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MAPS OF MANIFOLDS INTO THE PLANE WHICH LIFT TO STANDARD EMBEDDINGS IN CODIMENSION TWO

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RESUMO

Seja $f : M \to \mathbb{R}^2$ uma aplicação diferenciável de uma variedade fechada n-dimensional $n \geq 2$, no plano e seja $\pi_2^{n+2} : R^{n+2} \to R^2$ uma projeção ortogonal. Dizemos que f tem "standard lifting property", quando todo mergulho $\tilde{f} : M \to R^{n+2}$ com $\pi_2^{n+2} \circ \tilde{f} = f$ é "standard", num certo sentido. Neste artigo, obtemos condições suficientes para que uma aplicação genérica f tenha " standard lifting property" quando M é uma superfície fechada ou uma n-esfera de homotopia.

Maps of manifolds into the plane which lift to standard embeddings in codimension two

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Abstract. Let $f : M \to \mathbf{R}^2$ be a smooth map of a closed n-dimensional manifold ($n \geq 2$) into the plane and let $\pi_2^{n+2} : \mathbf{R}^{n+2} \to \mathbf{R}^2$ be an orthogonal projection. We say that f has the standard lifting property, if every embedding $\tilde{f} : M \to \mathbf{R}^{n+2}$ with $\pi_2^{n+2} \circ \tilde{f} = f$ is standard in a certain sense. In this paper we give some sufficient conditions for a generic smooth map f to have the standard lifting property when M is a closed surface or an n-dimensional homotopy sphere.

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1. Introduction

Let M be a smooth closed n-dimensional manifold (possibly disconnected or nonorientable). We assume that there is a notion of "standard" embeddings of M into the Euclidean space $\mathbf{R}^{n+2}$. For example, for $M = S^n$, an embedding is standard if its image bounds an embedded $(n+1)$-disk in $\mathbf{R}^{n+2}$. Let $\pi_p^{n+2} : \mathbf{R}^{n+2} \to \mathbf{R}^p$ be an orthogonal projection $(1 \leq p < n+2)$, which is fixed once for all for each pair $(n+2, p)$.

DEFINITION 1.1. A smooth map $f : M \to \mathbf{R}^p$ is said to have the *standard lifting property* (SLP for short), if every smooth embedding $\tilde{f} : M \to \mathbf{R}^{n+2}$ with $\pi_p^{n+2} \circ \tilde{f} = f$ is standard.

Note that the above definition does not depend on the choice of the particular projection π_p^{n+2}.

For example, it is known that Morse functions $f : S^n \to \mathbf{R}$ $(n \geq 1)$ with exactly two critical points have the SLP (for details, see Proposition 5.1). By a celebrated result of Scharlemann [**Sl**], Morse functions $f : S^2 \to \mathbf{R}$ with exactly four critical points also have the SLP. Furthermore, it is known that every embedding $f : S^n \to \mathbf{R}^{n+1}$ has the SLP and every immersion with normal crossings $f : S^1 \to \mathbf{R}^2$ with at most two crossings has the SLP. Some immersions of the torus into $\mathbf{R}^3$ which have the SLP have been studied in [**Sm**].

Note that, for a given smooth embedding $\tilde{f} : M \to \mathbf{R}^{n+2}$, we can isotope it slightly so that the map $\pi_p^{n+2} \circ \tilde{f}$ is *generic* in the sense of Mather [**M**]. Thus, it is natural to consider the problem of characterizing those generic maps which have the SLP.

The problem which we consider first in the present paper is to characterize those generic smooth maps $f : F \to \mathbf{R}^2$ which have the SLP with respect to an orthogonal projection $\pi_2^4 : \mathbf{R}^4 \to \mathbf{R}^2$, where F is a smooth closed surface (possibly disconnected or nonorientable). Generic maps of surfaces into $\mathbf{R}^2$ which we consider here are the class of excellent maps defined by Whitney [**W**]. More precisely, a smooth map $f : F \to \mathbf{R}^2$ is said to be *excellent* if it has *folds* and *cusps* as its singularities (for details, see [**W**, §4] or [**C**]). Note that folds arise along arcs and circles in F, while cusps arise as isolated points in F. Furthermore, the set of the singularities of f, called the *singular set* of f, denoted by Σ, is a finite disjoint union of (nonsingular) simple closed curves in F.

The standard embeddings of surfaces which we consider in the present paper are defined as follows. Let F be a closed surface each of whose components has even Euler characteristic. We say that a smooth embedding $\tilde{f} : F \to \mathbf{R}^4$ is *standard*, if its image bounds a disjoint union of embedded 3-dimensional handlebodies in $\mathbf{R}^4$, where a 3-*dimensional handlebody* is a 3-dimensional ball or a compact connected 3-manifold obtained by attaching some 1-handles (orientable or nonorientable) to a 0-handle.

Our first main result is the following.

THEOREM 1.2. *Let F be a smooth closed surface and $f : F \to \mathbf{R}^2$ a smooth excellent map. If f has no cusps and $\pi_1^2 \circ f : F \to \mathbf{R}$ is a Morse function with at most four critical points, then f has the SLP.*

Note that the conditions in the above theorem can easily be verified. In fact, as will be seen later, one can check the above conditions only by the discriminant of the excellent map f; i.e., by the map f restricted to the singular set Σ of f, which is a finite disjoint union of simple closed curves in F.

We note that, in the above theorem, the surface is diffeomorphic to the 2-sphere, the disjoint union of two 2-spheres, the torus, or the Klein bottle. The case of the (1-component) 2-sphere is a direct consequence of Scharlemann's result [**Sl**] cited above.

In the case of the 2-sphere, the above result is sharp in the sense that there are many examples of nonstandard embeddings $\tilde{f} : S^2 \to \mathbf{R}^4$ such that $\pi_2^4 \circ \tilde{f}$ is an excellent map without cusps and that $\pi_1^2 \circ \pi_2^4 \circ \tilde{f}$ is a Morse function with exactly *six* critical points. For details, see Remarks 3.2 and 3.3.

Using the excellent maps as in Theorem 1.2 as ingredients, we can construct a large class of excellent maps with the SLP by means of certain connected sum operations (for details, see Theorem 4.6 in §4).

As to higher dimensions, consider a smooth map $f : M \to \mathbf{R}^2$ of a smooth closed n-dimensional manifold into $\mathbf{R}^2$ ($n \geq 2$). If f is generic as discussed previously in this section, then f has folds and cusps as its singularities and each singularity has its own index (for example, see [**HL1**]). We say that a generic smooth map $f : M \to \mathbf{R}^2$ is *special generic*, if all its singularities are folds of extreme indices (or definite folds) (for example, see [**BdR**] or [**Sa2**]). Note that for $n = 2$, an excellent map is special generic if and only if it has no cusps. In [**Sa2**] those manifolds which admit special generic maps into the plane have been characterized; for example, the n-dimensional sphere S^n with $n \geq 2$ and an arbitrary homotopy n-sphere with $n \geq 7$ are such manifolds. Furthermore, every such manifold has a free fundamental group.

The second main result of the present paper is as follows.

THEOREM 1.3. *Let* $f : M \to \mathbf{R}^2$ *be a special generic map of a closed connected and simply connected* n*-dimensional manifold* M *into the plane* ($n \geq 2$). *If* $\pi_1^2 \circ f : M \to \mathbf{R}$ *has at most four critical points, then* f *has the SLP in the sense that if there exists an embedding* $\tilde{f} : M \to \mathbf{R}^{n+2}$ *with* $\pi_2^{n+2} \circ \tilde{f} = f$, *then* M *is diffeomorphic to the standard* n*-sphere* S^n *and* $\tilde{f}$ *is standard.*

The paper is organized as follows. In §2, we prepare basic lemmas necessary for the proof of Theorem 1.2. Some of the results may be folklore; however, to the authors' knowledge, there has been nothing explicitly written in the literature, so that we have included detailed proofs here. In §3, we prove Theorem 1.2 and give a slight generalization in the 2-sphere case (Proposition 3.5). In §4, we define certain kinds of connected sum operations of excellent maps of surfaces into the plane and give a method to construct a large class of excellent maps with the SLP (see Theorem 4.6). In §5, we discuss the higher dimensional case and prove Theorem 1.3.

Throughout the paper, all manifolds and maps are of class C^∞.

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2. Basic results

In this section, we prove the following.

PROPOSITION 2.1. *Let $\tilde{f} : S^2 \to \mathbf{R}^4$ be a smooth embedding such that $\pi_1^4 \circ \tilde{f} : S^2 \to \mathbf{R}$ is a Morse function with exactly two critical points. Then $\tilde{f}(S^2)$ bounds a 3-ball B embedded in $\mathbf{R}^4$ such that $(\pi_1^4)^{-1}(x) \cap B$ is either a point or diffeomorphic to the 2-disk for each $x \in \pi_1^4 \circ \tilde{f}(S^2)$.*

PROPOSITION 2.2. *Let $\tilde{f} : S_1^2 \cup S_2^2 \to S^3 \times [0,1]$ be a smooth embedding of the disjoint union of two 2-spheres such that $p \circ \tilde{f}|S_i^2$ is a Morse function with exactly two critical points with critical values 0 and 1 $(i = 1,2)$, where $p : S^3 \times [0,1] \to [0,1]$ is the projection to the second factor. Then $\tilde{f}(S_1^2 \cup S_2^2)$ bounds a disjoint union of embedded 3-balls $B_1^3 \cup B_2^3$ in $S^3 \times [0,1]$ such that $(B_1^3 \cup B_2^3) \cap p^{-1}(x)$ consists of two points for $x = 0, 1$ and is diffeomorphic to the disjoint union of two 2-disks for each x with $0 < x < 1$.*

REMARK 2.3. A similar result for embeddings of the union of an arbitrary number of 2-spheres is obtained in [**KSS**, Lemma 1.6], which is known as Horibe-Yanagawa's lemma. Note that they consider embeddings into $\mathbf{R}^3 \times [0,1]$ and that the disjoint union of 3-balls constructed in the proof of Horibe-Yanagawa's lemma lies in $\mathbf{R}^3 \times [0,\infty)$, while in Proposition 2.2 we consider embeddings into $S^3 \times [0,1]$ and the disjoint union of 3-balls lies in $S^3 \times [0,1]$.

For the proof of the above propositions we need the following lemmas.

LEMMA 2.4. *Let $j : S^1 \to \mathbf{R}^3$ be a standard embedding. Suppose that D_1 and D_2 are two smoothly embedded 2-disks in $\mathbf{R}^3$ bounded by $j(S^1)$. Then D_1 and D_2 are isotopic by an isotopy fixing $\partial D_1 = \partial D_2 = j(S^1)$.*

Proof. We embed $\mathbf{R}^3$ in S^3 as $S^3 = \mathbf{R}^3 \cup \{\infty\}$. Let N be a small tubular neighborhood of $j(S^1)$ and set $\bar{D}_i = D_i - \mathrm{Int} N$. We may assume that $\partial\bar{D}_1 \cap \partial\bar{D}_2 = \emptyset$ and $D_1 \cap D_2 \cap N = j(S^1)$, isotoping D_1 fixing ∂D_1 if necessary. Furthermore, we may assume that $\bar{D}_1$ and $\bar{D}_2$ intersect transversely. Then $\bar{D}_1 \cap \bar{D}_2$ is a (possibly empty) finite disjoint union of simple closed curves. If $\bar{D}_1 \cap \bar{D}_2 = \emptyset$, then $D_1 \cup D_2$ bounds an embedded 3-ball in $\mathbf{R}^3 = S^3 - \{\infty\}$ by the 3-dimensional Schoenflies Theorem (proved by J. W. Alexander), and D_1 and D_2 are isotopic by an isotopy fixing $j(S^1)$ in $\mathbf{R}^3$. If $\bar{D}_1 \cap \bar{D}_2 \neq \emptyset$, then there exists an innermost component C of $\bar{D}_1 \cap \bar{D}_2$ in $\bar{D}_2$ such that $D_2' \cap \bar{D}_1 = C$, where D_2' is the closed 2-disk in $\bar{D}_2$ bounded by C. Let the closed 2-disk in $\bar{D}_1$ bounded by C be denoted by D_1'. Then, since

$D_1' \cup D_2'$ is an embedded 2-sphere in S^3, it bounds two embedded 3-balls B and B' in S^3.

Choosing B appropriately, we may assume that $B \cap D_1 = D_1'$ and hence that B does not intersect $j(S^1)$ (see Figure 1). Then, using B, we can isotope D_1 in $\mathbf{R}^3$, fixing ∂D_1, so that the number of components of the intersection set is smaller than the original one, provided that B does not contain $\infty \in S^3$.

Figure 1 here

Suppose that B contains $\infty \in S^3$. Let us consider the 2-disk Δ with corner along C given by the union $A \cup D_2'$, where A is the closure of $D_1 - D_1'$. Then we can push Δ a little into B' and smooth the corners in a way to get a 2-disk D_3 such that $D_3 \cap \Delta = \partial D_1$. Then $D_1 \cup D_3$ bounds two 3-balls $\tilde{B}$ and $\tilde{B}'$ in S^3. We may assume that $B \subset \tilde{B}$ and $\tilde{B}' \subset B'$. Then, since $\tilde{B}'$ does not contain $\infty \in S^3$, we can isotope D_1 to D_3, fixing ∂D_1, using $\tilde{B}'$ so that the number of components of the intersection set is smaller than the original one.

Repeating this procedure, we can eliminate all the intersection circles isotoping D_1 to a 2-disk D' such that $D_2 \cap D' = \partial D_2 = \partial D' = \partial D_1$. Then we can isotope D' fixing $\partial D'$ to D_2 in $\mathbf{R}^3$ as discussed above. This completes the proof. ||

We note that the above result appears in [**LR**, Theorem 5.1]. However, their proof seems to be incomplete, since the 3-ball B (or B') in $S^3 - \{\infty\}$ as above may intersect with $j(S^1)$ as is seen in Figure 1.

LEMMA 2.5. *Suppose that $j : S^1 \cup S^1 \to S^3$ is a standard embedding; i.e., $j(S^1 \cup S^1)$ bounds a disjoint union of embedded 2-disks $D_1 \cup D_2$ in S^3. Suppose that S and S' are two embedded 2-spheres in $S^3 - j(S^1 \cup S^1)$ each of which separates the two components of $j(S^1 \cup S^1)$. Then S and S' are isotopic by an isotopy fixing $j(S^1 \cup S^1)$.*

Proof. By an isotopy, we may assume that S and S' intersect transversely. If $S \cap S' = \emptyset$, then $S \cup S'$ bounds an embedded $S^2 \times [-1,1]$ in $S^3 - j(S^1 \cup S^1)$ and we can isotope S' to S fixing $j(S^1 \cup S^1)$. Suppose that $S \cap S' \neq \emptyset$. Then $S \cap S'$ is a finite disjoint union of simple closed curves. Let C be an innermost component of $S \cap S'$ in S' which bounds an embedded 2-disk Δ' in S' with $\Delta' \cap S = C$. We may assume that the 3-ball B bounded by S in S^3 containing Δ' does not contain the first component of $j(S^1 \cup S^1)$. Then we can choose an embedded 2-disk Δ bounded by C in S such that the 3-ball B' bounded by the 2-sphere $\Delta \cup \Delta'$

contained in B does not contain the second component of $j(S^1 \cup S^1)$ and hence that $B' \cap j(S^1 \cup S^1) = \emptyset$. Then, using B', we can isotope S, fixing $j(S^1 \cup S^1)$, so that the number of components of the intersection set with S' is smaller than that of the original one. Repeating this procedure, we can eliminate all the intersection circles and can isotope S, fixing $j(S^1 \cup S^1)$, to S'. This completes the proof. ||

LEMMA 2.6. *Suppose that $j : S^1 \cup S^1 \to S^3$ is a standard embedding as in Lemma 2.5. If $j(S^1 \cup S^1)$ bounds another set of disjointedly embedded 2-disks $D_1' \cup D_2'$ in S^3, then $D_1 \cup D_2$ and $D_1' \cup D_2'$ are isotopic by an isotopy fixing $\partial D_1 \cup \partial D_2 = \partial D_1' \cup \partial D_2' = j(S^1 \cup S^1)$.*

Proof. Let S and S' be embedded 2-spheres in S^3 which separate the two components of $D_1 \cup D_2$ and $D_1' \cup D_2'$ respectively. Then by Lemma 2.5, we may assume that $S = S'$. Then applying Lemma 2.4 to each of the two components of $S^3 - S$, we get the result. ||

Proof of Proposition 2.1. Denote the two critical points of $\pi_1^4 \circ \tilde{f}$ by $c_1, c_2 (\in S^2)$, where $a_1 = \pi_1^4 \circ \tilde{f}(c_1) < \pi_1^4 \circ \tilde{f}(c_2) = a_2$. Note that $\pi_1^4 \circ \tilde{f}(S^2) = [a_1, a_2]$. Since $d\tilde{f}_{c_i}(T_{c_i}S^2) \subset (\pi_1^4)^{-1}(a_i)$ $(i = 1, 2)$, we can isotope $\tilde{f}$ by an isotopy whose support is contained in some small neighborhoods of c_i so that $\tilde{f}(\Delta_i) \subset H_{a_i}$ and that $\pi_1^4 \circ \tilde{f}|(S^2 - (\Delta_1 \cup \Delta_2))$ has no critical points for some closed 2-disk neighborhoods Δ_i of c_i in S^2 $(i = 1, 2)$, where $H_t = (\pi_1^4)^{-1}(t)$ for $t \in \mathbf{R}$. Then for every $b \in (a_1, a_2)$, the embedded circle $H_b \cap \tilde{f}(S^2)$ in H_b is isotopic to $\tilde{f}(\partial\Delta_1)$ in H_{a_1} when we identify H_b and H_{a_1} in a natural manner. (For example, use a vector field argument to show this.) Consider the trace of $\tilde{f}(\Delta_1)$ with respect to this isotopy. Then, when b approaches to a_2, we do not know if the trace approaches to $\tilde{f}(\Delta_2)$. However, using Lemma 2.4, we can arrange the isotopy so that the trace coincides with $\tilde{f}(\Delta_2)$ when $b = a_2$. The resulting trace B of $\tilde{f}(\Delta_1)$ is a smoothly embedded 3-ball in $\mathbf{R}^4$ bounded by $\tilde{f}(S^2)$ with the required property. This completes the proof. ||

REMARK 2.7. For an embedding $\tilde{f} : S^1 \to \mathbf{R}^3$ of the circle S^1 into $\mathbf{R}^3$, a result similar to Proposition 2.1 also holds. In this case, we can prove it in a much simpler way as follows. Set $\pi_1^3 \circ \tilde{f}(S^1) = [a_1, a_2]$. Then $\tilde{f}(S^1) \cap (\pi_1^3)^{-1}(t)$ consists of two points for all $t \in (a_1, a_2)$. Then we can connect the two points by a straight line segment in $(\pi_1^3)^{-1}(t)$. Then the union of the segments for all $t \in (a_1, a_2)$ together with $\tilde{f}(S^1)$ forms a 2-disk bounded by $\tilde{f}(S^1)$ with the required property.

Proof of Proposition 2.2. We proceed as in the proof of Proposition 2.1, starting with the images of two 2-disks contained in S_1^2 and S_2^2. Then, using Lemma 2.6 instead of Lemma 2.4, we can construct a disjoint union of embedded 3-balls $B_1 \cup$

$B_2 \subset S^3 \times [0,1]$ such that $\partial B_1 \cup \partial B_2 = \tilde{f}(S_1 \cup S_2)$ with the required property. This completes the proof. ||

3. Proof of Theorem 1.2

Let $\tilde{f} : F \to \mathbf{R}^4$ be an arbitrary embedding such that $\pi_2^4 \circ \tilde{f} = f$. We denote the singular set of $f : F \to \mathbf{R}^2$ by Σ: more precisely, $\Sigma = \{x \in F : \mathrm{rank} df_x < 2\}$, which is a finite disjoint union of simple closed curves in F (see [**W**]). Since the excellent map f does not have a cusp point, the critical points of $\pi_1^2 \circ f : F \to \mathbf{R}$ are exactly the critical points of $\pi_1^2 \circ f|\Sigma : \Sigma \to \mathbf{R}$, which is again a Morse function (for details, see [**Fu**], for example). Thus, in particular, the number of critical points is even and hence is equal to two or four by our assumption. If it is equal to two, then F must be diffeomorphic to the 2-sphere and the embedding $\tilde{f}$ is standard by Proposition 2.1. Thus, in the following, we assume that $\pi_1^2 \circ f$ has four critical points c_1, c_2, c_3 and $c_4 \in F$. By changing π_1^2 slightly and by reordering the critical points, we may assume that $t_1 < t_2 < t_3 < t_4$, where $t_i = \pi_1^2 \circ f(c_i)$ $(i = 1, 2, 3, 4)$ (see [**M**]).

Since f is excellent, the images of neighborhoods of c_1 and c_4 are as in Figure 2. As to c_2 and c_3, we have the four possibilities for the images of neighborhoods of c_2 and c_3 as in Figure 3 (1)–(4).

Figure 2 here

Figure 3 here

For $t \in T = [t_1, t_4] - \{t_1, t_2, t_3, t_4\}$, we set $X_t = (\pi_1^2 \circ f)^{-1}(t)$, which is a finite disjoint union of simple closed curves in F, since t is a regular value of $\pi_1^2 \circ f$. Furthermore, if t and t' belong to the same connected component of T, then X_t and $X_{t'}$ are naturally diffeomorphic to each other. Let $\pi : \mathbf{R}^2 \to \mathbf{R}$ be the orthogonal projection such that $\ker \pi$ and $\ker \pi_1^2$ are perpendicular to each other. Then $\varphi_t = \pi \circ f|X_t$ is a Morse function for $t \in T$, since f is an excellent map without cusps, and the set of critical points of φ_t coincides with $\Sigma \cap X_t$. Furthermore, if t and t' belong to the same connected component of T, then φ_t and $\varphi_{t'}$ have the same number of critical points on the corresponding connected components of X_t and $X_{t'}$, since f has no cusps. Set $s_i = (t_i + t_{i+1})/2$ for $i = 1, 2, 3$.

Suppose that the image by f of a neighborhood of c_3 is as in Figure 3 (2) or (4). Then, since the number of critical points of φ_{s_2} is equal to that of φ_{s_3}

minus two, φ_{s_2} has no critical point and hence the case of Figure 3 (4) does not occur. In the case of Figure 3 (2), F is diffeomorphic to the disjoint union of two 2-spheres and their images by $\tilde{f}$ are separated by the hyperplane $(\pi_1^2 \circ \pi_2^4)^{-1}(s_2)$. Furthermore, the Morse function $\pi_1^2 \circ f$ restricted to each component has exactly two critical points. Thus by Proposition 2.1, $\tilde{f}$ is standard. When the image by f of a neighborhood of c_2 is as in Figure 3 (1) or (3), a similar argument can be applied.

It remains the following four cases.

Case 1. The image by f of a neighborhood of c_2 is as in Figure 3 (2) and that of c_3 is as in Figure 3 (1).

In this case, it is easy to see that F is diffeomorphic to the disjoint union of two 2-spheres and that the Morse function $\pi_1^2 \circ f$ restricted to each component has exactly two critical points. By isotoping $\tilde{f}$, we may assume that the embedding $\tilde{f}$ satisfies the hypothesis of Proposition 2.2. Here we embed $S^3 \times [0,1]$ to $\mathbf{R}^4$ so that its image contains $\tilde{f}(F)$ and that the projection $\pi_1^2 \circ \pi_2^4$ is compatible with the projection $p : S^3 \times [0,1] \to [0,1]$ to the second factor when restricted to a neighborhood of $\tilde{f}(F)$. Then the result follows from Proposition 2.2.

Case 2. The image by f of a neighborhood of c_2 is as in Figure 3 (2) and that of c_3 is as in Figure 3 (3).

In this case F is diffeomorphic to the 2-sphere and the Morse function $\pi_1^2 \circ f$ has exactly four critical points. Then the result follows from the celebrated theorem of Scharlemann [**Sl**]. (In fact, in our situation, we do not need Scharlemann's theorem. One can prove the result more easily using an argument similar to that for Case 4 below.)

Case 3. The image by f of a neighborhood of c_2 is as in Figure 3 (4) and that of c_3 is as in Figure 3 (1).

An argument similar to that for Case 2 shows that $\tilde{f}$ is standard.

Case 4. The image by f of a neighborhood of c_2 is as in Figure 3 (4) and that of c_3 is as in Figure 3 (3).

In this case, F is diffeomorphic to the torus or the Klein bottle. By our hypothesis, $f(F) \cap (\pi_1^2)^{-1}(t_2 + \varepsilon)$ (resp. $f(F) \cap (\pi_1^2)^{-1}(t_3 - \varepsilon)$) consists of two segments for a sufficiently small $\varepsilon > 0$. Let $b_2 \in (\pi_1^2)^{-1}(t_2 + \varepsilon)$ (resp. $b_3 \in (\pi_1^2)^{-1}(t_3 - \varepsilon)$) be a point which separates the two segments. Then $X_{t_2+\varepsilon}$ (resp. $X_{t_3-\varepsilon}$) consists of two circles and the two components of $\tilde{f}(X_{t_2+\varepsilon})$ (resp. $\tilde{f}(X_{t_3-\varepsilon})$)

in the 3-dimensional hyperplane $H_2 = (\pi_1^2 \circ \pi_2^4)^{-1}(t_2+\varepsilon)$ (resp. $H_3 = (\pi_1^2 \circ \pi_2^4)^{-1}(t_3-\varepsilon)$) are separated by the plane $(\pi_2^4)^{-1}(b_2)$ (resp. $(\pi_2^4)^{-1}(b_3)$). Furthermore, $\pi \circ \pi_2^4 \circ \tilde{f}|X_{t_2+\varepsilon}$ (resp. $\pi \circ \pi_2^4 \circ \tilde{f}|X_{t_3-\varepsilon}$) restricted to each component is a Morse function with exactly two critical points. Thus $\tilde{f}(X_{t_2+\varepsilon})$ (resp. $\tilde{f}(X_{t_3-\varepsilon})$) is standard in H_2 (resp. in H_3); i.e., there exist disjointedly embedded 2-disks D_1 and D_2 in H_2 (resp. D_1' and D_2' in H_3) such that $\partial D_1 \cup \partial D_2 = \tilde{f}(X_{t_2+\varepsilon})$ (resp. $\partial D_1' \cup \partial D_2' = \tilde{f}(X_{t_3-\varepsilon})$). Note that D_1 and D_2 are constructed by using unions of straight line segments lying in $\ker \pi_2^4$ (see Remark 2.7).

Using H_2 and H_3, we decompose $\tilde{f}(F)$ into the union of four annuli. Then, using the four 2-disks D_1, D_2, D_1' and D_2', we construct disjointedly embedded four 2-spheres S_1, S_2, S_3 and S_4 in $\mathbf{R}^4$ such that $S_1 \cup S_3 \subset (\pi_1^2 \circ \pi_2^4)^{-1}((t_2+\varepsilon, t_3-\varepsilon))$, $S_2 \subset (\pi_1^2 \circ \pi_2^4)^{-1}([t_1, t_2+\varepsilon))$ and $S_4 \subset (\pi_1^2 \circ \pi_2^4)^{-1}((t_3-\varepsilon, t_4])$, where, for example, S_4 is constructed from $(\tilde{f}(F) \cap (\pi_1^2 \circ \pi_2^4)^{-1}([t_3-\varepsilon, t_4])) \cup D_1' \cup D_2'$ by smoothing the corners and by pushing it into $(\pi_1^2 \circ \pi_2^4)^{-1}((t_3-\varepsilon, t_4])$. Since the 2-disks D_1, D_2, D_1' and D_2' have been constructed by using straight line segments lying in $\ker \pi_2^4$, we may assume, by modifying S_i slightly, that $\pi_2^4|S_i$ are excellent maps without cusps $(i = 1, 2, 3, 4)$ and that $\pi_2^4|(S_1 \cup S_3)$ satisfies the condition in Case 1 above. Furthermore, by an isotopy of $\mathbf{R}^2$, which is naturally lifted to an isotopy of $\mathbf{R}^4$, we can arrange so that $(\pi_1^2 \circ \pi_2^4)|S_i$ $(i = 2, 4)$ is a Morse function with exactly two critical points. Then by the argument as in Case 1 together with Proposition 2.1, we see that there exist disjointedly embedded four 3-balls B_1, B_2, B_3 and B_4 in $\mathbf{R}^4$ such that $B_2 \subset (\pi_1^2 \circ \pi_2^4)^{-1}([t_1, t_2+\varepsilon))$, $B_4 \subset (\pi_1^2 \circ \pi_2^4)^{-1}((t_3-\varepsilon, t_4])$, and $\partial B_i = S_i$ $(i = 1, 2, 3, 4)$. Using the four 3-balls thus constructed, we can further construct four embedded 3-balls B_i' $(i = 1, 2, 3, 4)$ in $\mathbf{R}^4$ such that $B_2' \subset (\pi_1^2 \circ \pi_2^4)^{-1}([t_1, t_2+\varepsilon])$, $B_4' \subset (\pi_1^2 \circ \pi_2^4)^{-1}([t_3-\varepsilon, t_4])$, $B_1' \cap B_3' = \emptyset$, $\partial B_1' \cup \partial B_3' = (\tilde{f}(F) \cap (\pi_1^2 \circ \pi_2^4)^{-1}([t_2+\varepsilon, t_3-\varepsilon])) \cup D_1 \cup D_2 \cup D_1' \cup D_2'$, $\partial B_2' = (\tilde{f}(F) \cap (\pi_1^2 \circ \pi_2^4)^{-1}([t_1, t_2+\varepsilon])) \cup D_1 \cup D_2$, $\partial B_4' = (\tilde{f}(F) \cap (\pi_1^2 \circ \pi_2^4)^{-1}([t_3-\varepsilon, t_4])) \cup D_1' \cup D_2'$, $(B_1' \cup B_3') \cap B_2' = D_1 \cup D_2$ and $(B_1' \cup B_3') \cap B_4' = D_1' \cup D_2'$. Then the union $B = B_1' \cup B_2' \cup B_3' \cup B_4'$ is an embedded 3-dimensional handlebody (solid torus or solid Klein bottle) in $\mathbf{R}^4$ and satisfies $\partial B = \tilde{f}(F)$. Thus $\tilde{f}$ is standard. This completes the proof. ||

REMARK 3.1. Let $f : F \to \mathbf{R}^2$ be an excellent map of a closed surface into the plane. Then the condition that f has no cusps is not a necessary condition for f to have the SLP. For example, the excellent map $f : S^2 \to \mathbf{R}^2$ such that the image of its singular set is as shown in Figure 4 has two cusps (see [**HL2**, p.154]). Nevertheless f has the SLP, since it admits an orthogonal projection $\pi_1^2 : \mathbf{R}^2 \to \mathbf{R}$ such that $\pi_1^2 \circ f$ is a Morse function with exactly two critical points.

Figure 4 here

Remark 3.2. When $F = S^2$, Theorem 1.2 is sharp in the sense that there are many examples of nonstandard embeddings $\tilde{f} : S^2 \to \mathbf{R}^4$ such that $\pi_2^4 \circ \tilde{f}$ is an excellent map without cusps and that $\pi_1^2 \circ \pi_2^4 \circ \tilde{f}$ is a Morse function with exactly six critical points. More precisely, the following is true: given an integral polynomial $a(t)$ with $a(1) = 1$, there exists an embedding $\tilde{f} : S^2 \to \mathbf{R}^4$ such that $\pi_2^4 \circ \tilde{f} : S^2 \to \mathbf{R}^2$ is an excellent map without cusps, that $\pi_1^2 \circ \pi_2^4 \circ \tilde{f} : S^2 \to \mathbf{R}$ is a Morse function with exactly six critical points, and that the Alexander polynomial of $\tilde{f}(S^2)$ exists and is equal to $a(t)$. In fact, the embeddings constructed in [**Ki**, §3] are easily seen to satisfy the required properties. See also [**Y**, §5].

Remark 3.3. In the example mentioned in Remark 3.2, the Morse function $\pi_1^2 \circ \pi_2^4 \circ \tilde{f} : S^2 \to \mathbf{R}$ is C^∞ $\mathcal{A}$-equivalent to the height function as shown in Figure 5 (for the notion of C^∞ $\mathcal{A}$-equivalence, see Definition 4.1 of the next section). We can also construct examples of nonstandard embeddings $\tilde{f} : S^2 \to \mathbf{R}^4$ such that $\pi_2^4 \circ \tilde{f}$ is an excellent map without cusps and that $\pi_1^2 \circ \pi_2^4 \circ \tilde{f}$ is a Morse function with exactly six critical points in the following manner. Let k be a nonstandardly embedded circle in $\mathbf{R}^3$ such that the projection $\pi_1^3 : \mathbf{R}^3 \to \mathbf{R}$ to the last coordinate restricted to k is a Morse function with exactly four critical points (such knots are called *two-bridge knots*. See, for example, [**Sb**]). We may assume that $k \cap (\mathbf{R}^2 \times \{0\})$ consists of two points and that the projection π_1^3 restricted to $k' = k \cap \{(x, y, z) \in \mathbf{R}^3 : z \geq 0\}$ has exactly three critical points. Then let $\tilde{k}$ be the spun 2-sphere (or the rotated arc) constructed from k' (for example, see [**A**] or [**AC**, §3]. See also Figure 6). It is known that $\tilde{k}$ is a nonstandardly embedded 2-sphere in $\mathbf{R}^4$. Let $\tilde{f} : S^2 \to \mathbf{R}^4$ be an embedding such that $\tilde{f}(S^2) = \tilde{k}$. It is easy to see that for an appropriate projection $\pi_2^4 : \mathbf{R}^4 \to \mathbf{R}^2$, the map $\pi_2^4 \circ \tilde{f}$ is an excellent map without cusps and that the image of its singular set consists of three concentric circles in the plane (see Figure 6). Furthermore, for an arbitrary orthogonal projection π_1^2, the composition $\pi_1^2 \circ \pi_2^4 \circ \tilde{f}$ is a Morse function which has exactly six critical points and which is C^∞ $\mathcal{A}$-equivalent to the height function as shown in Figure 7.

Figure 5 here

Figure 6 here

Figure 7 here

REMARK 3.4. As to another notion of "standard" embeddings of surfaces in $\mathbf{R}^4$, see [**HK**]. In fact, the embeddings which are standard in our sense are also standard in the sense of [**HK**]; however, the converse is not true for embeddings of nonorientable surfaces.

Using Horibe-Yanagawa's lemma [**KSS**, Lemma 1.6] as mentioned in Remark 2.3, we can also prove the following.

PROPOSITION 3.5. (1) *Let $f : S_1^2 \cup \cdots \cup S_r^2 \to \mathbf{R}^2$ be a smooth excellent map of the disjoint union of r 2-spheres into the plane ($r \geq 1$). Suppose that f has no cusps and that $\pi_1^2 \circ f$ is a Morse function with exactly two critical points on each component S_i^2 ($1 \leq i \leq r$). Then f has the SLP.*

(2) *Let $f : S^2 \to \mathbf{R}^2$ be a smooth excellent map. Suppose that f has no cusps and that $\pi_1^2 \circ f$ is a Morse function with critical points $c_0, c_1, \ldots, c_{2r+1}$ ($r \geq 1$), where c_0 has index two, $c_1, \ldots, c_r$ have index one, $c_{r+1}, \ldots, c_{2r+1}$ have index zero, and the critical values corresponding to the critical points of index one coincide with each other (see Figure 8). Then f has the SLP.*

Figure 8 here

Proof. (1) This is a direct consequence of Horibe-Yanagawa's lemma.

(2) Set $a_0 = \pi_1^2 \circ f(c_0)$ and $a_1 = \pi_1^2 \circ f(c_1) = \cdots = \pi_1^2 \circ f(c_r)$. By an argument similar to that in the proof of Theorem 1.2, we see that for a sufficiently small $\varepsilon > 0$, $f(S^2) \cap (\pi_1^2)^{-1}(a_1 - \varepsilon)$ is a disjoint union of $(r+1)$ segments. Then, using Horibe-Yanagawa's lemma, we can construct 3-balls $B_0, B_1, \ldots, B_{r+1}$ embedded in $\mathbf{R}^4$ such that $B_i \cap B_j = \emptyset$ for $1 \leq i < j \leq r+1$, $B_0 \subset (\pi_1^2 \circ \pi_2^4)^{-1}([a_1 - \varepsilon, a_0])$, $B_1 \cup \cdots \cup B_{r+1} \subset (\pi_1^2 \circ \pi_2^4)^{-1}((-\infty, a_1 - \varepsilon])$, and that $B_0 \cap (B_1 \cup \cdots \cup B_{r+1})$ is a union of disjoint $(r+1)$ 2-disks embedded in the hyperplane $(\pi_1^2 \circ \pi_2^4)^{-1}(a_1 - \varepsilon)$ bounded by $\tilde{f}(S^2) \cap (\pi_1^2 \circ \pi_2^4)^{-1}(a_1 - \varepsilon)$. Then the union $B = B_0 \cup B_1 \cup \cdots \cup B_{r+1}$ is a 3-ball embedded in $\mathbf{R}^4$ such that $\partial B = \tilde{f}(S^2)$. Thus the embedding $\tilde{f}$ is standard. This completes the proof. ||

REMARK 3.6. Let $\tilde{f} : S^2 \to \mathbf{R}^4$ be an embedding. Suppose that there exists an orthogonal projection $\pi_1^4 : \mathbf{R}^4 \to \mathbf{R}$ such that $\pi_1^4 \circ \tilde{f}$ is a Morse function with $(2r+2)$ critical points ($r \geq 1$) as in Proposition 3.5. Then, Hosokawa [**H**] claimed that $\tilde{f}$

is standard. As Scharlemann points out [**Sl**, p.125], Hosokawa's proof contained an error (see also [**HK**, p.243]), and we do not know if the embedding $\tilde{f}$ is standard or not until now, except the case $r = 1$, which has been affirmatively answered by Scharlemann [**Sl**].

4. A generalization of Theorem 1.2

In this section, we generalize Theorem 1.2 to a class of excellent maps of surfaces into the plane which are obtained by certain connected sum operations from excellent maps as in Theorem 1.2 and Proposition 3.5.

In the following, we use the following definition.

DEFINITION 4.1. Let $h_i : X_i \to Y_i$ $(i = 1, 2)$ be smooth maps between smooth manifolds. We say that h_1 and h_2 are *C^∞ $\mathcal{A}$-equivalent* if there exist diffeomorphisms $\Phi : X_1 \to X_2$ and $\varphi : Y_1 \to Y_2$ such that $h_2 = \varphi \circ h_1 \circ \Phi^{-1}$.

Let $g_i : F_i \to \mathbf{R}^2$ $(i = 1, 2)$ be excellent maps of closed surfaces into the plane. Let $\pi_1^2 : \mathbf{R}^2 \to \mathbf{R}$ be a generic orthogonal projection such that $\pi_1^2 \circ g_i$ are Morse functions (see [**M**]). Assume that m_1 is the unique point in F_1 which takes the absolute minimum a_1 of $\pi_1^2 \circ g_1$ and that M_2 is the unique point in F_2 which takes the absolute maximum A_2 of $\pi_1^2 \circ g_2$. Let $\varepsilon > 0$ be so small that $\pi_1^2 \circ g_1$ contains no critical values in the interval $(a_1, a_1 + 2\varepsilon)$ and that $\pi_1^2 \circ g_2$ contains no critical values in the interval $(A_2 - 2\varepsilon, A_2)$. Note that $\Delta_1 = (\pi_1^2 \circ g_1)^{-1}([a_1, a_1 + \varepsilon])$ and $\Delta_2 = (\pi_1^2 \circ g_2)^{-1}([A_2 - \varepsilon, A_2])$ are 2-disk neighborhoods of m_1 in F_1 and M_2 in F_2 respectively. Then we can construct an excellent map $g : F_1 \sharp F_2 \to \mathbf{R}^2$ such that $F_1 \sharp F_2 = (F_1 - \mathrm{Int}\Delta_1) \cup (F_2 - \mathrm{Int}\Delta_2)$ is the connected sum of the surfaces F_1 and F_2 obtained by gluing $F_1 - \mathrm{Int}\Delta_1$ and $F_2 - \mathrm{Int}\Delta_2$ along their boundaries, that $g|(F_1 - \mathrm{Int}\Delta_1)$ and $g|(F_2 - \mathrm{Int}\Delta_2)$ are C^∞ $\mathcal{A}$-equivalent to $g_1|(F_1 - \mathrm{Int}\Delta_1)$ and $g_2|(F_2 - \mathrm{Int}\Delta_2)$ respectively, that $\pi_1^2 \circ g|(F_1 - \mathrm{Int}\Delta_1)$ and $\pi_1^2 \circ g|(F_2 - \mathrm{Int}\Delta_2)$ are C^∞ $\mathcal{A}$-equivalent to $\pi_1^2 \circ g_1|(F_1 - \mathrm{Int}\Delta_1)$ and $\pi_1^2 \circ g_2|(F_2 - \mathrm{Int}\Delta_2)$ respectively, and that $(\pi_1^2 \circ g(F_1 - \mathrm{Int}\Delta_1)) \cap (\pi_1^2 \circ g(F_2 - \mathrm{Int}\Delta_2)) = \pi_1^2 \circ g(\partial\Delta_1) = \pi_1^2 \circ g(\partial\Delta_2)$.

DEFINITION 4.2. We say that an excellent map $f : F_1 \sharp F_2 \to \mathbf{R}^2$ is obtained by a *mini-max connected sum operation* from the excellent maps $f_1 : F_1 \to \mathbf{R}^2$ and $f_2 : F_2 \to \mathbf{R}^2$ if f is C^∞ $\mathcal{A}$-equivalent to an excellent map g constructed as above from excellent maps g_1 and g_2 which are C^∞ $\mathcal{A}$-equivalent to f_1 and f_2 respectively.

We can generalize the above operation as follows. In the construction above, we assume that $\pi_1^2 \circ g_1$ takes the minimum value a_1 at the points $m_1, \ldots, m_r$

($r \geq 1$), that $\pi_1^2 \circ g_2$ takes the maximum value A_2 at the points $M_1, \ldots, M_r$, that $g_1(m_1), \ldots, g_1(m_r)$ are all distinct, and that $g_2(M_1), \ldots, g_2(M_r)$ are all distinct ($r \geq 1$). Then using disjoint 2-disk neighborhoods Δ_j^1 of m_j in F_1 and Δ_j^2 of M_j in F_2 ($j = 1, 2, \ldots, r$), we can construct an excellent map $g^r : F_1 \sharp_r F_2 \to \mathbf{R}^2$ by a method similar to the above, where $F_1 \sharp_r F_2 = (F_1 - \mathrm{Int}(\Delta_1^1 \cup \cdots \cup \Delta_r^1)) \cup (F_2 - \mathrm{Int}(\Delta_1^2 \cup \cdots \cup \Delta_r^2))$.

DEFINITION 4.3. We say that an excellent map $f : F_1 \sharp_r F_2 \to \mathbf{R}^2$ is obtained by an *r-generalized mini-max connected sum operation* from the excellent maps $f_1 : F_1 \to \mathbf{R}^2$ and $f_2 : F_2 \to \mathbf{R}^2$ if f is C^∞ $\mathcal{A}$-equivalent to an excellent map g^r constructed as above from excellent maps g_1 and g_2 which are C^∞ $\mathcal{A}$-equivalent to f_1 and f_2 respectively.

LEMMA 4.4. *Suppose that $f_i : F_i \to \mathbf{R}^2$ ($i = 1, 2$) are excellent maps of closed surfaces into the plane which have the SLP. Then we have the following.*

(1) *Every excellent map $f : F_1 \sharp F_2 \to \mathbf{R}^2$ obtained by a mini-max connected sum operation from f_1 and f_2 has the SLP.*

(2) *If F_1 and F_2 are connected, then every excellent map $f : F_1 \sharp_r F_2 \to \mathbf{R}^2$ obtained by an r-generalized mini-max connected sum operation from f_1 and f_2 ($r \geq 2$) has the SLP.*

Proof. (1) It is obvious that a smooth map which is C^∞ $\mathcal{A}$-equivalent to a map with the SLP also has the SLP. Thus we may assume that $f = g, f_1 = g_1$ and $f_2 = g_2$, where g is the excellent map constructed as above from g_1 and g_2. Let $\tilde{g} : F_1 \sharp F_2 \to \mathbf{R}^4$ be an arbitrary embedding such that $\pi_2^4 \circ \tilde{g} = g$. Set $y = (\pi_1^2 \circ g(F_1 - \mathrm{Int}\Delta_1)) \cap (\pi_1^2 \circ g(F_2 - \mathrm{Int}\Delta_2)) = \pi_1^2 \circ g(\partial\Delta_1) = \pi_1^2 \circ g(\partial\Delta_2)$ and $H = (\pi_1^2 \circ \pi_2^4)^{-1}(y)$, which is a 3-dimensional hyperplane in $\mathbf{R}^4$. In the following, we assume that $\pi_1^2 \circ g(F_1 - \mathrm{Int}\Delta_1) \subset [y, \infty)$ and $\pi_1^2 \circ g(F_2 - \mathrm{Int}\Delta_2) \subset (-\infty, y]$.

It is easily seen that $C = \tilde{g}(F_1 \sharp F_2) \cap H$ is a standardly embedded circle in H, since there exists an orthogonal projection $\pi : H \to \mathbf{R}$ such that $\pi|C$ is a Morse function with exactly two critical points (see Remark 2.7). Thus there exists a 2-disk D embedded in H such that $\partial D = C$. Then using an argument similar to the Case 4 of the proof of Theorem 1.2, we can "decompose" the embedding $\tilde{g}$ into two embeddings $\tilde{g_i} : F_i \to \mathbf{R}^4$ which are lifts of g_i. In particular, there exists an embedding $b : I \times D^2 \to \mathbf{R}^4$ ($I = [-1, 1]$) such that $b(I \times D^2) \cap H = b(\{0\} \times D^2) = D$, $b(I \times D^2) \cap \tilde{g_1}(F_1) = b(\{1\} \times D^2), b(I \times D^2) \cap \tilde{g_2}(F_2) = b(\{-1\} \times D^2)$, and that $\tilde{g}(F_1 \sharp F_2)$ coincides with $(\tilde{g_1}(F_1) - b(\{1\} \times D^2)) \cup b(I \times \partial D^2) \cup (\tilde{g_2}(F_2) - b(\{-1\} \times D^2))$ with the corners smoothed. Since $\tilde{g_i}$ are standard embeddings by our hypothesis,

$\tilde{g}_i(F_i)$ bound disjoint unions V_i $(i = 1, 2)$ of 3-dimensional handlebodies such that $V_1 \subset (\pi_1^2 \circ \pi_2^4)^{-1}((y, \infty))$ and $V_2 \subset (\pi_1^2 \circ \pi_2^4)^{-1}((-\infty, y))$.

Note that $b((-1,1) \times D^2)$ may intersect $V_1 \cup V_2$. However, we can isotope $\tilde{g}(F_1 \sharp F_2)$ so that $b((-1,1) \times D^2) \cap (V_1 \cup V_2) = \emptyset$ as follows. Set $\mathbf{R}^4_+ = (\pi_1^2 \circ \pi_2^4)^{-1}((y, \infty))$ and $\mathbf{R}^4_- = (\pi_1^2 \circ \pi_2^4)^{-1}((-\infty, y))$. Since $b(I \times D^2) \cap H = b(\{0\} \times D^2)$, there exists a 4-ball Δ_1 embedded in $\mathbf{R}^4_+$ containing V_1 in its interior such that $b(I \times D^2) \cap \partial\Delta_1 = b(\{\varepsilon\} \times D^2)$ for some $\varepsilon \in (0, 1)$. Then we can isotope $\Delta_1 \cup b(I \times D^2) \cup \tilde{g}_2(F_2)$ in $\mathbf{R}^4$ fixing $\tilde{g}_2(F_2)$ so that, after the isotopy, $b(I \times D^2) \cap V_2 = b(\{-1\} \times D^2)$ and that $\Delta_1 \cup b([0, 1] \times D^2)$ is in the original position. Then we take an embedded 4-ball Δ_2 in $\mathbf{R}^4_-$ containing V_2 in its interior such that $b(I \times D^2) \cap \partial\Delta_2 = b(\{\varepsilon'\} \times D^2)$ for some $\varepsilon' \in (-1, 0)$ and we isotope $\Delta_2 \cup b(I \times D^2) \cup \tilde{g}_1(F_1)$ in $\mathbf{R}^4$ fixing $\tilde{g}_1(F_1)$ so that, after the isotopy, $b(I \times D^2) \cap V_1 = b(\{1\} \times D^2)$ and that $\Delta_2 \cup b([-1, 0] \times D^2)$ is in the original position. Then, after all these isotopies, V_1 and V_2 are in their original positions, $b(I \times D^2)$ is isotoped, and $b((-1,1) \times D^2) \cap (V_1 \cup V_2) = \emptyset$.

Then $(V_1 \cup V_2) \cup b(I \times D^2)$ with the corners smoothed is a disjoint union of embedded 3-dimensional handlebodies bounded by $\tilde{g}(F_1 \sharp F_2)$ and hence $\tilde{g}$ is standard.

(2) Let $\tilde{g}^r : F_1 \sharp_r F_2 \to \mathbf{R}^4$ be an arbitrary embedding such that $\pi_2^4 \circ \tilde{g}^r = g^r$. We use the same argument and notation as in (1). Then we see that there exist embeddings $b_j : I \times D^2 \to \mathbf{R}^4$ $(j = 1, \dots, r)$ such that $b_j(I \times D^2) \cap b_k(I \times D^2) = \emptyset$ for $j \neq k$, $b_j(I \times D^2) \cap H = b_j(\{0\} \times D^2)$, $b_j(I \times D^2) \cap \tilde{g}_1(F_1) = b_j(\{1\} \times D^2), b_j(I \times D^2) \cap \tilde{g}_2(F_2) = b_j(\{-1\} \times D^2)$ and that $\tilde{g}^r(F_1 \sharp_r F_2)$ coincides with $(\tilde{g}_1(F_1) - (\cup_{j=1}^r b_j(\{1\} \times D^2))) \cup (\cup_{j=1}^r b_j(I \times \partial D^2)) \cup (\tilde{g}_2(F_2) - (\cup_{j=1}^r b_j(\{-1\} \times D^2)))$ with the corners smoothed.

By isotoping b_j slightly, we may assume that $b_j((-1, -1+\delta) \times D^2)$ and $b_j((1-\delta, 1) \times D^2)$ do not intersect with $V_1 \cup V_2$ for a sufficiently small $\delta > 0$ and that $b_j((-1,1) \times \{c\})$ intersect with $V_1 \cup V_2$ transversely at finitely many points, where V_i $(i = 1, 2)$ are disjoint unions of 3-dimensional handlebodies bounded by F_i and c denotes the center of D^2. We may further assume that $b_j((-1,1) \times D^2) \cap (V_1 \cup V_2) = b_j(\{s_1, \dots, s_m\} \times D^2)$ for some $s_1, \dots, s_m \in (-1, 1)$.

When F_i is orientable, V_i is necessarily orientable, and we fix an orientation for V_i. We also fix an orientation for $\mathbf{R}^4$. Then every intersection point of $b_j((-1,1) \times \{c\})$ and V_i has its own sign $(= \pm 1)$ determined by the orientations of $b_j((-1,1) \times \{c\})$, V_i and $\mathbf{R}^4$. By isotoping b_j on a neighborhood of $\{-1, 1\} \times D^2$, we may assume that the sum of the signs of all the intersection points is equal to zero for each j. When F_i is nonorientable, we may assume that the number of the intersection points is even for each j.

Since F_i is connected by our assumption, the complement of $\tilde{g}_i(F_i)$ in $\mathbf{R}^4$ has an abelian fundamental group, which is isomorphic to $\mathbf{Z}$ if F_i is orientable and to $\mathbf{Z}_2$ otherwise.

Suppose that F_1 is orientable. If $b_j((0,1)\times\{c\})$ intersects V_1, then there exists a pair of intersection points $b_j(s,c)$ and $b_j(s',c)$ $(0<s<s'<1)$ of $b_j((0,1)\times\{c\})$ and V_1 with opposite signs of intersection such that $b_j((s,s')\times\{c\})\cap V_1=\emptyset$. Since V_1 is connected, there exists an embedded arc α in V_1 with end points $b_j(s,c)$ and $b_j(s',c)$. We may assume that $\alpha\cap b_j(I\times D^2)$ is contained in $b_j(\{s,s'\}\times D^2)$ and that α does not intersect $b_k(I\times D^2)$ for all k different from j. Then, since the simple closed curve $\beta=\alpha\cup b_j([s,s']\times\{c\})$ has zero linking number with $\tilde{g}_1(F_1)$, it is null-homotopic in $\mathbf{R}^4_+-\tilde{g}_1(F_1)$. Hence there exists an embedded 2-disk D_β in $\mathbf{R}^4_+-\tilde{g}_1(F_1)$ with $\partial D_\beta=\beta$. We may assume that $D_\beta\cap b_j(I\times D^2)$ is contained in $b_j([s,s']\times D^2)$ and that D_β does not intersect $b_k(I\times D^2)$ for all k different from j. Then by using D_β, we can isotope b_j, fixing the complement of a neighborhood of $[s,s']\times D^2$, so that the number of intersection points of $b_j((-1,1)\times\{c\})$ and V_1 decreases by two. Repeating this procedure, we can eliminate all the intersections of $b_j((-1,1)\times\{c\})$ and V_1. Then by a further isotopy, we may assume that $b_j((-1,1)\times D^2)$ does not intersect V_1 $(j=1,2,\ldots,r)$.

When F_1 is nonorientable, we can do the same procedure, by ignoring the signs of the intersections.

Now we apply the same procedure as above to $b_j((-1,0)\times D^2)$, $\tilde{g}_2(F_2)$ and V_2. Then the resulting embeddings b_j satisfy $b_j((-1,1)\times D^2)\cap(V_1\cup V_2)=\emptyset$ $(j=1,2,\ldots,r)$. Then $(V_1\cup V_2)\cup(\cup_{j=1}^r b_j(I\times D^2))$ with the corners smoothed is an embedded 3-dimensional handlebody bounded by $\tilde{g}^r(F_1\sharp_r F_2)$ and hence $\tilde{g}^r$ is standard. This completes the proof. ||

DEFINITION 4.5. We say that an excellent map $f:F\to\mathbf{R}^2$ of a closed surface into the plane is *elementary* if it satisfies the condition of Theorem 1.2 or that of Proposition 3.5 (1) or (2).

The following is a direct consequence of Lemma 4.4, Theorem 1.2 and Proposition 3.5.

THEOREM 4.6. (1) *Let $f:F\to\mathrm{R}^2$ be an excellent map which is obtained by a finite iteration of mini-max connected sum operations from elementary excellent maps. Then f has the SLP.*

(2) *Let $f : F \to \mathrm{R}^2$ be an excellent map which is obtained by a finite iteration of generalized mini-max connected sum operations from elementary excellent maps with connected sources. Then f has the SLP.*

EXAMPLE 4.7. The excellent maps of surfaces into the plane such that the images of their singular sets are as shown in Figures 9 and 10 are examples of excellent maps which are obtained by (generalized) mini-max connected sum operations from elementary excellent maps.

Figure 9 here

Figure 10 here

REMARK 4.8. In the proof of Theorem 1.2, we could prove Case 4 alternatively, using Theorem 4.6 (2) and the Cases 1 and 2 of the proof of Theorem 1.2.

REMARK 4.9. Let $f : T^2 \to \mathbf{R}^2$ be a smooth excellent map without cusps such that $\pi_1^2 \circ f : T^2 \to \mathbf{R}$ is a Morse function with exactly six critical points. Then, in some cases, f decomposes as a (generalized) mini-max connected sum of elementary maps, and consequently it has the SLP. However, there are other cases where f does not so decomposes. In such cases, we do not know if f has the SLP or not.

REMARK 4.10. Using an idea similar to the definition of mini-max connected sum operations for excellent maps, we can also define mini-max connected sum operations for Morse functions. Then a result corresponding to Theorem 4.6 *does not hold* for such operations for Morse functions. The example constructed in Remark 3.3 gives a counter example. In general, the middle level circle, which corresponds to $\tilde{g}(F_1 \sharp F_2) \cap H \subset H$ in the proof of Lemma 4.4 (1), is not standard as is seen in that example, and the argument as in the proof of Lemma 4.4 cannot be applied.

5. Higher dimensional results

First we prove the following proposition, which might be folklore. However, we include a short proof here.

PROPOSITION 5.1. *Let $\tilde{f} : S^n \to \mathbf{R}^{n+2}$ $(n \geq 1)$ be a smooth embedding such that $\pi_1^{n+2} \circ \tilde{f} : S^n \to \mathbf{R}$ is a Morse function with exactly two critical points. Then the embedding f is standard. Consequently, every Morse function $S^n \to \mathbf{R}$ with exactly two critical points has the SLP.*

We note that the above proposition for $n = 1$ and $n \geq 4$ has been obtained by Ferus [**Fe**]. Here we give a simpler proof as follows.

Proof of Proposition 5.1. For $n = 1$ and 2, the assertion is proved in Remark 2.7 and Proposition 2.1 respectively. Thus we assume that $n \geq 3$. We use the same notations as in the proof of Proposition 2.1. By the same argument, we may assume that $\tilde{f}(\Delta_i) \subset (\pi_1^{n+2})^{-1}(a_i)$ and $\pi_1^4 \circ \tilde{f}|(S^n - (\Delta_1 \cup \Delta_2))$ has no critical points for some closed disk neighborhoods Δ_i of c_i in S^n $(i = 1, 2)$. Then it is easy to see that both $(\pi_1^4)^{-1}((-\infty, a_0]) - \tilde{f}(S^n)$ and $(\pi_1^4)^{-1}([a_0, \infty)) - \tilde{f}(S^n)$ are homotopy equivalent to S^1, where $a_0 = \frac{1}{2}(a_1 + a_2)$ (for example, see [**Ke**]). This implies that the complement of $\tilde{f}(S^n)$ in the one point compactification $\mathbf{R}^{n+2} \cup \{\infty\}$ of $\mathbf{R}^{n+2}$ is homotopy equivalent to S^1. Then by [**JL1**] for $n \geq 4$ and by [**Sn**] for $n = 3$ (see also [**Sa1**] for $n = 3$), the embedding $\tilde{f}$ is standard. ||

Note that a result similar to the above proposition holds also for embeddings of homotopy n-spheres, when $n \geq 7$.

When $\pi_1^{n+2} \circ \tilde{f}$ has four critical points, we have the following[1].

PROPOSITION 5.2. *Let $\tilde{f} : M \to \mathbf{R}^{n+2}$ be a smooth embedding of a closed n-dimensional manifold M $(n \geq 3)$ such that $\pi_1^{n+2} \circ \tilde{f} : S^n \to \mathbf{R}$ is a Morse function with exactly four critical points of indices $0, i, i+1$ and n $(0 \leq i \leq n-1)$, where M is diffeomorphic to the standard n-sphere for $n \leq 6$ and is a homotopy n-sphere for $n \geq 7$. If $i \leq (n-3)/2$ or $i \geq (n+1)/2$, then M is diffeomorphic to the standard n-sphere and the embedding $\tilde{f}$ is standard.*

Proof. By a result of Kearton [**Ke**] (see also [**KL**]), we see that the complement E of $\tilde{f}(M)$ in the one point compactification $S^{n+2} = \mathbf{R}^{n+2} \cup \{\infty\}$ of $\mathbf{R}^{n+2}$ has the same homotopy groups as S^1 up to the dimension i. Note that Kearton works

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[1] The authors are deeply indebted to Mituhiro Sekine for kindly bringing Kearton's paper [**Ke**] to their attention.

in the piecewise linear category with M piecewise linearly homeomorphic to S^n; however, his arguments work also in our setting, since our M is piecewise linearly homeomorphic to S^n. Thus, if $i \geq (n+1)/2$, then the complement E is homotopy equivalent to S^1, M is diffeomorphic to the standard n-sphere, and $\tilde{f}$ is standard by results of Levine (see [**JL1**] and [**JL2**, §23]). When $i \leq (n-3)/2$, consider $-\pi_1^4 \circ \tilde{f}$ instead of $\pi_1^4 \circ \tilde{f}$. This completes the proof. ||

Proof of Theorem 1.3. For $n = 2$, the result follows from [**Sl**] (or Theorem 1.2).

Suppose $n \geq 3$. By the characterization of n-dimensional closed manifolds which admit special generic maps into the plane ([**BdR**], [**Sa2**]), we see that M is diffeomorphic to the standard n-sphere for $n \leq 6$ and is a homotopy n-sphere for $n \geq 7$.

Using the same argument as in the proof of Theorem 1.2, we see that the number of critical points of the Morse function $\pi_1^2 \circ f$ is always even. When the Morse function has exactly two critical points, the result follows from Proposition 5.1. When the Morse function has exactly four critical points, by considering the Stein factorization of the special generic map $f : M \to \mathbf{R}^2$ (see [**Sa2**, §4]), we see that the image $f(M)$ coincides with the image of an immersion of the 2-disk into $\mathbf{R}^2$. Then, replacing π_1^2 by $-\pi_1^2$ if necessary, we see that the indices of the four critical points are equal to $0, n-1, n$ and n respectively. Then the result follows from Proposition 5.2, since $n - 1 \geq (n+2)/2$. This completes the proof. ||

REMARK 5.3. As in Remarks 3.2 and 3.3, Theorem 1.3 is sharp in the sense that there are many examples of nonstandard embeddings $\tilde{f} : S^n \to \mathbf{R}^{n+2}$ ($n \geq 2$) such that $\pi_2^{n+2} \circ \tilde{f}$ is a special generic map and that $\pi_1^2 \circ \pi_2^{n+2} \circ \tilde{f}$ is a Morse function with exactly six critical points. Such examples can be constructed by imitating the construction in [**Ki**, §3].

REMARK 5.4. Let us consider the following problem. Given a smooth excellent map (or a special generic map) $f : M \to \mathbf{R}^2$ of a closed n-dimensional manifold M into the plane ($n \geq 2$), does there always exist an embedding $\tilde{f} : M \to \mathbf{R}^{n+2}$ such that $\pi_2^{n+2} \circ \tilde{f} = f$? The answer to this question is negative in general. For example, it is known that every homotopy n-sphere Σ^n with $n \geq 5$ admits a special generic map $f : \Sigma^n \to \mathbf{R}^2$ into the plane (see [**Sa2**]). If Σ^n is an exotic n-sphere (i.e., if Σ^n is not diffeomorphic to the standard n-sphere S^n), then there does not exist an embedding $\tilde{f} : \Sigma^n \to \mathbf{R}^{n+2}$ such that $\pi_2^{n+2} \circ \tilde{f} = f$ by Theorem 1.3.

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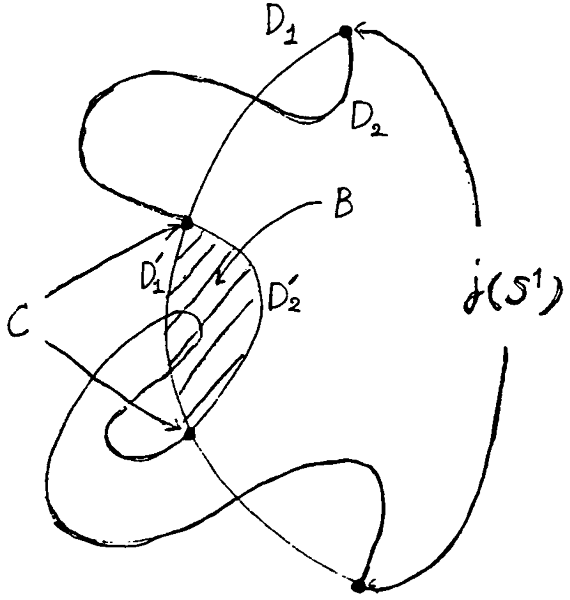

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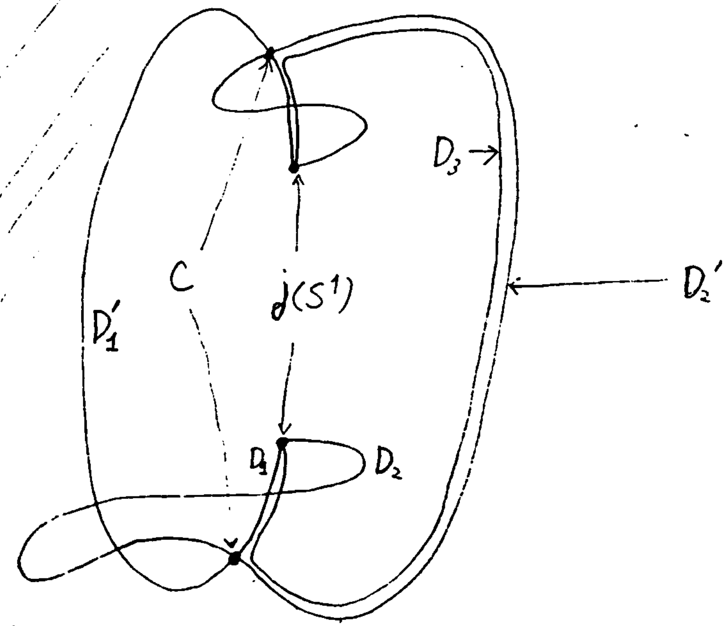

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Figure 1

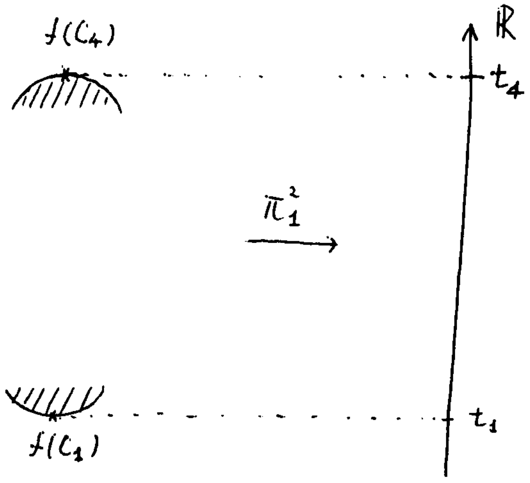


Figure 2

Figure 3

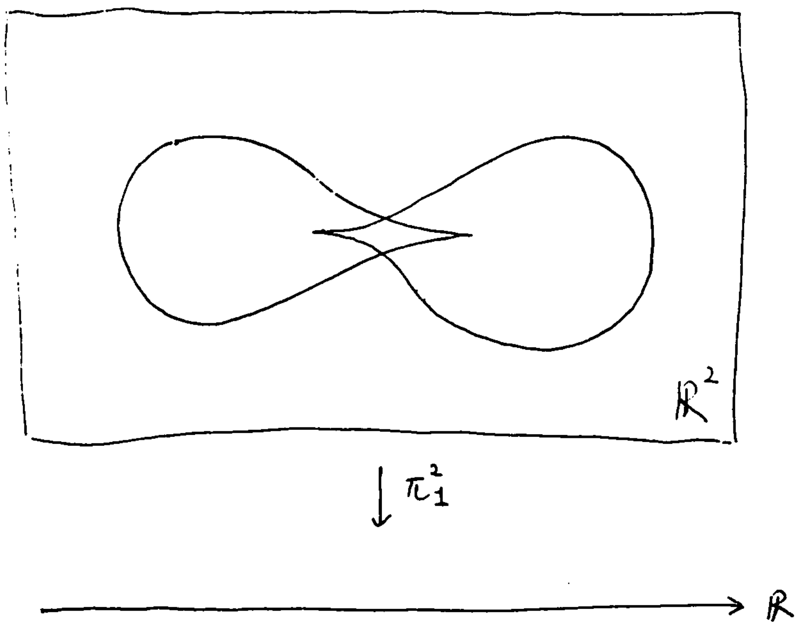


Figure 4

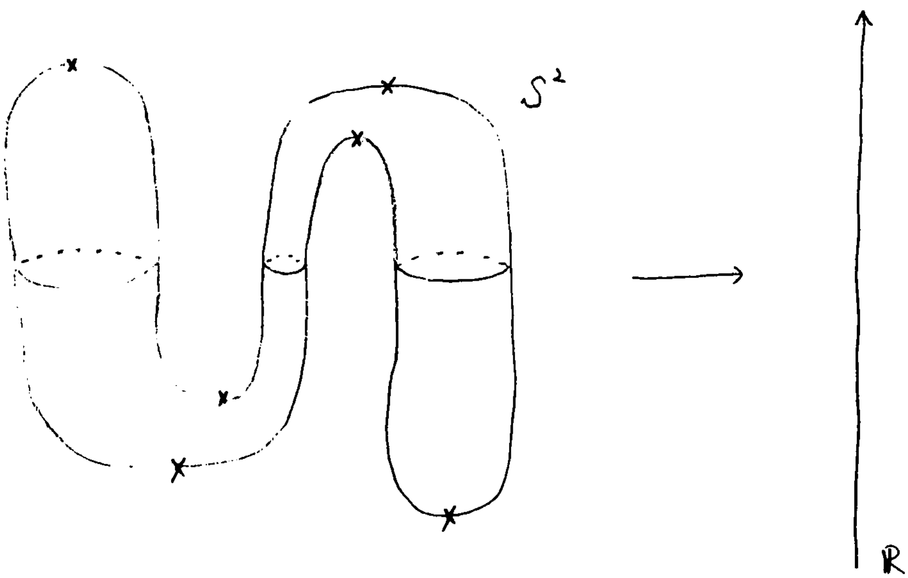
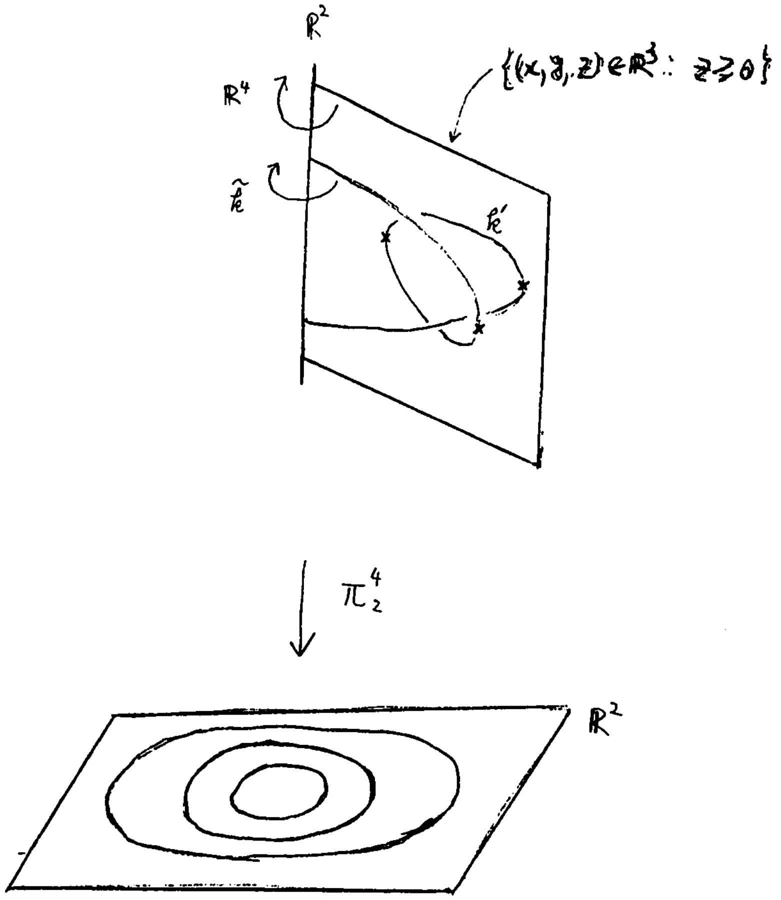


Figure 5



$\mathbb{R}^2$
$\{(x,y,z)\in\mathbb{R}^3 : z \geq 0\}$
$\mathbb{R}^4$
$\tilde{k}$
k'
π_2^4
$\mathbb{R}^2$

Figure 6

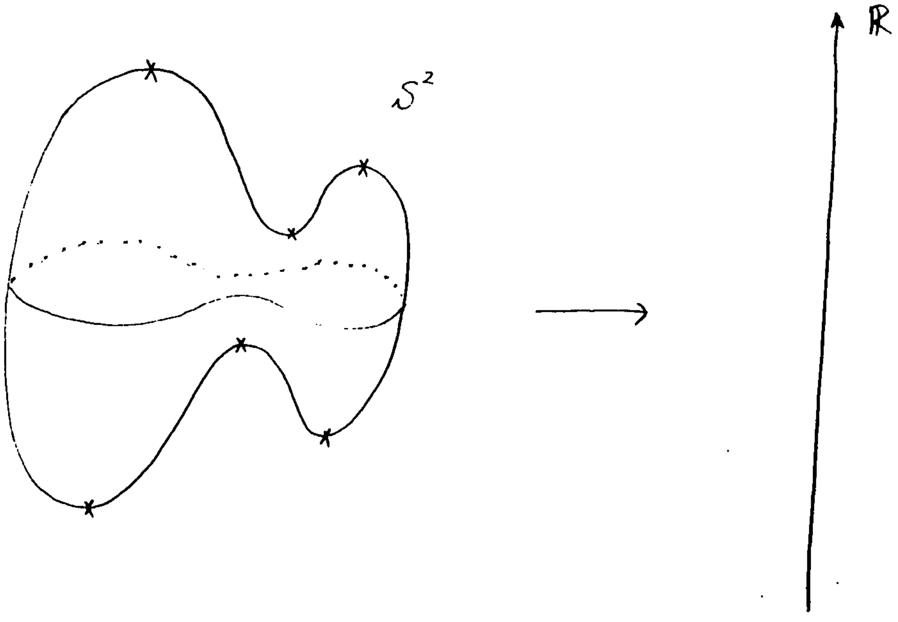


Figure 7

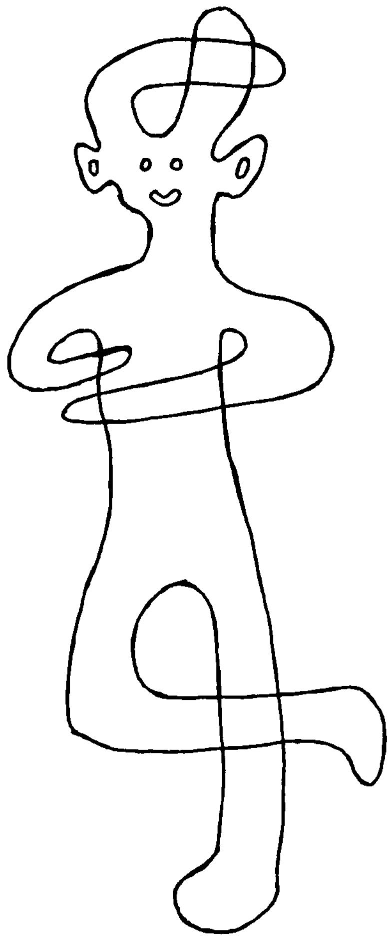


Figure 9

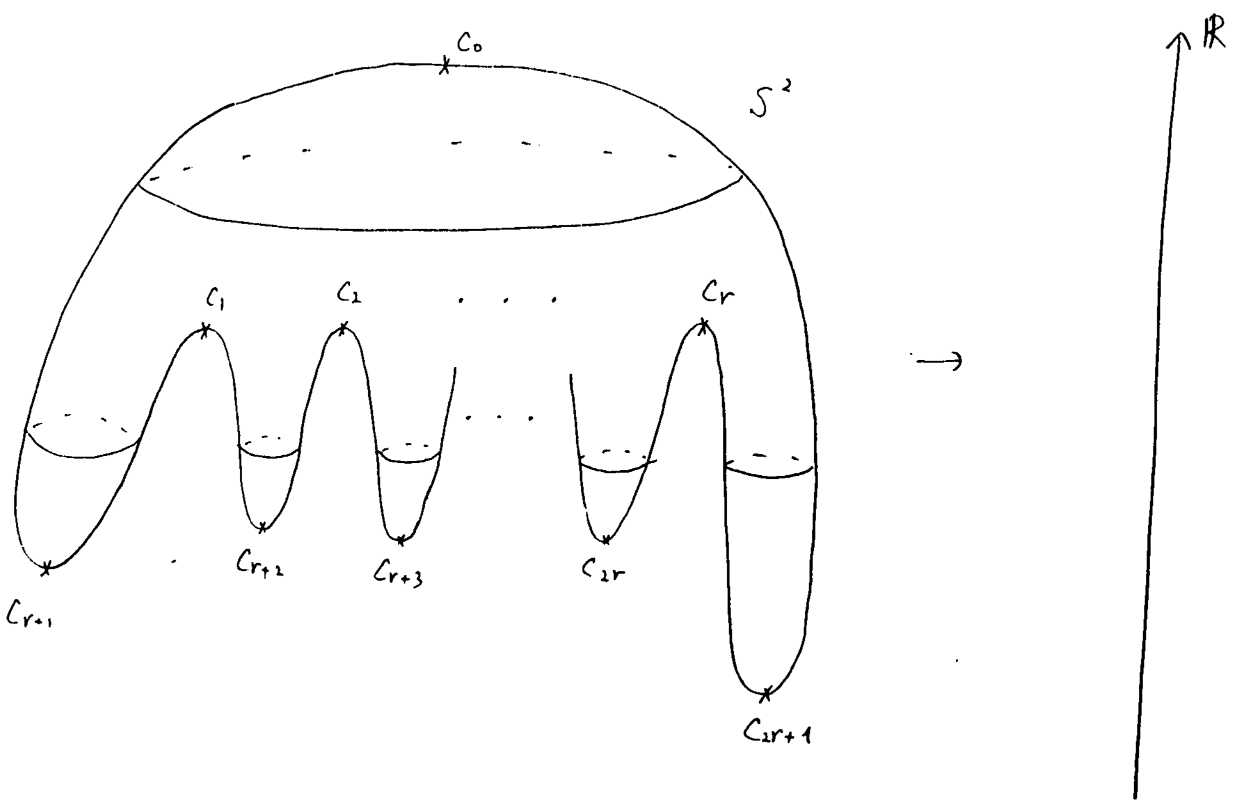


Figure 8

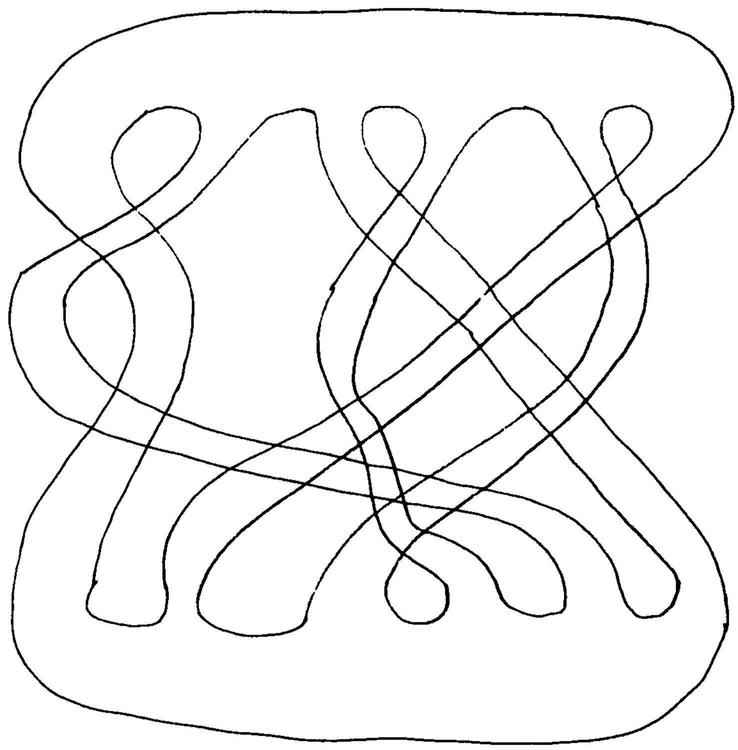


Figure 10

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