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METASTABLE IMMERSION WITH THE NORMAL BORDISM APPROACH

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ABSTRACT. Let $f : M \to N$ be a continuous map between two closed n-manifolds such that $f_* : H_*(M, \mathbb{Z}_2) \to H_*(N, \mathbb{Z}_2)$ is an isomorphism. Suppose that M immerses in $\mathbb{R}^{n+k}$ for $5 \leq n < 2k$. Then N also immerses in $\mathbb{R}^{n+k}$. We use techniques of normal bordism theory to prove this result.

1. INTRODUCTION

In this paper we are concerned with the following immersion problem:

Let M and N be closed smooth connected n-dimensional manifolds and let $f : M \to N$ be a continuous map. Suppose that M immerses in $\mathbb{R}^{n+k}$, for some k, with $5 \leq n < 2k$. Under which conditions on f does N immerse in $\mathbb{R}^{n+k}$?

For the special case where f is a homotopy equivalence between M and N, if M immerses in $\mathbb{R}^{n+k}$ for some $k \geq \left[\frac{n}{2}\right] + 2$, Glover and Mislin ([GM]) proved that N also immerses in $\mathbb{R}^{n+k}$. They also proved that in the case where M and N are connected, simple, orientable and closed smooth manifolds and the 2-localizations M_2 and N_2 are homotopy equivalent, if M immerses in $\mathbb{R}^{n+k}$ for some $k \geq \left[\frac{n}{2}\right] + 1$ then N immerses in $\mathbb{R}^{n+2[k/2]+1}$.

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Later, Glover and Homer ([GH1],[GH2]) proved the same result under weaker hypotheses on M and N, for example, nilpotent manifolds in ([GH1])(in fact they considered the more general problem where M immerses in another manifold.)

In our work we do not require that M and N are nilpotent manifolds and in ours results the dimension of the space where M and N are immersed is better than one in ([GM]). We use a normal bordism approach to investigate this problem. We prove the following main results:

Theorem A: *Let M and N be closed smooth connected n-manifolds and let $f : M \to N$ be a continuous map such that*

$$f_* : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$$

is an isomorphism for $i \geq 0$.

Then if M immerses in $\mathbb{R}^{n+k}$ for $5 \leq n < 2k$, so does N.

Theorem B: *Let $f : \widetilde{M} \to M$ be a finite regular covering of a closed connected smooth n-manifold M.*

Suppose $\pi = \dfrac{\pi_1(M)}{f_\# \pi_1(\widetilde{M})}$ is of odd order and operates trivially on $H_*(\widetilde{M}, \mathbb{Z}_2)$. Then if $\widetilde{M}$ immerses in $\mathbb{R}^{n+k}$ for $5 \leq n < 2k$, so does M.

Theorem C: *Let M and N be closed smooth connected n-manifolds, $f : M \to N$ a continuous map and $H = f_\# \pi_1(N)$. Suppose the inclusion $i : H \to \pi_1(N)$ induces isomorphisms in homology with Z_2 coefficient, the index $[\pi_1(N), H]$ is finite and odd,* *and $\pi_i(N,M)$ is odd torsion group for $1 < i \leq n$. Then if M immerses in $\mathbb{R}^{n+k}$ for $5 \leq n < 2k$, so does N.*

The work is divided into 4 sections. In section 2 we review The Normal Bordism Theory. In section 3 we proof Theorems A and B. In section four we show that for a family of spaces, larger than the family of Nilpotent spaces, we can replace the homology hypotheses given in Theorem A by the correspondent one in homotopy. This is proposition 4.1. Finally in section 5 we analyze algebraic conditons in terms of the fundamental group of the spaces, in order to have an isomorphism of homology with Z_2 coefficients. Then we proof Theorem C. Finally we give an example where we have an isomorphism in homotopy modulo finite odd groups but isomorphism of homology with Z_2 coefficients doesn't happen.

2. Normal Bordism

Let M be a closed smooth n-manifold and let X be a connected smooth $(n+k)$-manifold.

Let us consider $h : M \to X$ a map with $\dim \nu = k + \ell$, for ℓ large. Here ν denotes the stable normal bundle of h.

Then by Hirsch [H], h is homotopic to an immersion if and only if there is a monomorphism from $M \times \mathbb{R}^\ell$ to ν or equivalently if and only if geom $\dim(\nu) \leq k$.

In ([K2],[K3]), Koschorke defines an invariant $\omega_k(\nu) \in \Omega_{n-k-1}(M \times P^\infty, \phi)$ which is an obstruction to the existence of a monomorphism from $M \times \mathbb{R}^\ell$ into ν.

We recall that $\phi = \lambda \otimes (\nu - \varepsilon^{\ell}) - TM$ is a virtual vector bundle over $M \times P^{\infty}$, where λ denotes the canonical line bundle over the real projective space P^{∞} and $\Omega_{n-k-1}(M \times P^{\infty}, \phi)$ is a normal bordism group. For the definition and more details about normal bordism see ([K1] or [S].)

So geom $\dim(\nu) \leq k$ if and only if $\omega_k(\nu) = 0$, provided $n < 2k$. ([K2],[K3])

Salomonsen [S] defines a vector bundle $\pi_M : \widetilde{V}_k(\Psi) \to M$ such that the existence of a cross section $s : M \to \widetilde{V}_k(\Psi)$ implies that geom $\dim(\Psi) \leq k$. Here Ψ is a virtual bundle over M. In order to study whether cross sections exist we consider $(\pi_M)_* : \Omega_n(\widetilde{V}_k(\Psi), TM^0) \to \Omega_n(M, TM^0)$, where TM^0 is the virtual vector bundle $TM - \varepsilon^n$. We note that if geom $\dim(\Psi) \leq k$ then $(\pi_M)_*$ is onto. We also can consider the following exact sequence, for $5 \leq n < 2k$ ([S]).

$$\text{(I)} \qquad \cdots \to \Omega_n(\widetilde{V}_k(\Psi), TM^0) \xrightarrow{(\pi_M)_*} \Omega_n(M, TM^0) \xrightarrow{\gamma_M} \Omega_{n-k-1}(M \times P^{\infty}, \phi) \to \cdots$$

The mapping γ_M is defined by the construction of the sequence and $\phi = -(n-k-1)\lambda - \lambda \otimes \Psi + TM^o$.

Let $[M] = [M, 1_M, t_M] \in \Omega_n(M, TM^0)$ be the fundamental class of M where

$$t_M : TM \oplus \varepsilon^n \cong \varepsilon^n \oplus TM$$

is the isomorphism which interchange factors. If we consider $\Psi = h^* TX - \varepsilon^k \oplus TM$, we have that $\gamma_M[M] = \omega_k(\nu)$.

For the next lemma we recall that $F(q)$ is the monoid of degree 1 or -1 pointed maps of S^q and $F = \cup F(q)$. For a connected finite CW complex X, let $\alpha \in [X, BF]$ be a stable bundle over X and we define α_p as the composition of α and the canonical map $BF \longrightarrow (BF)_p$, where $(BF)_p$ is the p-localization of BF (p prime or 0).

T denotes the Thom complex and $\mathcal{C}$ is the class of all torsion groups where the torsion is odd.

Lemma 2.1: *Let $f : M \to N$ be a mapping between closed connected smooth n-manifolds such that*

$$f_* : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$$

is an isomorphism for $i \geq 0$.

Then we have:

1)$f_* : \Omega_n(M, f^*TN^0) \longrightarrow \Omega_n(N, TN^0)$ is a $\mathcal{C}$- isomorphism.

2)$f^*(\beta_2) = \alpha_2$, where $\alpha = \nu_M$ and $\beta = \nu_N$ are the stable normal bundle over M and N, respectively.

Proof :1) Let $\varphi = \varepsilon^n - TN$ be a virtual bundle over N. We observe that $\varphi \cong (\varepsilon^n - TN) \oplus (\nu_N^{k+\ell} - \nu_N^{k+\ell}) = (\varepsilon^n \oplus \nu_N^{k+\ell}) - (TN \oplus \nu_N^{k+\ell}) = (\varepsilon^n \oplus \nu_N^{k+\ell}) - \varepsilon^{n+k+\ell}$, for ℓ large and $\nu_N^{k+\ell}$ denotes the stable normal bundle of N. Let us denote $\varepsilon^n \oplus \nu_N^{k+\ell}$ by β^p, where $p = n + k + l$.

The Thom's construction gives the following isomorphisms:

$$\Omega_i(M, -f^*\beta^0) \cong \pi_{i+p}^S(T(f^*(\beta))) = \pi_{i+p+t}(T(\varepsilon^t \oplus f^*(\beta))$$

and

$$\Omega_i(\overline{N}, -\beta^0) \cong \pi_{i+n+t}(\overline{T}(\varepsilon^t \oplus \beta)).$$

Now the proof follows as in the proof of Lemma 2 ([S]). □

2) We prove this lemma using the same techniques of ([GM]) in the proposition 4.1. Let us define $\theta \in [M, BF]$ by $\theta_p = \alpha_p, \quad p \neq 2$ and $\theta_2 = f^*(\beta_2)$. We observe that for $p \neq 2, \quad T(\theta_p)$ is S-reducible, because $\alpha_p = (\nu_M)_p$.

Since f_* is an isomorphism for $i \geq 0$ we have that $(T(f^*\beta))_2$ and $(T(\beta))_2$ are homotopy equivalent. But $(T(f^*\beta))_2 = (T(f^*\beta_2))_2 = (T(\theta_2))_2 = (T(\theta))_2$ and since $T(\beta)$ is S- reducible, it follows that $T(\theta)$ is S-reducible at 2. Therefore $T(\theta)$ is S-reducible and by proposition 5.6([Sp]), we have that $\theta = \alpha$.

□

3. Proofs of Theorems A and B

Proof of Theorem A: Let us consider the following commutative diagram

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
\Omega_n(\widetilde{V}_k(\Psi_M), f^*TN^0) & \xrightarrow{G_*} & \Omega_n(\widetilde{V}_k(\Psi_N), TN^0) \\
\downarrow (\pi'_M)_* & \circlearrowleft \quad \circlearrowright & \downarrow (\pi_N)_* \\
\Omega_n(M, f^*TN^0) & \xrightarrow{f_*} & \Omega_n(N, TN^0) \\
\downarrow \gamma_M & \circlearrowleft \quad \circlearrowright & \downarrow \gamma_N \\
\Omega_{n-k-1}(M \times P^\infty, \phi_M) & \xrightarrow{F_*} & \Omega_n(N \times P^\infty, \phi_N) \\
\downarrow & & \downarrow
\end{array}$$

where the vertical exact sequences are obtained from (I) and G_* and F_* are given in [S], $\Psi_M = \varepsilon^{n+k} - TM \oplus \varepsilon^k$ and $\Psi_N = \varepsilon^{n+k} - TN \oplus \varepsilon^k$. We observe that $(\pi'_M)_*$ is the induced map of π_M in normal bordism groups with virtual bundle f^*TN^o.

Since f_* is a $\mathcal{C}$- isomorphism by the Lemma 2.1 , there exists an odd number ℓ_1 such that

$$\ell_1 \cdot [N] = f_*(x), \quad \text{for some} \quad x \in \Omega_n(M, f^*TN^0).$$

If we prove that $(\pi'_M)_*$ is a $\mathcal{C}$-epimorphism, then there exists an odd number ℓ_2 such that $\ell_2 \cdot x = (\pi'_M)_*(y)$ where $y \in \Omega_n(\widetilde{V}_k(\Psi_M), TM^0)$.

Using the commutative diagram we have that $(\pi_N)_* G_*(y) = f_*(\pi'_M)_*(y) = f_*(\ell_2 \cdot x) = \ell_2 \cdot \ell_1[N]$ and so $\ell_2 \cdot \ell_1 \cdot \gamma_N[N] = 0$.

Since the elements of the image of γ_N have order a power of 2 and $\ell_2 \cdot \ell_1$ is an odd number, $\gamma_N[N] = 0$ and N immerses in $\mathbb{R}^{n+k}$.

Then all we need to prove is that if M immerses in $\mathbb{R}^{n+k}$ then $(\pi'_M)_*$ is a $\mathcal{C}$-epimorphism and this is what we will do .

We recall that $\alpha = \nu_M$ and $\beta = \nu_N$ as in the Lemma 2.1.

We use the following commutative diagram equivalent to the top commutative diagram above.

$$\begin{array}{ccc}
\Pi^S_{n+p}(T(f^*\widehat{\beta})) & \xrightarrow{G_*} & \Pi^S_{n+p}(T(\widehat{\beta})) \\
\Big\downarrow (\pi'_M)_* & \circlearrowleft \quad \circlearrowright & \Big\downarrow (\pi_N)_* \\
\Pi^S_{n+p}(T(f^*\beta)) & \xrightarrow{f_*} & \Pi^S_{n+p}(T(\beta))
\end{array}$$

where $\widehat{\beta}$ denotes the pull back of β by π_N.

We observe that

$$(T(f^*\beta))_2 = (T(f^*\beta_2))_2 = (T(\alpha_2))_2 = (T(\alpha))_2$$

where the second equality is given by Lemma 2.1. We also have that $(T(f^*\widehat{\beta}))_2 = (T(\widehat{\alpha}))_2$, where $\widehat{\alpha}$ denotes the pullback of α by π_M.

Now, if M immerses in $\mathbb{R}^{n+k}$ then

$$(\pi_M)_* : \Omega_n(\widetilde{V}_k(\Psi_M), TM^0) \to \Omega_n(M, TM^0)$$

is an epimorphism and then

$$\Pi^S_{n+p}((T(\widehat{\alpha}))_2) \to \Pi^S_{n+p}((T(\alpha))_2)$$

is a $\mathcal{C}$-epimorphism or equivalently $\Pi^S_{n+p}((Tf^*\widehat{\beta})_2) \to \Pi^S_{n+p}((Tf^*\beta)_2)$ is a $\mathcal{C}$-epimorphism and the result follows. □

Proof of Theorem B: If π has odd order and since the coefficient group of the homology group is $\mathbb{Z}_2$, the divisibility condition of Theorem 10.8.8. ([HW]) is satisfied.

Therefore $H_*(\widetilde{M}, \mathbb{Z}_2) \approx H_*(M, \mathbb{Z}_2)$ and the result follows by Theorem A. □

4. Applications

Throughout this paragraph we will consider M ,N connected closed and smooth n-manifolds and $f : M \longrightarrow N$ a continuous map.

Let us consider the cases:

1. f is a homotopy equivalence.

2. M and N are nilpotent orientable and f is such that the induced $f_2 : M_2 \longrightarrow N_2$ from the 2-localizations M_2 of M into N_2 of N is a homotopy equivalence.

3. M and N are $\mathcal{C}$-nilpotent manifolds and f is such that

$$f_{\#1} : \pi_1(M) \to \pi_1(N) \text{ is a } \mathcal{C}\text{-epimorphism}$$

and

$$f_{\#i} : \pi_i(M) \to \pi_i(N) \text{ is a } \mathcal{C}\text{-isomorphism}$$

for $i < n$ and a $\mathcal{C}$-epimorphism for $i = n$. For details about nilpotent spaces see [HMR] and for $\mathcal{C}$-nilpotent spaces see [G2].

4. $f : M \to N$ is such that

$$f^1_{\#} : \pi_1(M) \to \pi_1(N) \text{ is an epimorphism}$$

and

$f^i_{\#} : \pi_i(M) \to \pi_i(N)$ is a $\mathcal{C}$-isomorphism for $i < n$ and a $\mathcal{C}$-epimorphism for $i = n$.

5. f is an orientable map such that

$$f_* : H_i(M, \tilde{\mathbf{Z}}) \to H_i(N, \tilde{\mathbf{Z}})$$

is an isomorphism for $i \geq 0$, where $\tilde{\mathbf{Z}}$ denotes the twisted integer coefficients over M, respectively N, associated to $w_1(M)$, respectively $w_1(N)$.

In all the cases above we have that if M immerses in $\mathbb{R}^{n+k}$ for $5 \leq n < 2k$, so does N. Under the hypotheses of cases 1 or 2, the result follows immediately from Theorem A.

For the third and the fourth cases, we use the Proposition 3.5 ([G2]) and Theorem 1.1 ([BG]), respectively in order to show that $f_* : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ is an isomorphism for $i \geq 0$. So by Theorem A the result follows.

The last case follows by Theorem 3.5 ([A]).

The Theorem B has an interesting consequence:

Corollary 4.1: *If $H_i(M, \mathbb{Z}_2) = 0$ or $\mathbb{Z}_2$, for all i, and if G is an odd order group which acts freely on M then if M immerses in $\mathbb{R}^{n+k}$ for $5 \leq n < 2k$, so does $\frac{M}{G}$.* □

5. The mod 2 Homology Isomorphism Condition

In this section we will assume that $f_{i\#} \cdot \pi_i(M) \to \pi_i(N)$ satisfies the hypotheses: $f_{i\#}$ is a $\mathcal{C}$-isomorphism for $1 < i < n$, $\mathcal{C}$-epimorphism for $i = n$, $\mathcal{C}$-injective for $i = 1$ and the index $[\pi_1(N), f_{1\#}(\pi_1(M))]$ is finite and odd. We will study the question of to decide when the map $f : M \to N$ among two closed n-dimensional manifolds induces $f_{i*} : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ an isomorphism for $0 \leq i \leq n$. We will also provide an example where this does not happen. As before, $\mathcal{C}$ stands for the class of all torsion groups where the torsion is odd.

Proposition 5.1. *Given $f : M \to N$ as above, there is a finite cover $p : \overline{N} \to N$ and a lifting*

$$\begin{array}{ccc} & \overset{\bar f}{\nearrow} & \overline{N} \\ & & \downarrow \\ M & \xrightarrow[f]{} & N \end{array}$$

such that $\bar f_{*i} : H_i(M, \mathbb{Z}_2) \to H_i(\overline{N}, \mathbb{Z}_2)$ is an isomorphism for $i \geq 0$.

Proof: Let $\overline{N}$ be the cover which corresponding to the subgroup $f_{1\#}(\pi_1(M))$ and $\bar f$ a lift of f. By Theorem 1.1 of [BG] the result follows. □

Corollary 5.2. If $H_i(M, \mathbb{Z}_2) = 0$ or $\mathbb{Z}_2$, for all i, then if M immerses in $\mathbb{R}^{n+k}$ for $5 \leq n < 2k$, so does N.

Proof: This follows from Proposition 5.1 and Corollary 4.1 $\square$

For the next proposition let us consider the commutative diagram

$$\begin{array}{ccc}
\widetilde{\overline{N}} & \xrightarrow{\widetilde{p}} & \widetilde{N} \\
\downarrow & & \downarrow \\
\overline{N} & \xrightarrow{p} & N \\
\downarrow & & \downarrow \\
K(\pi_1(\overline{N}), 1) & \xrightarrow{p'} & K(\pi_1(N), 1)
\end{array}$$

where $\widetilde{\overline{N}}, \widetilde{N}$ are the universal covers of $\overline{N}, N$ respectively and $K(\pi, 1)$ are Eilenberg-MacLane spaces.

Proposition 5.3. *The induced homomorphisms $p_{i*} : H_i(\overline{N}, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ are isomorphisms for all $i \geq 0$ if $p'_{i*} : H_i(\pi_1(\overline{N}), \mathbb{Z}_2) \to H_i(\pi_1(N), \mathbb{Z}_2)$ is an isomorphism for all $i \geq 0$, where H_* means group homology.*

Proof: Since $\widetilde{\overline{N}}$ and $\widetilde{N}$ are simply connected and $\widetilde{p}_\# : \pi_i(\widetilde{\overline{N}}) \to \pi_i(\widetilde{N})$ is $\mathcal{C}$-isomorphism for $i \geq 0$, we have that $\widetilde{p}_{i*} : H_i(\widetilde{\overline{N}}, \mathbb{Z}_2) \to H_i(\widetilde{N}, \mathbb{Z}_2)$ is an isomorphism. Now we consider the Serre spectral sequence of the fibrations and the induced map. By the spectral mapping theorem, the result follows. $\square$

Remarks:

- 1 The converse of proposition 5.2 is also true, but will not be used here.

• 2 This Proposition generalizies the Proposition 10.8.8 of [HW].

From now on let us denote $\overline{H} = \pi_1(\overline{\overline{N}})$, $\overline{G} = \pi_1(\overline{N})$ and $j : \overline{H} \mapsto \overline{G}$ the inclusion. We have that the index [G,H] is odd.

Proposition 5.4. *The homomorphism* $j_{i*} : H_i(H, \mathbb{Z}_2) \to H_i(G, \mathbb{Z}_2)$ *is epimorphism for* $i \geq 0$.

Proof: It is equivalent to show that the induced map in cohomology is injective. Then we consider the composite of the corestriction with the restriction (See [W]) $\tau : H^i(G, \mathbb{Z}_2) \to H^i(H, \mathbb{Z}_2) \to H^i(G, \mathbb{Z}_2)$, which is multiplication by $\ell = [G, H]$. Since ℓ is odd this is an isomorphism. So follows that the first map is injective and hence the result. □

Corollary 5.5. *If G is finite of order odd, then* $H_i(G, \mathbb{Z}_2) = 0$, $i > 0$.

Proof: We take H the trivial group. By Proposition 5.3 follows $H_i(G, \mathbb{Z}_2) = 0$, $i > 0$. □

Proof of Theorem C: The result follows directly from Propositions 5.1, 5.2 and Theorem A. □

In general one can not expect to have $j_{i*} : H_i(H, \mathbb{Z}_2) \to H_i(G, \mathbb{Z}_2)$ isomorphism for all $i > 0$.

Example 5.6 (Algebraic). Let us consider the following extension $0 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to G \to \mathbb{Z}_3 \to 0$ where the action $\omega : \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})$ is given by

$\omega(1) = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and the extension corresponds to the nontrivial element of $H^2(\mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \approx \mathbb{Z}_3$. That $H^2(\mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}) \simeq Z_3$, it is a standart calculation using [M] chapter IV Section 7. So let $H = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

We will use the Lyndon-Hochschild-Serre spectral sequence, in order to compute $H_*(G, \mathbb{Z}_2)$. The E^2 term is $E^2_{p,q} = H_p(\mathbb{Z}_3, H_q(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}; \mathbb{Z}_2))$.

So

$$E^2_{p,q} = \begin{cases} 0 & q \geq 4 \\ H_p(\mathbb{Z}_3, \mathbb{Z}_2) & q = 0, 3 \\ H_p(\mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) & q = 1 \\ H_p(\mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) & q = 2 \end{cases}$$

where the local coefficient is given by $\omega_1(1) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\omega_2(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, for the cases q=1, q=2 respectively.

Again, using [M] chapter IV section 7, we get

$$H_p(\mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & p = 0 \\ 0 & p \neq 0 \end{cases}$$

where the action ω is either ω_1 or ω_2 . So we get

$$H_*(G,\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & * = 0,3 \\ H_1(H\,,\,\mathbb{Z}_2,\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2) \approx \mathbb{Z}_2 \\ H_2(H\,,\,\mathbb{Z}_2,\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2) \approx \mathbb{Z}_2\,. \end{cases}$$

Remarks:

- 1 It is not difficult to see that the group G is not nilpotent but it is certainly infra-abelian.
- 2 If $\omega : Q \to SL(n,\mathbb{Z})$ has the property that the reduced action $\omega_2 : Q \to SL(n,\mathbb{Z}_2)$ is nilpotent, then we have that $H_i(\underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{n},\,\mathbb{Z}_2) \to H_i(G,\mathbb{Z}_2)$ is an isomorphism for all i , where G is any extension of $\underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{n}$ by $Q \in \mathcal{C}$ with action ω. By [G1] Proposition 5, we have that ω_2 nilpotent implies necessarily that ω_2 it is trivial, since every element of Q is torsion of odd order. Now by Theorem IX.7 of [N], we can conclude that ω_2 implies ω also trivial. This means that the hypothesis ω_2 be nilpotent is not stronger than ω is trivial.

Example 5.7 (Geometric). Finally we will show how to realize geometrically the Example 5.5. For let us consider any action $\omega : Q \to Aut(\underbrace{\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}}_{n})$. Consider the action of Q on $\mathbb{R}^n$, $Q\times\mathbb{R}^n \to \mathbb{R}^n$ given by the linear transformation which have matrices $\omega(Q)$. This action certainly induces an action on the Torus $T^n = \underbrace{S^1\times\cdots\times S^1}_{n}$, which we also denote by ω. Now let M be a manifold where Q acts freely. So there is a free action $Q\times T^n\times M \to T^n\times M$ given by $(q,x,y)\to(q\cdot x, q\cdot y)$. Therefore, we have the map $T^n\times M \to \frac{T^n\times M}{Q}$, where $\frac{T^n\times M}{Q}$ is the orbit space, which is a finite regular cover of a compact manifold.

So we get a short exact sequence

$$1 \to \pi_1(T^n \times M) \to \pi_1\left(\frac{T^n \times M}{Q}\right) \to Q \to 1$$

which is

$$1 \to \mathbb{Z}^n \oplus \pi_1(M) \to G \to Q \to 1$$

where $G = \pi_1\left(\frac{T^n \times M}{Q}\right)$. Now we apply the above procedure for the Example 5.5. For let ω be the action given by the upper left corner 2x2 submatriz of the 3x3 matriz given in 5.5. Let M be the circle S^1 where Z_3 acts freely by rotating of 120 degrees. So we get a finite cover $T^3 \to N$ where the three manifold N has homology given by $H_i(N, \mathbb{Z}_2) \approx \mathbb{Z}_2$ for $3 \geq i \geq 0$. Certainly $H_i(T^3, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ is not an isomorphism for all i and the result follows. Observe that $H^i(N, \mathbb{Z}_2) \approx \mathbb{Z}_2$ for $3 \geq i \geq 0$ which is the same as the $H^*(RP^3, \mathbb{Z}_2)$, where RP^3 is the 3-projective space. It is not hard to show that the cohomology ring structures of these two spaces are different.

References

[A] M.F.Atiyah. Thom Complexes.*Proc. London Math.* (3) 11 (1961) 291-310.

[BG] C. Biasi and D.L. Gonçalves. Equivalent conditions for n-equivalences modulo a class $\mathcal{C}$ of groups. *Bol. Soc. Brasil. Mat.*, 27 (1996) 187-198.

[G1] D.L. Gonçalves. Nilpotent actions, Preprint series Department of Mathematics- IME - University of S. Paulo - Brasil (1980).

[G2] ————————. Generalized classes of groups, $\mathcal{C}$-nilpotent spaces and "The Hurewicz Theorem". *Math. Scand.*, 53 (1983), 39-61.

[GH1] H. Glover and W. Homer. Immersing manifolds and 2-equivalence. *Lectures Notes in Math.* 657 (1978) 194-197. Springer Verlag.

[GH2] ——————. Metastable immersion, span and the two-type of a manifold. *Canadian Math. Bull,* 29 (1986) 20-24.

[GM] H. GLOVER AND G. MISLIN. Immersion in the matastable range and 2-localization. *Proceedinds of the Am. Math. Soc.* 43 (1974) 443-448.

[H] M. HIRSCH. Immersions of manifolds. *Trans. AMS.* 93 (1959) 242-276.

[HMR] P.J. HILTON; G. MISLIN AND J. ROITBERG. *Localization of nilpotent groups and spaces.* North-Holland Math-Holland Publ. Co. Amsterdam, Oxford; American Elsevier Publ. Co., I nc. New York, (1975).

[HW] P.J. HILTON AND S. WYLIE. *Homology Theory An Introdution to Algebraic Topology.* Cambridge University Press (1960).

[K1] U. KOSCHORKE. Vector Fields and Other Vector Bundle Morphism - A Singularity Approach. *Lecture Notes in Mathematics* 847 (1981) - Springer Verlag.

[K2] ——————. The Singularity Method and Immersions of m-manifolds into Manifolds of dimensions $2m-2$, $2m-3$ and $2m-4$. *Lecture Notes in Mathematics* 1350 (1988) - Springer Verlag.

[K3] ——————. *Nonstable and stable monomorphisms of Vector Bundles.* Topology and its Applications. 75 (1997) 261-286.

[M] S. MACLANE. *Homology.* Grundlehren Math. Wiss. 114, Springer Verlag, Berlin-Heidelberg-New York (1963).

[N] M. NEWMAN. *Integral Matrices.* Academic Press, New York and London (1972).

[S] H.A. SALOMONSEN. Bordism and Geometric Dimension. *Math. Scand.* 32 (1973) 87-111.

[Sp] M.SPIVAC. Spaces Satisfying Poincaré Duality. *Topology.* 6 (1967) 77-101.

[W] E. WEISS. *Cohomology of Groups.* Cambridge University Press (1969).

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