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GLOBAL ATTRACTORS FOR PROBLEMS WITH MONOTONE OPERATORS

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ABSTRACT. While the theory of existence of global attractors for semilinear parabolic equations has already been completed then quasilinear case still contains many open problems. Our aim in this paper is to study existence of global attractors for quasilinear parabolic problems with monotone principal part from an abstract point of view. The results are applied to degenerated parabolic problems of second and higher order.

1. INTRODUCTION

In this paper we consider the existence of global attractors (see [HA, p.39]) for problems of the form

$$\begin{cases} \frac{du}{dt}(t) + A(u(t)) + B(u(t)) = 0, \ t > 0, \\ u(0) = u_0 \in H, \end{cases} \tag{1}$$

where A is a maximal monotone operator and B is a globally Lipschitz map on a Hilbert space H.

To motivate the study done here we point out a few features of semilinear parabolic problems which will be different for the quasilinear problems under consideration in this paper. For simplicity of presentation we do this using the easiest possible examples.

Let $\Omega \subset R^n$ be a bounded smooth domain in R^n. Assuming that $A = \Delta_D$ denotes the Laplace operator with homogeneous Dirichlet boundary condition and that $B(u) = \lambda u$, $\lambda \in R$, $g \in L^2(\Omega)$, we obtain the problem:

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$$\begin{cases} u_t = \Delta u + \lambda u + g, \ t > 0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \\ u(0) = u_0 \in L^2(\Omega). \end{cases} \tag{2}$$

If λ_1 denotes the first eigenvalue of $-\Delta_D$, the problem (2) with $\lambda < \lambda_1$ has a trivial compact global attractor consisting of the only solution of stationary problem

$$\begin{cases} \Delta u + \lambda u + g = 0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega. \end{cases} \tag{3}$$

On the other hand, if $\lambda \geq \lambda_1$, then the problem (2) no longer has a compact global attractor.

In contrast to this we will prove that the problem

$$\begin{cases} u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda u + g, \ t > 0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \\ u(0) = u_0 \in L^2(\Omega), \end{cases} \tag{4}$$

with $p > 2$, has a compact global attractor for all values of λ.

This examples show that the dissipation properties of the p-Laplacian, for $p > 2$, are much stronger than the corresponding properties of the linear Laplacian. Such properties are related to the fact that the main part of (4) exhibits a nonlinear diffusion which is very large for large values of the gradient of the solution. Also for small gradients of the solution the diffusion is small and this may produce interesting and complicated dynamics.

In fact we prove existence of global attractors for a class of abstract problems of the form (1) which includes (4) as well as many other second and higher order examples of more complicated nature.

2. Abstract results

In this section we prove the abstract theorem concerning existence of the global attractor for the semigroup associated with (1) in $cl_H(D(A_H))$ (with the metric inherited from H) under some extra hypotheses on the monotone operator A.

The following hypotheses are known in the literature (cf. [BR], [BA], [LI], [TE]) and will be used to obtain existence and smoothness of solutions to (1).

H 1. (i) *Let V be a reflexive Banach space such that*

$$V \subset H \subset V^*,$$

with continuous inclusions and with V^* denoting the topological dual of V. Assume in addition that V is dense in H.

(ii) *Let A be a nonlinear, monotone, coercive and hemicontinuous operator such that $A : V \to V^*$ (defined on all of V).*

(iii) *Let $B : H \to H$ be a globally Lipschitz map.*

Define the set

$$D(A_H) := \{v \in V;\ A(v) \in H\}$$

and consider the operator $A_H : D(A_H) \subset H \to H$ given by

$$A_H(u) = A(u) \ \text{ for } \ u \in D(A_H).$$

We denote by $\langle \cdot, \cdot \rangle$ the inner product in H and by $\langle \cdot, \cdot \rangle_{V^*,V}$ the duality between V^* and V.

Recall the definitions of *strong* and *weak* solutions to (1).

Definition 1.

- *A function $u \in C([0,T]; H)$ is a* strong solution *to* (1) *if u is absolutely continuous in any compact subinterval of* $(0,T)$, *$u(t) \in D(A_H)$ for a. a. $t \in (0,T)$. and*

$$\frac{du}{dt}(t) + A(u(t)) + B(u(t)) = 0 \ \text{ for a. a. } \ t \in (0,T).$$

- *A function $u \in C([0,T]; H)$ is called a* weak solution *to* (1) *if there is a sequence $\{u_n\}$ of strong solutions convergent to u in $C([0,T]; H)$.*

We have:

Proposition 1. *If* **H1** *holds, then the equation* (1) *defines a semigroup of nonlinear operators $\{T(t) : cl_H(D(A_H)) \to cl_H(D(A_H)),\ t \geq 0\}$, where for each $u_0 \in cl_H(D(A_H))$*

$$t \to T(t)u_0 \tag{5}$$

is the global weak solution of (1) *starting at u_0. This semigroup is such that*

$$R^+ \times cl_H(D(A_H)) \ni (t, u_0) \to T(t)u_0 \in cl_H(D(A_H))$$

is a continuous map. Additionally, if $u_0 \in D(A_H)$ then $u(\cdot) = T(\cdot)u_0$ is a Lipschitz continuous strong solution of (1).

Proof. Observe that under the assumption **H1** the operator A will be maximal monotone (see [BR, Ex. 2.3.7, p. 26]). Also, as a result of [BR, Remark 3.14, p. 106] we have immediately the existence and regularity of global solutions to (1). For the joint continuity one observes that:

$$\sup_{t\in[0,T]} \|T(t)u_0 - T(t)v_0\|_H \le C(T)\ \|u_0 - v_0\|_H,$$

which follows from the usual estimates using the equation (1) and monotonicity of A. The proof is completed. □

For further results we will need additional assumptions on A. The following hypothesis can be used to obtain the density of the domain of A, even though this is sometimes a trivial problem. It is also helpful to show that the semigroup is compact and dissipative in the topology of H.

H 2. *There are constants $\omega_1, \omega_2 > 0$, $c_1 \in R$ and $p \ge 2$ such that for all $v \in V$ the following two conditions hold:*

$$\langle Av, v\rangle_{V^*,V} \ge \omega_1 \|v\|_V^p + c_1, \tag{6}$$

$$\|Av\|_{V^*} \le \omega_2(1 + \|v\|_V^{p-1}). \tag{7}$$

Lemma 1. *If* **H1***(i),(ii) and* **H2** *hold, then the domain $D(A_H)$ is dense in H.*

Proof. Let u be an arbitrary element of H and $\varepsilon \in (0,1)$. Let $u_\varepsilon = (1+\varepsilon A_H)^{-1}(u)$. Since

$$u_\varepsilon + \varepsilon A_H(u_\varepsilon) = u, \tag{8}$$

then

$$\|u_\varepsilon\|_H^2 + \varepsilon\langle A_H(u_\varepsilon), u_\varepsilon\rangle = \langle u, u_\varepsilon\rangle$$

and using the condition (6) we find that

$$\|u_\varepsilon\|_H^2 + \varepsilon\omega_1\|u_\varepsilon\|_V^p \le \|u\|_H\|u_\varepsilon\|_H - \varepsilon c_1. \tag{9}$$

Hence the norm $\|u_\varepsilon\|_H$ is bounded and also

$$\varepsilon\|u_\varepsilon\|_V^p \le const, \tag{10}$$

where *const* is independent of $\varepsilon \in (0,1)$. ¿From (8) and (7), (10) we get further

$$\|u_\varepsilon - u\|_{V^*} = \varepsilon\|A_H(u_\varepsilon)\|_{V^*} \le \omega_2\varepsilon(1 + \|u_\varepsilon\|_V^{p-1})$$

$$\le \omega_2\varepsilon\Big(1 + \Big(\frac{const}{\varepsilon}\Big)^{\frac{p-1}{p}}\Big) \le \omega_2(\varepsilon + \varepsilon^{\frac{1}{p}} const^{\frac{p-1}{p}}).$$

This ensures that $u_\varepsilon \to u$ in V^*. Since H is Hilbert space and $\{u_\varepsilon\}$ is bounded in H, any sequence $\{u_{\varepsilon_n}\}$ has a subsequence weakly convergent in H. As a consequence u_ε converges to u weakly in H when

$\varepsilon \to 0$. Using (9) we observe that $y_\varepsilon := \|u_\varepsilon\|_H$ satisfies quadratic inequality

$$y_\varepsilon^2 - \|u\|_H y_\varepsilon + \varepsilon c_1 \leq 0.$$

This shows that $\limsup_{\varepsilon \to 0} \|u_\varepsilon\|_H \leq \|u\|_H$ and therefore u_ε converges to u strongly in H (cf. [BR, Proposition 1.4, p.14]). The proof is completed. □

Lemma 2. *Let K be a continuous map in a metric space X and W be a dense subset of X. Then the following two conditions are equivalent:*

(i) for each open ball $B_X(r)$ the image $K\big(B_X(r) \cap W\big)$ is precompact in X,

(ii) for each bounded subset $\mathcal{B}$ of X, the image $K(\mathcal{B})$ is precompact in X.

Proof. Implication $(ii) \to (i)$ is obvious. To prove $(i) \to (ii)$ denote by $d(\mathcal{B})$ the diameter of $\mathcal{B}$ and take $v_0 \in \mathcal{B}$. We have:

$$\mathcal{B} \subset \{v \in X;\ \rho(v_0, v) < d(\mathcal{B}) + 1\} =: \tilde{\mathcal{B}}$$

and also

$$\mathcal{B} \subset cl_X(\tilde{\mathcal{B}} \cap W).$$

Since K is continuous, then

$$K(\mathcal{B}) \subset K\big(cl_X(\tilde{\mathcal{B}} \cap W)\big) \subset cl_X K(\tilde{\mathcal{B}} \cap W). \tag{11}$$

¿From (i) the set $cl_X K(\tilde{\mathcal{B}} \cap W)$ is compact in X and hence $cl_X K(\mathcal{B})$ is compact as the result of (11). The proof is completed. □

Remark 1. This lemma shows that it suffices to check compactness of the semigroup $\{T(t)\}$ associated with (1) considering initial data u_0 from dense subset of the phase space $cl_H(D(A_H))$; in particular from $D(A_H)$. This observation will be used in the proof of Lemma 4.

Lemma 3. *If* **H1**, **H2** *hold and $p > 2$, then for any $u_0 \in D(A_H)$ and all $T > 0$ we have that:*

$$\int_0^T \|u(s)\|_V^p ds \leq C_1(\|u_0\|_H, T), \tag{12}$$

$$\int_0^T \|\frac{du}{dt}(s)\|_{V^*}^\theta ds \leq C_2(\|u_0\|_H, T), \tag{13}$$

where C_1, C_2 are locally bounded functions and $\theta = \frac{p}{p-1}$.

Proof. According to Proposition 1 we have a weak solution $u(t)$ of the problem (1). Consider $u_0 \in D(A_H)$. By [BR, Theorem 2.17] the corresponding to u_0 weak solution satisfies:

$$u_0 \in C([0,T]; H) \text{ for all } T > 0.$$

Therefore, for any B Lipschitz from H into H, we have:

$$f(t) := B(u(t)) \in C([0,T]; H) \subset L^\theta(0,T; V^*).$$

Applying [BA, Theorem 2.6, p. 140] we verify that

$$u \in C([0,T]; H) \cap L^p(0,T; V),$$
$$\frac{du}{dt} \in L^\theta(0,T; V^*).$$

We are now able to prove (12), (13).

¿From equation (1) and inequality (6) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u\|_H^2 = -\langle A(u), u\rangle_{V^*,V} - \langle B(u), u\rangle_H$$
$$\leq -\omega_1\|u\|_V^p - c_1 + L\|u\|_H^2 + const\|u\|_H$$
$$\leq -\frac{\omega_1}{2}\|u\|_V^p + const_1,$$

where L is the Lipschitz constant for $B : H \to H$ and $p > 2$. Integrating over $(0,T)$ we find:

$$\|u(T)\|_H^2 + \omega_1 \int_0^T \|u(s)\|_V^p ds \leq \|u_0\|_H^2 + const_1 T. \tag{14}$$

This shows that (12) is satisfied. Also

$$\|u\|_{L^\infty(0,T;H)} \leq const_2(\|u_0\|_H, T). \tag{15}$$

It follows further from equation (1) and inequality (7) that

$$\|\frac{du}{dt}\|_{V^*}^\theta \leq const_3\Big(\|A(u)\|_{V^*}^\theta + \|B(u)\|_{V^*}^\theta\Big)$$
$$\leq const_4\Big(1 + \|u\|_V^p + \|u\|_H^\theta\Big).$$

Integrating over $(0,T)$ and using (12), (15) we come to (13). The proof is completed. □

Lemma 4. *Let $\{T(t)\}$ be a semigroup associated with (1) on $cl_H(D(A_H))$. Assume* **H1(i)**, *(12) and (13), for some $T > 0$, $p > 1$, $\theta > 1$ and the compactness of the embedding $V \subset H$. Then $T(t) : cl_H(D(A_H)) \to cl_H(D(A_H))$ is a compact map for each $t > 0$.*

Proof. To prove compactness of the semigroup $\{T(t)\}$ it suffices, according to Lemma 2, to consider bounded subsets of H having the form $\mathcal{B} = B_H(r) \cap D(A_H)$ ($B_H(r)$ being the ball in H with radius r centered at zero). Fix $T > 0$ according to our assumptions and define the subset $\tilde{\mathcal{B}} \subset C([0,T]; H)$:

$$\tilde{\mathcal{B}} := \{T(\cdot)u_0;\ u_0 \in \mathcal{B}\},$$

where $u(\cdot) = T(\cdot)u_0 \in C([0,\infty); H)$ denotes a weak solution (5) of (1) resulting from Proposition 1.

Let us introduce further a Banach space

$$W := \{v \in L^p(0,T;V);\ \frac{dv}{dt} \in L^\theta(0,T;V^*)\}$$

(with p as in **H2**) endowed with the norm

$$\|v\|_W := \|v\|_{L^p(0,T;V)} + \|\frac{dv}{dt}\|_{L^\theta(0,T;V^*)}.$$

As a consequence of (12), (13) the set $\tilde{\mathcal{B}}$ is bounded in the norm of W. Therefore, from [LI, Theorem 5.1, Chapt. 1]

$$\tilde{\mathcal{B}} \text{ is precompact in } L^p(0,T;H). \tag{16}$$

Take any sequence $\{u_n\} \subset \mathcal{B}$ and consider the sequence $\{T(\cdot)u_n\} \subset \tilde{\mathcal{B}}$. From (16) there is a subsequence $\{T(\cdot)u_{n_k}\}$ of $\{T(\cdot)u_n\}$ and $v_0 \in L^p(0,T;H)$ such that

$$\left(\int_0^T \|T(s)u_{n_k} - v_0(s)\|_H^p ds\right)^{\frac{1}{p}} \to 0 \text{ when } k \to \infty. \tag{17}$$

Hence the sequence $\{\|T(\cdot)u_{n_k} - v_0(\cdot)\|_H\}$ of real functions $\|T(\cdot)u_{n_k} - v_0(\cdot)\|_H : (0,T) \to R$ converges to zero in $L^p(0,T;R)$ and, in particular, there is a subsequence $\{\|T(\cdot)u_{n_{k_l}} - v_0(\cdot)\|_H\}$ such that

$$\|T(\cdot)u_{n_{k_l}} - v_0(\cdot)\|_H \to 0 \text{ a. e. on } (0,T). \tag{18}$$

Now for any $t > 0$ using (18) we have

$$\exists_{\tau\in(0,t)}\ T(\tau)u_{n_{k_l}} \to v_0(\tau) \text{ in } H.$$

Therefore

$$T(t)u_{n_{k_l}} = T(t-\tau)T(\tau)u_{n_{k_l}} \to T(t-\tau)v_0(\tau),$$

which proves that the sequence $\{T(t)u_n\}$ has a convergent subsequence. □

Lemma 5. *Let* **H1**, (6) *be satisfied. If* $p > 2$ *then the semigroup* $\{T(t)\}$ *associated with* (1) *is bounded dissipative (as in* [HA, p. 38] *) in* $cl_H(D(A_H))$.

Proof. It suffices to consider initial data $u_0 \in D(A_H)$. ¿From equation (1), assumption (6) and Lipschitz continuity of B it follows that:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|_H^2 = -\langle A(u), u\rangle_{V^*,V} - \langle B(u), u\rangle_H \\
&\leq -\omega_1\|u\|_V^p - c_1 + const(\|u\|_H + 1) \\
&\leq -\frac{\omega_1}{2}\|u\|_V^p + const' \\
&\leq -\frac{\omega_1}{2}e^{-p}\|u\|_H^p + const'.
\end{aligned}$$

Hence the function $y(t) := \|u(t)\|_H^2$ satisfies the differential inequality

$$y'(t) \leq -\omega_1 e^{-p} y^{\frac{p}{2}}(t) + 2const'.$$

Therefore, from [TE, Lemma 5.1, p. 163], we get

$$y(t) = \|u\|_H^2 \leq \left(\frac{2const'}{\omega_1 e^{-p}}\right)^{\frac{2}{p}} + \left(\omega_1(\frac{p}{2} - 1)t\right)^{-\frac{2}{p-2}}.$$

This shows that the set $\{u_0 \in cl_H(D(A_H)) : \|u_0\|_H \leq r\}$ *attracts* (see [HA, p.36]) bounded subsets of $cl_H(D(A_H))$ in the $H-$norm for each $r \geq \left(\frac{2const'}{\omega_1 e^{-p}}\right)^{\frac{2}{p}}$. The proof is completed. □

As a consequence of Proposition 1, Lemmas 2-5 and Hale's theory of dissipative systems [HA, Theorem 3.4.8] we conclude immediately that:

Theorem 1. *Let* **H1**, (6), (13) *be satisfied,* $p > 2$ *and* V *be compactly embedded in* H. *Then the semigroup* $\{T(t)\}$ *associated with* (1) *has a global attractor in* $cl_H(D(A_H))$.

Remark 2. Note that the above result is also true for $p = 2$ provided that $Le^2 < \omega_1$ is satisfied, where e is the embedding constant for $V \subset H$.

3. Applications

3.1. **Example 1:** Let Ω be an open, bounded domain in R^n with smooth boundary $\partial\Omega$ and let $H = L^2(\Omega)$. Consider the following nonlinear second order partial differential equation

$$u_t = \mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u}) - |\mathrm{u}|^{\rho-1}\mathrm{u} + \mathrm{f}(\mathrm{u}) \tag{19}$$

with homogeneous Dirichlet boundary condition, where $p > 2$ and $\rho > 1$.

Next we rewrite the above problem in the abstract setting. Let A_1 and A_2 be the nonlinear operators defined by

$$D(A_1) = \{u \in W_0^{1,p}(\Omega);\ \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) \in L^2(\Omega)\},$$

$$A_1(u) = -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)$$

and

$$D(A_2) = \{u \in L^{\rho+1}(\Omega); |u|^{\rho-1}u \in L^2(\Omega)\},$$

$$A_2(u) = |u|^{\rho-1}u.$$

Define the operator A in $D(A_1) \cap D(A_2)$ as

$$A(u) = A_1(u) + A_2(u).$$

Let $f : R \to R$ be a globally Lipschitz function and B be the Nemitskii operator defined by $-f$ on $L^2(\Omega)$. For the above defined operators consider below the problem (1) in H.

Let $V = W_0^{1,p}(\Omega) \cap L^{\rho+1}(\Omega)$. We then have

Lemma 6. *The Banach space V normed by $\|\cdot\|_V = \|\cdot\|_{W_0^{1,p}(\Omega)} + \|\cdot\|_{L^{\rho+1}(\Omega)}$ is reflexive.*

Proof. The proof follows from the Eberlein-Shmulyan theorem and the characterization of V^* given in [G-Z]. □

To apply the abstract results we first need to verify condition **H1**. It is clear that condition **H1**(i) is satisfied. Now $A : V \to V^*$ by

$$\langle A(u), v\rangle_{V^*,V} = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega |u|^{\rho-1}uv dx \quad \text{for all} \quad v \in V. \tag{20}$$

It is a standard result from monotone operator theory that A is a monotone, coercive and hemicontinuous nonlinear operator defined on V with $D(A_H)$ dense in H, where

$$D(A_H) = \{v \in V;\ -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{\rho-1}u \in H\}.$$

Also B is trivially seen to be a Lipschitz map from H into itself.

To apply Theorem 1 we need also to verify (6). It follows from (20) and the Young inequality that

$$\begin{aligned}\langle A(u), u\rangle_{V^*,V} &= \|\nabla u\|^p_{L^p(\Omega)} + \|u\|^{\rho+1}_{L^{\rho+1}(\Omega)} \\ &\geq \omega_1\|u\|^\eta_V + c_1 \quad \text{for all} \quad u \in V,\end{aligned}$$

with $\eta = \min\{p, \rho+1\}$. Finally to verify (13) observe that for $u_0 \in D(A_H)$ we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_H^2 &\leq -\|\nabla u\|_{L^p(\Omega)}^p - \|u\|_{L^{\rho+1}(\Omega)}^{\rho+1} + L\|u\|_{L^2(\Omega)}^2 + c \\ &\leq -\omega(\|\nabla u\|_{L^p(\Omega)}^p + \|u\|_{L^{\rho+1}(\Omega)}^{\rho+1}) + c'. \end{aligned}$$

Integrating from 0 to T we obtain that

$$\frac{1}{2}\|u(T)\|_H^2 + \omega\int_0^T (\|\nabla u(s)\|_{L^p(\Omega)}^p + \|u(s)\|_{L^{\rho+1}(\Omega)}^{\rho+1})ds \leq C_1(\|u_0\|_H, T). \tag{21}$$

Also, using Hölder inequality, we have

$$\begin{aligned} \|A_1(u)\|_{W^{-1,p'}(\Omega)} &\leq \|\nabla u\|_{L^p(\Omega)}^{p-1}, \\ \|A_2(u)\|_{L^{\frac{\rho+1}{\rho}}(\Omega)} &\leq \|u\|_{L^{\rho+1}(\Omega)}^{\rho}. \end{aligned} \tag{22}$$

Now, from equation (19), estimates (22) and [G-Z, Chapt. I, §5] it follows that

$$\begin{aligned} \|\frac{du}{dt}\|_{V^*} &\leq \|A_1(u) + A_2(u)\|_{V^*} + \|B(u)\|_{V^*} \\ &\leq \|A_1(u)\|_{W^{-1,p'}(\Omega)} + \|A_2(u)\|_{L^{\frac{\rho+1}{\rho}}(\Omega)} + L\|u\|_H + const \\ &\leq c(\|u\|_{W_0^{1,p}(\Omega)}^{p-1} + \|u\|_{L^{\rho+1}(\Omega)}^{\rho} + 1). \end{aligned}$$

Therefore, for $\theta = \min\{\frac{p}{p-1}, \frac{\rho+1}{\rho}\}$ condition (13) holds:

$$\int_0^T \|\frac{du}{dt}\|_{V^*}^\theta dt \leq c'\int_0^T (\|u\|_{W_0^{1,p}(\Omega)}^{(p-1)\theta} + \|u\|_{L^{\rho+1}(\Omega)}^{\rho\theta} + 1)dt \leq C_2(\|u_0\|_H, T).$$

Theorem 2. *The problem* (19) *has a global attractor in* $L^2(\Omega)$.

Remark 3. The case of $(2m)^{th}$-order nonlinear operators as in [BA, p. 144] can be treated similarly with minor changes in the proofs.

3.2. **Example 2:** Let Ω be as in Example 1 and Δ_D as in the introduction. Consider the following nonlinear second order partial differential equation

$$u_t = \mathrm{div}(|\mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u}) + \mathrm{f}(\mathrm{u}) \tag{23}$$

with homogeneous Dirichlet boundary condition, $p > 2$ and $f : R^+ \to R^+$ globally Lipschitz continuous.

Next we write an abstract week formulation of the above problem (see [LI, Ex. 3.2, p. 191]). Consider $H = H^{-1}(\Omega)$ endowed with the inner product given by the extension to $H \times H$ of the bilinear form

$$\langle u, v\rangle = \left(u, (-\Delta_D)^{-1} v\right)_{L^2(\Omega)}, \quad u, v \in L^2(\Omega).$$

For $V = L^p(\Omega)$ and $p > 2$ we have that

$$V \subset H \subset V^*$$

with continuous and dense inclusions (here H is identified with its dual).

If, naively, we try to obtain the estimates needed in Section 2 we proceed as follows:

$$\langle \operatorname{div}(|u|^{p-2}\nabla u), v\rangle = \int_\Omega \operatorname{div}(|u|^{p-2}\nabla u)[(-\Delta_D)^{-1}v]dx$$

$$= \frac{1}{p-1}\int_\Omega \Delta(|u|^{p-2}u)[(-\Delta_D)^{-1}v]dx$$

$$= -\frac{1}{p-1}\int_\Omega \nabla(|u|^{p-2}u)\nabla[(-\Delta_D)^{-1}v]dx + \frac{1}{p-1}\int_{\partial\Omega}\frac{\partial}{\partial n}[|u|^{p-2}u][(-\Delta_D)^{-1}v]d\sigma$$

$$= -\frac{1}{p-1}\int_\Omega \nabla(|u|^{p-2}u)\nabla[(-\Delta_D)^{-1}v]dx$$

$$= -\frac{1}{p-1}\int_\Omega |u|^{p-2}uv dx - \frac{1}{p-1}\int_{\partial\Omega}|u|^{p-2}u\frac{\partial}{\partial n}[(-\Delta_D)^{-1}v]d\sigma.$$

Then the form

$$a(u, v) = \frac{1}{p-1}\int_\Omega |u|^{p-2}uv dx$$

satisfies

$$a(u, u-v) - a(v, u-v) \geq 0$$

and

$$a(u, u) = \frac{1}{p-1}\|u\|^p_{L^p(\Omega)}.$$

Therefore, the operator $A : V \to V^*$ defined by $a(\cdot,\cdot)$ is monotone and satisfies

$$\langle A(u), u\rangle_{V^*,V} = \frac{1}{p-1}\|u\|^p_{L^p(\Omega)},$$
$$\|A(u)\|_{V^*} \leq \frac{1}{p-1}\|u\|^{p-1}_{L^p(\Omega)}.$$

If $A_1 : V \to V^*$ is the operator defined by the extension of the form

$$a_1(u, v) = \frac{1}{p-1}\int_{\partial\Omega}|u|^{p-2}u\frac{\partial}{\partial n}[(-\Delta_D)^{-1}v]d\sigma$$

then

$$\|A_1(u)\|_{V^*} \leq c\|u\|^{p-1}_{L^p(\Omega)}.$$

However we are not able to prove monotonicity.

Since we are interested in solving the Dirichlet boundary value problem we disregard the operator A_1 and consider the problem:

$$\begin{cases} \frac{du}{dt}(t) + A(u(t)) + B(u(t)) = 0, \ t > 0, \\ u(0) = u_0 \in H \end{cases} \tag{24}$$

with B being the extension to H of the Nemitskii operator given by $-f$ in $L^2(\Omega)$.

Now the problem (24) with A given by the form $a(\cdot,\cdot)$ defines a semigroup on $cl_H(D(A_H))$. Since conditions **H1** and **H2** are also satisfied we have, by Lemma 1, that $cl_H(D(A_H)) = H$ and by Theorem 1 that:

Theorem 3. *The problem* (24) *has a global attractor in* $H^{-1}(\Omega)$.

Next, we briefly describe why a sufficiently regular solution must satisfy (23) and the boundary condition.

If $v \in \Delta_D(C_0^\infty(\Omega))$ it follows from (24) that

$$\begin{aligned} \int_\Omega \frac{du}{dt}[(-\Delta_D)^{-1}v]dx &= -\frac{1}{p-1}\int_\Omega |u|^{p-2}uvdx + \int_\Omega f(u)[(-\Delta)^{-1}v]dx \\ &= \frac{1}{p-1}\int_\Omega \Delta(|u|^{p-2}u)[(-\Delta_D)^{-1}v]dx + \int_\Omega f(u)[(-\Delta_D)^{-1}v]dx \end{aligned}$$

so that

$$u_t = \frac{1}{p-1}\Delta(|u|^{p-2}u) + f(u)$$

almost everywhere in Ω. To verify that the boundary condition is also satisfied we shall use the following:

Lemma 7. *For given* $\phi_1 \in C_0^\infty(\partial\Omega)$ *there exists a function* $v \in L^p(\Omega)$ *such that*

$$\frac{\partial}{\partial n}[(-\Delta_D)^{-1}v] = \phi_1 \quad in \ \ \partial\Omega.$$

Proof. Note that for any $g \in L^p(\Omega)$, $\phi, \phi_1 \in C_0^\infty(\partial\Omega)$ the unique solution of the elliptic boundary value problem

$$\begin{cases} \Delta^2 w = g \ \text{ in } \ \Omega, \\ w = \phi \ \text{ in } \ \partial\Omega, \\ \frac{\partial w}{\partial n} = \phi_1 \ \text{ in } \ \partial\Omega, \end{cases}$$

belongs to $W^{4,p}(\Omega)$. Then $v = -\Delta_D w$ is a required function. □

If we now assume that $v \in L^p(\Omega)$ we have

$$\begin{aligned}\int_\Omega \frac{du}{dt}[(-\Delta_D)^{-1}v]dx &= -\frac{1}{p-1}\int_\Omega |u|^{p-2}uv dx \\ &- \frac{1}{p-1}\int_{\partial\Omega}|u|^{p-2}u\frac{\partial}{\partial n}[(-\Delta_D)^{-1}v]d\sigma + \int_\Omega f(u)[(-\Delta_D)^{-1}v]dx.\end{aligned}$$

Since the solution satisfies (23) almost everywhere in Ω we have that

$$\int_{\partial\Omega}|u|^{p-2}u\frac{\partial}{\partial n}[(-\Delta_D)^{-1}v]d\sigma = 0.$$

Therefore from Lemma 7 $u = 0$ in $\partial\Omega$.

These computations justify the choice of A.

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