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ON REAL SINGULARITIES WHICH FIBER AS COMPLEX SINGULARITIES

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RESUMO

Singularidades reais que possuem fibração de Milnor são raras, e não ocorrem em geral. Um método para construir tais singularidades é discutido em [8,9].

Neste artigo, usamos resultados da teoria de singularidades para transformar a propriedade de "satisfazer uma condição de Milnor", em um problema de determinação finita dos germes de aplicações que definem as singularidades. Estes resultados são aplicados às famílias discutidas em [8,9] para mostrar que deformações de ordem suficientemente alta dessas singularidades também satisfazem à condição de Milnor. Os resultados de [6,7] são então utilizados para a obtenção de estimativas precisas da ordem de tais perturbações para o caso das singularidades homogêneas e quase-homogêneas. Finalmente, na última seção, resultados semelhantes são discutidos para campos vetorias de " tipo Morse", no sentido de [1].

ON REAL SINGULARITIES WHICH FIBER AS COMPLEX SINGULARITIES

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ABSTRACT. Real singularities which have an associated "Milnor fibration" are rare, in the sense that there are not "too many" of them. A method for constructing such singularities was found in [8,9]. In this article we use results on singularity theory to translate the property of "satisfying Milnor's condition" into a problem of finite determinacy of map-germs. We then apply these results to the singularities of [8,9] to show that these form a "robust" family, in the sense that sufficiently high order deformations of them also satisfy Milnor's condition. We then use the results of [6,7] to give precise estimates on the orders of the perturbations that we can allow in the homogeneous and weighted homogeneous cases. Finally, in the last section we make similar considerations for the vector fields of "Morse type", in the sense of [1]. These are germs of holomorphic vector fields whose phase portrait has a very nice topological description.

1. INTRODUCTION

It is well known [5] that if

$$f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)\,,$$

is a holomorphic function with a critical point at 0, then for every sufficiently small sphere $\mathcal{S}_\varepsilon$ centered at 0, the function

$$\phi = \frac{f}{\|f\|} : \mathcal{S}_\varepsilon - K \to \mathcal{S}^1\,,$$

is the projection map of a locally trivial fiber bundle, where $K = f^{-1}(0) \cap \mathcal{S}_\varepsilon$ is the link of 0. This is the **Milnor fibration** of f. Milnor also proved, in the last chapter of his book, a fibration theorem

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for real singularities. He proved that if

$$f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)\,, \quad n > p\,.$$

is a real analytic map such that its Jacobian matrix Df has rank p on a punctured neighbourhood of $0 \in \mathbb{R}^n$, then one has a locally trivial fibre bundle

$$\phi : \mathcal{S}_\varepsilon - K \to \mathcal{S}^{p-1}\,,$$

where $\mathcal{S}_\varepsilon$ is now a small sphere in $\mathbb{R}^n$ and $K = f^{-1}(0) \cap \mathcal{S}_\varepsilon$ is the link of 0. Alas, the hypothesis of Df having maximal rank everywhere near 0 is too strong, and as Milnor pointed out in his book, it is difficult to find examples of such functions. In fact Milnor asks whether there exist "non-trivial" examples when $p = 2$, others than the holomorphic ones. This was answered positively by Looijenga in [4] by proving an existence theorem, but he did not give explicit singularities which satisfy Milnor's hypothesis. A method for constructing such singularities was recently given in [8,9]: Given real analytic vector fields F and X in $\mathbb{C}^n \cong \mathbb{R}^{2n}$, one has the real analytic map,

$$\psi_{F,X}(z) = < F(z), X(z) > = \sum_{i=1}^{n} F_i(z) \cdot \overline{X}_i(z)\,,$$

and it turns out that there are infinite families of such vector fields for which the function $\psi_{F,X}$ satisfies Milnor's hypothesis, for all $n > 1$.

In this note we prove that satisfying Milnor's condition at 0 is a "robust" phenomenon for vector fields F and X, in the sense that if these vector fields are such that the function $\psi_{F,X}$ satisfies Milnor's condition, then any perturbation of them of sufficiently high order will produce vector fields for which the corresponding funtion ψ also satisfies Milnor's condition.

In the last section, we consider vector fields " of Morse type", in the sense of [1]. This is a weaker condition than satisfying Milnor's condition. This means essentially that the function $\psi_{F,X}$ is a submersion restricted to a punctured neighbourhood of 0 in the (singular) variety $M = {\psi_{F,X}}^{-1}(0)$. For the sake of geometry, we restrict to the case envisaged in [1], where X is the radial vector field $(z_1, z_2, \dots, z_n)$ (up to a linear automorphism given by a hermitian matrix) and the vector field F is holomorphic. In this case the variety M is the set of points where the holomorphic foliation defined by F is tangent to the foliation defined by all spheres centered at the origin. Vector fields of Morse type have a nice description of their phase portrait ([1]), since in this case the function "distance to 0" is a Morse function on each leaf of F.

2. About Milnor's Condition

Let $C(n,p)$ be the space of smooth map-germs from $(\mathbb{R}^n,0)$ into $(\mathbb{R}^p,0)$, and let $\mathcal{J}^k(n,p)$ be the set of k-jets $\{j^k\}$ of elements of $C(n,p)$. Let $\mathcal{R}$ be the group of germs of C^∞-diffeomorphisms $(\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$; $\mathcal{R}$ acts on $C(n,p)$ by composition on the right. An element $f \in C(n,p)$ is k-$\mathcal{R}$-determined if the $\mathcal{R}$-orbit of f contains all germs whose k-jet at 0 coincides with the k-jet of f. Similarly, one has the group C^l-$\mathcal{R}$, of local diffeomorphisms of class C^l, $l > 0$ or homeomorphisms if $l = 0$, which acts on the corresponding space $C^l(n,p)$. We will consider rather the induced equivalence relation on $C(n,p)$ and the corresponding notion of k-C^l-$\mathcal{R}$-determinacy. One says that f is C^l-$\mathcal{R}$-finitely determined if it is k-C^l-$\mathcal{R}$-determined for some k.

Let $\mathcal{I}_\mathcal{R}(f)$ be the ideal of $C(n)$ generated by the $p \times p$-minors of the Jacobian matrix of f and $N_\mathcal{R}(f) = det\{(df_x)(df_x)^t\}$ = sum of squares of $p \times p$-minors of df_x. We say that $N_\mathcal{R}(f)$ satisfies a Lojasiewicz condition of order r (> 0) if there exists a constant $c > 0$ such that $N_\mathcal{R}(f) \geq c|x|^r$.

The condition that f be C^l-$\mathcal{R}$-finitely determined for $0 \leq l < \infty$ is equivalent to the condition that $N_\mathcal{R}(f)$ satisfies a Lojasiewicz condition for some r. Moreover, this inequality provides precise estimates for the degree of C^l-$\mathcal{R}$-determinacy of f. This result was discovered by Kuo in [2], for the case p=1. The extension to higher dimensions was considered by many authors.(See [11] for a complete account on the subject, and [6] for precise estimates for the degree of C^l-determinacy of f)

Definition 2.1. Let $f \in C(n,p)$ be analytic. We say that f satisfies Milnor's condition at 0 if its Jacobian Matrix Df has rank p everywhere on a punctured neighbourhood of $0 \in \mathbb{R}^n$.

The theorem below is implicit in [11] (see also [6], Proposition 2.4.d):

Theorem 2.1. *Let $f \in C(n,p)$ be analytic. Then f satisfies Milnor's condition at 0 if and only if f is C^l-$\mathcal{R}$-finitely determined for every $l \in [0,\infty)$.*

Proof:

For analytic germs, $N_\mathcal{R}(f)$ satisfies a Lojasiewicz condition at zero if and only if the variety of the ideal $\mathcal{I}_\mathcal{R}(f)$ reduces to 0 ([11], Lemma 6.2), and this is clearly equivalent to the Milnor's condition for f. Now, the result follows for instance, from [6], proposition 2.4.

Let F and X be germs of real analytic vector fields in $(\mathbb{C}^n, 0)$, and define the *real analytic* map $\psi_{F,X} \in C(2n, 2)$ given by

$$\psi_{F,X} = \sum_{i=1}^{n} Fi(z) \cdot \overline{X}_i(z) \ .$$

The two vector fields F and X are complex orthogonal at a point z if and only if $\psi_{F,X}$ vanishes. Otherwise this function is, in some sense, measuring how far these two vectors are from being orthogonal. More precisely, if we let X^T be the real orthogonal complement of $X(z)$, so that this is a $(2n-1)$-dimensional real vector space, then the complex line spanned by $F(z)$ intersects this hyperplane in a real line, and the argument of the number $i\bar{\psi}(z)$ is the angle by which we must rotate the vector $F(z)$ in its complex line to make it be real orthogonal to $X(z)$.

Applying the above result to $\psi_{F,X}$, we can state the following:

Corollary 2.2. *Suppose that the map-germ $\psi_{F,X}$ satisfies Milnor's condition at 0. Let F_* and X_* be analytic vector fields obtained, respectively, by adding to F and X terms of sufficiently high degree. Then the corresponding function ψ_{F_*,X_*} satisfies Milnor's condition at 0. Moreover, there exists a germ of homeomorphism $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\psi_{F_*,X_*} = \psi_{F,X} \circ h^{-1}$.*

3. The Weighted Homogeneous Case

Definition 3.1. Given $(r_1, \ldots, r_n : d_1, \ldots, d_p)$, positive integers, an element $f \in C(n, p)$ is *weighted homogeneous of type* $(r_1, \ldots, r_n : d_1, \ldots, d_p)$ if for all $\lambda \in \mathbb{R} - \{0\}$ one has:

$$F(\lambda x_1, \ldots, \lambda x_n) = (\lambda^{d_1} f_1(x), \lambda^{d_2} f_2(x), \ldots \lambda^{d_p} f_p(x))$$

The numbers $r_1, \ldots, r_n$ are *the weights* of $x_1, \ldots, x_n$. The integer $d = d_1 + \cdots + d_p$, is *the total degree* of f. (Note: These integers are not unique!.)

Definition 3.2. i) Given weights $(r_1, \ldots, r_n)$ for $(x_1, \ldots, x_n)$, and a monomial $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$, $\alpha = \alpha_1 + \cdots + \alpha_n$, we define its *filtration* by:

$$fil(x^\alpha) = \sum_{i=1}^{n} \alpha_i r_i \ .$$

ii) Given $f \in C(n, 1)$, its filtration is:

$$fil(f) = \inf_{\alpha} \left\{ fil(x^\alpha) \mid \left(\frac{\partial^\alpha f}{\partial x^\alpha} \right)(0) \neq 0 \right\}$$

where $\dfrac{\partial^\alpha f}{\partial x^\alpha}$ means all the partial derivatives of f with respect to the x_i's, with total degree α.

iii) Given $f = (f_1, \dots, f_p) \in C(n,p)$, with $fil(f_i) = d_i$, we say that f has *filtration* $fil(f) = (d_1, \dots, d_p)$.

We will be considering complex valued, real analytic functions, from $\mathbb{C}^n$ into $\mathbb{C}$. These can be regarded as functions into $\mathbb{R}^2$: Let

$$f : \mathbb{C}^n \to \mathbb{C} ,$$

be real analytic. Its real and imaginary parts are,

$$f_1 = \operatorname{Re} f = \frac{1}{2}(f + \bar{f}) , \quad f_2 = \operatorname{Im} f = \frac{1}{2}(f - \bar{f}) .$$

So, f_1 is weighted homogeneous if and only if f_2 is weighted homogeneous, and in this case they have the same weights and total degree.

We now consider vector fields F and X such that the function $\psi_{F,X}$ is weighted homogeneous.

Example 3.1. i) Let

$$F(x,y) = (x^p, y^q) , \quad X(x,y) = (x^r, y^s) ,$$

then $\psi(x,y) = x^p\bar{x}^r + y^q\bar{y}^s$, is weighted homogeneous of type $(q+s, p+r : (p+r)(q+s))$. The corresponding function $\psi_{F,X}$ satisfies Milnor's condition at 0 if and only if $p \neq r$ and $q \neq s$, by [9].

ii) Let F be as above, but take $X(x,y) = (y^s, x^r)$, with $p > r$ and $q > s$, then $\psi_{F,X}$ is weighted homogeneous of type $(q+s, p+r : (p+s)(q+r)$, but the function $\psi_{F,X}$ satisfies Milnor's condition at 0 if and only if $r, s = 0, 1$. If $r = s = 0$, then $\psi_{F,X}$ is actually holomorphic, so this follows from [5]. If $r = s = 1$, this is proved in [8]. If $r = 0$ and $s = 1$, one has,

$$\psi_{F,X}(x,y) = x^p\,\bar{y} + y^q ,$$

and it is an exercise to show that its Jacobian matrix has rank 2 everywhere away from 0. It is also an exercise to show that if either r or s is > 1, then $\psi_{F,X}$ does not satisfy Milnor's condition at 0.

iii) More generally, if $F(z_1, \dots, z_n)$ is any vector field of the form

$$F(z) = \left(k_1\, z_{i_1}^{a_1}, \dots, k_n\, z_{i_n}^{a_n}\right) ,$$

where the a_i's are integers > 1, the k_i's are non-zero constant complex numbers and $z_{i_1}, \dots, z_{i_n}$ is some (any) permutation of the components of z, and if we take $X(z) = (\lambda_1\, z_1, \dots, \lambda_n\, z_n)$, $\lambda_i \in \mathbb{C} - \{0\}$, then the function ψ is weighted homogeneous and it satisfies Milnor's condition, by [8]. Moreover, by [8], in this case the projection map of the corresponding fibre bundle is the obvious one

$$\phi = \frac{f}{\|f\|} ,$$

c.f. [3].

We remark that the fact that F and X be weighted homogeneous is not sufficient, neither necessary, to guarantee that $\psi_{F,X}$ is weighted homogeneous.

Theorem 3.1. *Let F and X be germs of real analytic vector fields in $(\mathbb{C}^n, 0)$ such that the map $\psi_{F,X}$ is weighted homogeneous and satisfies Milnor's condition at 0. Let F_t and X_t be analytic vector fields of the form,*

$$F_t(z,t) = (F_1(z) + \theta_1(z,t), \dots , F_n(z) + \theta_n(z,t)) ,$$
$$X_t(z,t) = (X_1(z) + \rho_1(z,t), \dots , X_n(z) + \rho_n(z,t)) ,$$

with $F_0 = F$ and $X_0 = X$. Assume further that each θ_i and each ρ_i has total filtration $> d$ in $(z_1, \dots , z_n)$, $|t| \leq 1$, or filtration $= d$, for sufficiently small values of $|t|$, where d is the total weight of $\psi_{F,X}$. Then the corresponding function $\psi_{F,X}$ satisfies Milnor's condition at 0.

Proof:
With the hypothesis, ψ_{F_t,X_t} will be a perturbation of $\psi_{F,X}$ of filtration $\geq d$. Hence, the result follows from [7], Proposition 2.2.

Example 3.2. If $F(x,y) = (x^p, y^q)$, $X(x,y) = (x^r, y^s)$, then we can take perturbations of filtration $> (p+r)(q+s)$, or small perturbations of filtration $(p+r)(q+s)$. In fact one can be more precise: By [6], Proposition 2.2, if F_1, F_2, X_1 and X_2 are as above, then the statements of the theorem are true whenever we take $\theta_1, \theta_2, \rho_1, \rho_2$ such that

$$fil(\theta_1) \geq p \cdot (q+s) \quad , \quad fil(\theta_2) \geq q \cdot (p+r)$$
$$fil(\rho_1) \geq r \cdot (q+s) \quad , \quad fil(\rho_2) \geq s \cdot (p+r) .$$

4. The Homogeneous Case

If F and X are homogeneous vector fields of degrees, say d and e respectively. Then the function

$$\psi(z) = \Sigma F_i(z) \cdot \overline{X}_i(z) ,$$

is homogeneous of degree $d+e$. However, the function ψ can be homogeneous even if F and X are not homogeneous. For instance, consider the vector fields in $\mathbb{C}^2$ defined by:

$$F(x,y) = (x^p, y^q) , \quad X(x,y) = (x^r, y^s) .$$

Then $\psi(x,y) = x^p\bar{x}^r + y^q\bar{y}^s$, which is homogeneous whenever $p+r = q+s$.

Theorem 4.1. *Let $F = (F_1, \dots, F_n)$ and $X = (X_1, \dots, X_n)$ be real analytic vector fields in $\mathbb{C}^n$ such that the function ψ is homogeneous of total degree d. Let F_* and X_* be analytic perturbations of F and X, i.e. analytic families of vector fields of the form,*

$$F_t(z,t) = (F_1(z) + \theta_1(z,t), \dots, F_n(z) + \theta_n(z,t)) \ ,$$
$$X_t(z,t) = (X_1(z) + \rho_1(z,t), \dots, X_n(z) + \rho_n(z,t)) \ ,$$

with $F_0 = F$ and $X_0 = X$. Assume further that each θ_i and each ρ_i involves only terms in $(z_1, \dots, z_n)$ of degree $> d$, $|t| \leq 1$, or degree $= d$ for sufficiently small values of $|t|$. Then if $\psi_{F,X}$ satisfies Milnor's condition at 0, then for each value of t, ψ_{F_*,X_*} also satisfies Milnor's condition at 0.

We note that if F and X are homogeneous, then $\psi_{F,X}$ is homogeneous. In this case 4.1 can be refined:

Theorem 4.2. *If F is homogeneous of degree d_1, X is homogeneous of degree d_2, and $\psi_{F,X}$ satisfies Milnor's condition at 0, then for any perturbation F_* of F of degree $> d_1$ in $(z_1, \dots, z_n)$ and any perturbation X_* of X of degree $> d_2$ in these variables, ψ_{F_*,X_*} also satisfies Milnor's condition at 0. The same statement holds for sufficiently small homogeneous perturbations of F and X of degree d_1 and d_2, respectively.*

Proof:
With the hypothesis, $\psi_{F,X}$ is homogeneous of degree $d_1 + d_2$. It follows from [6] that $\psi_{F,X}$ is (d_1+d_2)-$C^1 - \mathcal{R}$-determined, and every sufficiently small perturbation of degree $d_1 + d_2$ is $C^0 - \mathcal{R}$-equivalent to $\psi_{F,X}$. Hence, the result follows.

Example 4.1. If X is of the form $(\lambda_1 z_1 \, , \, \lambda_2 z_2)$, $\lambda_1 \, , \, \lambda_2 \in \mathbb{C}$, and F_t is any vector field of the form

$$F_t(z_1, z_2) = \left(z_1^k + t\theta_1(z_1, z_2) \ , \ z_2^k + t\theta_2(z_1, z_2)\right) \ ,$$

where θ_1 and θ_2 contain only terms of total degree $\geq k$, then ψ_{F_t,X_t} satisfies Milnor's condition at 0.

5. Vector fields of Morse type

There is a particular case which is specially interesting. This is when the germ of the vector field F at 0 is holomorphic with an isolated zero at the origin, and X is the radial vector field $X(z) = (z_1, \dots, z_n)$. Then the variety

$$M = \{z \in \mathbb{C}^n \mid < F(z), X(z) >= 0\} \ ,$$

is the set of points where the vector field F is tangent to the spheres centered at 0. M is defined by two real analytic equations, so that generically, M will have real codimension 2, and it will be singular at $0 \in \mathbb{C}^n$. If $M=0$ or has real codimension 2 and is smooth away from the origin, then F was defined in [1] to be *of Morse type.* It was shown that in these cases one has a nice topological description of the phase portrait of F near the singular point. The solutions of F are all transversal to M and each point in M is either a local minimal point in its leaf, or else a saddle point; $M - \{0\}$ has finitely many components, each consisting entirely of local minimal points or saddle points. Away from a tubular neighbourhood of $M - \{0\}$, the leaves are immersed copies of the plane R^2 or cylinders $R^2 - \{0\}$, and the limit set of these leaves is the origin. It is clear that if the function $\psi_{F,X}$ satisfies Milnor's condition at 0 then F is of Morse type.

The same statements hold if we deform the metric in $\mathbb{C}^n$ and consider the spheres corresponding to some other hermitian metric. Given a Riemannian metric ρ, we say that F is a vector field of Morse type with respect to ρ if $M = \{z \in \mathbb{C}^n, 0 / < F(z), \bar{A}.z >= 0\}$, is a $2n-2$-smooth manifold away from zero, or reduces to 0, where A is the hermitian matrix defining ρ.

Theorem 5.1. *Let F be an analytic vector field of Morse type with respect to some Riemannian metric ρ. Then :*

(i) There exists an integer k such that every vector field G with $j^kF(0) = j^kG(0)$, is of Morse type with respect to ρ.

(ii) When F is homogeneous of degree k, any deformation G of F by terms of degree $> k$ is of Morse type with respect to ρ. The same is true if G is a sufficiently small deformation of F by terms of degree k. Moreover, in all cases, the varieties ${\psi_{F,X}}^{-1}(0)$ and ${\psi_{G,X}}^{-1}(0)$ are homeomorphic.

Proof:

The result will follow, as before, by connecting the geometric definition with the finite determinacy of the real analytic map-germ ψ. More precisely, we know that F is of Morse type if and only if $\psi_{F,X}$ restricted to a neighbourhood of $M - 0$ is a submersion. It follows from [11] that this last condition is equivalent to the finite determinacy of $\psi_{F,X}$, with respect to Mather's contact group $\mathcal{K}$. This proves (i).

The proof of (ii) follows from [6], Corollary 3.14.

Corollary 5.2. *Let F be a homogeneous vector field in $\mathbb{C}^n, 0$. Then, the property of being of Morse type with respect to some Riemannian metric ρ is robust with respect to small perturbations of the metric.*

Proof:

A small deformation of the metric will produce a small deformation of $\psi_{F,X}$ of the same degree, and the result will follow again by [6], Corollary 3.14.

We do not know whether the above result is true for weighted homogeneous vector fields, but we can see in the next example that the previous arguments do no apply in this case.

Example 5.1. (see [1] and [10])

Given integers $p \geq q > 2$, let F be the vector field in $\mathbb{C}^2$ defined by $F(z_1, z_2) = (-qz_2^{q-1}, pz_1^{p-1})$. The solutions of F are the fibers of the polynomial $f(z_1, z_2) = z_1^p + z_2^q$. The variety M of contacts consists of the two axis $\{z_1 = 0\}$, $\{z_2 = 0\}$ and the variety $M' = M_1' \cap M_2'$, where

$$M_1' = \{z \in \mathbb{C}^2 \,/\, q|z_2|^{q-2} = p|z_1|^{p-2}\},$$

$$M_2' = \{z \in \mathbb{C}^2 / q\, arg(z_2) = p\, arg(z_1)\},$$

$M - \{0\}$ is smooth of codimension 2, so F is of Morse type. In fact, it follows that (see Example 3.5 in [1]) M' is a cone over a torus knot of type (p,q), so that the pair $(D, D \cap M')$ is homeomorphic to $(D, D \cap f^{-1}(0))$ by [5], where D is the unit ball.

Let ϵ be a positive real number, $\epsilon < 1$, and $A = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}$ be a perturbation of the usual hermitian metric in $\mathbb{C}^2$.

Then $\psi_{F,A.z} = -qz_2^{q-1}\bar{z}_1 + pz_1^{p-1}\bar{z}_2 - \epsilon(qz_2^{q-1}\bar{z}_2 + pz_1^{p-1}\bar{z}_1)$. When $\epsilon = 0$, this map-germ is weighted homogeneous of type $(q-2, p-2, pq-(p+q))$, but, with respect to the same weights, the filtrations of the terms $-\epsilon(qz_2^{q-1}\bar{z}_2)$ and $-\epsilon(pz_1^{p-1}\bar{z}_1)$ are $pq - 2q$ and $pq - 2p$, respectively.

Thus, $\psi_{F,A.z}$ is a perturbation of the weighted homogeneous map-germ $\psi_{F,X}$ by terms of higher and lower filtration than its weighted degree, hence the previous results do not apply.

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