2b_1\beta) \\ A_2 &= \xi(a_2\xi^2 + (2b_2 + a_1)\xi\beta + 2b_1\beta^2) \\ B_1 &= \alpha^2\xi b_1 \\ B_2 &= \alpha\xi(\beta b_1 + \xi b_2). \end{aligned}$$

If $b_1 \neq 0$ we can set $A_1 = 0$ and reduce the 1-jet to one of the following: $(y, \pm x + b_0y, 0)$, $(0, x + y, 0)$ or $(0, x, 0)$.

If $b_1 = 0$ but $a_1 + 2b_2 \neq 0$ we can set $A_2 = 0$ and reduce to $(x, b_0 y, 0)$ or $(0, y, 0)$. In the case $b_1 = a_1 + 2b_2 = 0$, we can reduce to $(x + y, -\frac{1}{2}y, 0)$, $(y, 0, 0)$, $(x, -\frac{1}{2}y, 0)$ or $(0, 0, 0)$.

We remark that the germs $(0, x, 0)$ and $(0, y, 0)$ are equivalent to $(y, y, 0)$. (The lifted field of BDEs with 1-jet $(x, -\frac{1}{2}y, 0)$ vanish on the whole exceptional fibre, so this case is degenerate and is not studied here.) We have established the following normal forms in the space of 1-jets.

Proposition 5.1 *The 1-jet of a BDE without constant term whose discriminant has a singularity of type worse than Morse is equivalent to one of the following orbits:*

1. $ydy^2 + 2(\pm x + b_0 y)dxdy$,
2. $ydy^2 + 2ydxdy$,
3. $(x + y)dxdy$,
4. $xdy^2 + 2b_0 ydxdy$
5. $(x + y)dy^2 - ydxdy$
6. ydy^2
7. *0.*

Note that the hypotheses of Proposition 4.7 hold only for the cases 1, 4 (when $b_0 \neq 0$) and 5.

We shall consider those BDEs whose discriminant have simple singularities; It follows that those whose 1-jet is 0 are excluded in our study. We remark that the 1-jets 1-5 yield BDEs with discriminant having a singularity of type A_k (except when $b_0 = 0$ in case 4 where D_k and E_6 singularities could occur). The 1-jet in case 6 yields BDEs with discriminants having singularities of type D_k, E_6 or E_7. The E_8 singularity does not occur in this context. This can be established in a fairly straightforward way by setting $(b^2 - ac) = l^3 + m^5$ for some functions a, b, c, l, m, writing these as a sum of their homogeneous parts, and deducing that the linear parts of l and m are dependent. A more general discussion of discriminant curves, and their deformations is postponed to a second paper.

We shall treat cases 1 and 2 in some detail and summarise in Table 1 the calculations for the remaining cases. We need the following technical result.

Lemma 5.2 *For generic values of a_0, b_0, c_0 the codimension of the ideal $(y^2 + a_0 x^k, y^2 + b_0 x^l y + c_0 x^m)$ is given by $min\{2k, 2m, k + 2l\}$.*

The proof follows by a straightforward computation.

5.1 The 1-jet $ydy^2 + 2(\pm x + b_0 y)dxdy$

Proposition 5.3 *Suppose a BDE has 1-jet $ydy^2 + 2(\pm x + b_0 y)dxdy$. Then for almost all values of b_0 the BDE can be reduced by a formal diffeomorphism to the form*

$$ydy^2 + 2(\pm x + b_0 y)dxdy + c(y)dx^2 = 0$$

where c is a formal power series in y with zero 1-jet.

Proof Assume that we can reduce the $(k-1)$-jet to $(y, \pm x+b_0y, g(y))$, and write the k-jet of the BDE in the form $(y+a_k(x,y), \pm x+b_0y+b_k(x,y), g(y)+c_k(x,y))$ where all subscripts j refer to homogeneous polynomials in (x,y) of degree j. Consider a change of coordinates of the form

$$x = X + p_k(X,Y), \ y = Y + q_k(X,Y).$$

We can also multiply the BDE by a non zero function of the form $1+r_{k-1}(X,Y)$. The coefficients of the new BDE are (we denote (X,Y) again by (x,y)) of the form $(y + A_k(x,y), \pm x + b_0y + B_k(x,y), g(y) + C_k(x,y))$ with

$$\begin{aligned} A_k &= a_k + q_k + 2(\pm x + b_0y)\tfrac{\partial p_k}{\partial y} + 2y\tfrac{\partial q_k}{\partial y} + yr_{k-1} \\ B_k &= b_k \pm p_k + b_0q_k + y\tfrac{\partial q_k}{\partial x} + (\pm x + b_0y)(\tfrac{\partial p_k}{\partial x} + \tfrac{\partial q_k}{\partial y} + r_{k-1}) \\ C_k &= c_k + 2(\pm x + b_0y)\tfrac{\partial q_k}{\partial x}. \end{aligned}$$

Writing $q_k = \Sigma_{i=0}^{i=k} q_k^i x^{k-i}y^i$, we can choose appropriate $q_k^0, ..., q_k^{k-1}$ so as to eliminate all monomials divisible by x in C_k. (The matrix of this transformation has a non-zero determinant when $b_0 \neq 0$.) We would like to set $A_k = B_k = 0$. This yield a linear system of $2(k+1)$ equations with $2(k+1)$ unknows, the unknowns being the coefficients of the polynomial p_k and r_{k-1} together with the coefficient q_k^k of y^k in q_k. The determinant of the matrix of the system is a non-zero polynomial in b_0. Therefore for b_0 away from the roots of this polynomial we can set $A_k = B_k = 0$, and the result follows.

Proposition 5.4 *(i) The surface M of a BDE whose N-jet (N large enough) is equivalent to*

$$ydy^2 + 2(\pm x + b_0y)dxdy + c(y)dx^2$$

has a singularity of type A_k and its discriminant a singularity of type A_{k+1} where $k = \mu(c)$.

(ii) The multiplicity of the BDE is $\mu(\delta)+2$. The lifted field has 3 zeros on the exceptional fibre, two of which have multiplicity 1 and the remaining zero (the singular point of M) has multiplicity $\mu(\delta)$.

Proof: (i) Working in the N-jet space, we assume that the BDE has coefficients $(y, \pm x + b_0y, c(y))$. Then the surface of the equation is given by the zero set of

$$F = yp^2 + 2(\pm x + b_0y)p + c(y).$$

This has one singularity on the exceptional fibre at $p = 0$ which is clearly of type A_k with $k = \mu(c)$. The discriminant $\delta = (\pm x + b_0y)^2 - yc(y)$ on the other hand has a singularity of type A_{k+1}.

(ii) We have

$$\begin{aligned} \delta &= (\pm x + b_0y)^2 - yc(y) \\ \delta_x &= \pm 2(\pm x + b_0y) \\ \delta_y &= 2b(\pm x + b_0y) - c(y) + yc'(y) \end{aligned}$$

We seek the codimension of the ideal $(\delta, y\delta_x - (\pm x + b_0y)\delta_y)$ (Lemma 4.6). The Newton polyhedra of the ideal generated by $\delta(X,y)$ and $y\delta_x(X,y) - X\delta_y(X,y)$, where $X = \pm x + b_0 y$, is non degenerate for $b_0 \neq 0$, so its codimension is given by Kouchnirenko's formula [18]. It is equal to $k+4$ with $k = \mu(c)$. As the codimension of the ideal $(y, \pm x + b_0 y)$ is equal to 1, it follows from Lemma 4.6 that the multiplicity of the BDE is $k+3$.

We would like to determine the contribution of each singular point of the lifted field to the multiplicity of the BDE. These singularities are the roots of the cubic $(F_x + pF_y)(0,0,p) = p(p^2 + 2b_0 p \pm 1)$. We need to compute the multiplicity of $(F, F_p, F_x + pF_y)$ at the roots. It is not hard to show that at $p = 0$ the multiplicity is $\mu(\delta)$ and at the other two roots it is equal to 1.

5.2 The 1-jet $ydy^2 + 2ydxdy$

Proposition 5.5 *Suppose a BDE has 1-jet $ydy^2 + 2ydxdy$. Then the BDE can be transformed formally to the form*

$$(y + a(x))dy^2 + 2(y + b(x))dxdy + c(x)dx^2 = 0$$

where a, b, c are formal power series with zero 1-jets.

Proof: Assume that the reduction is completed at the $(k-1)$-jet level. Then using changes of coordinates as in the previous case and multiplying by $1 + r_{k-1}$ we obtain a new BDE with coefficients $(y + a_{k-1}(x) + A_k(x,y), y + b_{k-1}(x) + B_k(x,y), c_{k-1} + C_k(x,y))$ with

$$\begin{aligned} A_k &= a_k + q_k + 2y(\tfrac{\partial p_k}{\partial y} + \tfrac{\partial q_k}{\partial y}) + y r_{k-1} \\ B_k &= b_k + q_k + y\tfrac{\partial q_k}{\partial x} + y(\tfrac{\partial p_k}{\partial x} + \tfrac{\partial q_k}{\partial y}) + y r_{k-1} \\ C_k &= c_k + 2y\tfrac{\partial q_k}{\partial x}. \end{aligned}$$

It is clear that we cannot eliminate the x^k monomials in A_k, B_k and C_k. We can however eliminate all monomials divisible by y by a suitable choice of p_k, q_k, r_{k-1}. This follows in the same way as in Proposition 5.3.

Proposition 5.6 *(i) The surface M of a BDE whose N-jet (N large enough) is equivalent to*

$$(y + a(x))dy^2 + 2(y + b(x))dxdy + c(x)dx^2$$

has two singularities on the exceptional fibre at $(0,0,0)$ and $(0,0,-2)$. The singularity is of type A_p at $(0,0,0)$ and A_q at $(0,0,-2)$, where $p = \mu(c)$ and $q = \mu(a - b + c/4)$. The discriminant has a singularity of type A_k with $k = p + q + 1$.

(ii) The multiplicity of such a BDE is $\mu(\delta) + \mu(c) + 2$. The lifted field has a double root at $(0,0,0)$ with multiplicity $2\mu(c) + 2$ and a root at $(0,0,-2)$ with multiplicity $\mu(a - b + c/4) + 1$.

Proof: (i) We have $F = (y + a(x))p^2 + 2(y + b(x))p + c(x)$, $F_x(0,0,p) = 0$, $F_p(0,0,p) = 0$ and $F_y(0,0,p) = p^2 + 2p$ so that F is singular at $p = -2$ and $p = 0$. At $p = 0$ we can write $F = (2y + 2b + (y + a)p)p + c$, which is equivalent to $yp + c$.

It is clear that this is an $A_{\mu(c)}$-singularity. At $p = -2$ by a change of variable $p = q - 2$ we can write F on the form $(y+a)q^2 + 2(-y+b-2a)q + 4(a-b+c/4)$. This is an $A_{\mu(a-b+c/4)}$-singularity.

The discriminant is given by $\delta = (y+b)^2 - c(y+a)$. Setting $y = y - b$, yields $\delta = y^2 - cy - c(a-b)$. A further change of variable reduces δ to $y^2 - c(a-b) - c^2/4$. This is an A_k-singularity with $k = \mu(c) + \mu(a - b + c/4) + 1$.

(ii) We consider the Newton polyhedra of the ideal $(\delta, (y+a)\delta_x - b\delta_y)$ whose principal part can be written in the form given in Lemma 5.2. The result then follows using Lemma 5.2 and Lemma 4.6.

The cubic giving the zeros of the lifted field is $p^2(p+2)$. Its roots coincide with the singularities of the surface M. A calculation shows that the multiplicity of $(F, F_p, F_x + pF_y)$ is $2\mu(c) + 2$ at 0 and $\mu(a - b + c/4) + 1$ at -2.

The calculations for the remaining 1-jets follow in the same way as for the two cases above. We summarise the results in Table 1, where the symbol $\wedge$ represents the minimum function of two numbers, m_r (resp. m_s) denotes the contribution of the zeros of the lifted field at regular (resp. singular) points of M and $m(BDE)$ is the multiplicity of the BDE.

Table 1 here

Remarks about Table 1: (i). We have shown by direct calculation that the Milnor number of the discriminant $\mu(\delta)$ is the sum of the Milnor numbers of the singularities of M (see Table 1). For all cases in Table 1 where $F = 0$ has a single singularity this follows from Proposition 4.10 (a consequence of a theorem of Wall). We hope to produce a proof of a more general result which will cover all cases in the near future.

(ii). The surface M has one singularity on the exceptional fibre for all the cases in Table 1 except in 2 and 3 where it has two.

(iii). The zeros of the lifted field on the exceptional fibrė are the roots of the cubic $\phi = F_x + pF_y$. In case 1 this cubic has two roots at regular points of M each making a contribution of one to the multiplicity of the BDE. In cases 3,4,5,6 the cubic has 1 root at a regular point of M. In case 2 ϕ has a double and a simple root both occuring at singular points of M. In 7,8,9,10 the cubic has a triple root which coincides with the singularity of M. It can be seen from Table 1 that the sum of the mutiplicities m_r and m_s is the multiplicity m(BDE) of the BDE.

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Table 1.

Formal normal form	δ	M	m_r	m_s	$m(BDE)$
1.$(y, \pm x + b_0 y, c(y))$	$A_{\mu(c)+1}$	$A_{\mu(c)}$	1, 1	$\mu(\delta)$	$\mu(\delta) + 2$
2.$(y + a(x), y + b(x), c(x))$	A_{p+q+1} $p = \mu(c)$ $q = \mu(a - b + \frac{c}{4})$	$A_p + A_q$	–	$q + 1,\ 2p + 2$	$\mu(\delta) + \mu(c) + 2$
3.$(a(x), x + y, c(x))$	A_{p+q+1} $p = \mu(a)$ $q = \mu(c)$	$A_p + A_q$	$(p \wedge q) + 1$	$p + 1,\ q + 1$	$\mu(\delta) + \mu(a) + 2$ $\wedge$ $\mu(\delta) + \mu(c) + 2$
4.$(x, b_0 y + b(x), c(x))$	$A_{\mu(c)+1}$	$A_{\mu(c)}$	1	$\mu(\delta) + 2\mu(b) + 1$ $\wedge$ $2\mu(\delta)$	$\mu(\delta) + 2\mu(b) + 2$ $\wedge$ $2\mu(\delta) + 1$
5.$(x, b(y), c(x, y))$ $c_{xy}(0,0) \neq 0$	D_{k+1} $k = \begin{array}{c} 2\mu(b) + 2 \\ \wedge \\ 2\mu(c(0, y)) + 1 \end{array}$	A_k	$\mu(b) + 1$	$\mu(\delta) + 2$	$\mu(\delta) + \mu(b) + 3$
6.$(x, y^2 + b(y), c_0 x^2 + c(x, y))$	E_6	D_5	2	8	10
7.$(x + y, -\frac{1}{2}y + b(x), c(x))$	$A_{\mu(c)+1}$	$A_{\mu(c)}$	–	$\mu(\delta) + 2\mu(b) + 2$ $\wedge$ $2\mu(\delta) + 1$	$\mu(\delta) + 2\mu(b) + 2$ $\wedge$ $2\mu(\delta) + 1$
8.$(y + a(x), b(x), x^2 + c(x, y))$	D_{k+3} $k = \mu(x^2 + c)$	$k = 1 : A_3$ $k > 1 : D_{k+2}$	–	$\mu(\delta) + 4$	$\mu(\delta) + 4$
9.$(y + a(x), b(x), xy + c(x, y))$	D_{k+1}	D_k	–	$\mu(\delta) + \mu(c) + 4$ $\wedge$ $2\mu(\delta)$	$\mu(\delta) + \mu(c) + 4$ $\wedge$ $2\mu(\delta)$
10.$(y + a(x), b(x), y^2 + c(x, y))$	$\mu(b) = 2 :\ E_6$	D_5	–	11	11
	$\mu(b) > 2 :\ E_7$	E_6	–	13	13

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