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PARABOLIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS AND CRITICAL NONLINEARITIES

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Resumo

Provamos a existência, unicidade e regularidade de soluções para equações de calor semilineares com condição de fronteira não linear. Estudamos estes problemas com dados iniciais em $L^q(\Omega)$, $W^{1q}(\Omega)$, $1<q<\infty$ ou medidas e com não linearidades de crescimento crítico.

Parabolic Problems with Nonlinear Boundary Conditions and Critical Nonlinearities

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Abstract

We prove existence, uniqueness and regularity of solutions for heat equations with nonlinear boundary conditions. We study these problems with initial data in $L^q(\Omega)$, $W^{1,q}(\Omega)$, $1 < q < \infty$ or measures and with critically growing nonlinearities.

1 Introduction

This paper is devoted to the study of existence, uniqueness and regularity of solutions of parabolic problems with nonlinear boundary conditions and critical nonlinearities. Let $\Omega \subset I\!R^N$ be a bounded smooth domain, $N \geq 1$ and consider the problem

$$\begin{cases} u_t = \Delta u + f(u) \text{ in } \Omega \\ \dfrac{\partial u}{\partial n} = g(u) \text{ in } \Gamma := \partial\Omega \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $f, g : I\!R \to I\!R$ are locally Lipschitz functions.

We study (1.1) with initial data in $L^q(\Omega)$, $W^{1,q}(\Omega)$, $1 < q \leq N$ or even in spaces of measures. In these spaces some growth conditions on f and g must be imposed and we are mostly interested on obtaining results for the fastest possible growth, called critical growth. In particular, for initial data in $L^q(\Omega)$, we will show that solution exist for $f(u)$ growing like the power $|u|^{\rho_1}$, for $\rho_1 \leq \frac{N+2q}{N}$ and $g(u)$ growing like the power $|u|^{\rho_1}$, for $\rho_1 \leq \frac{N+q}{N}$.

For the case of homogeneous linear boundary conditions and critically growing nonlinearity f, this problem has been considered in the L^q-settting in, [18, 19, 8, 5], in the $W^{1,q}$–settting in [5] and in the space of measures in [9].

For the case of nonlinear boundary condition of the type $\partial u/\partial n = g(u)$ there are results in the $W^{1,q}$– setting, for $q > N$ (see [3]). These results are obtained with no restrictions in the growth of

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f and g thanks to the fact that $W^{1,q}(\Omega) \subset C(\overline{\Omega})$. The H^1-setting with subcritical nonlinearities is treated in [10]. There are also results when the nonlinearity g is monotone in [7, 11]. For other related results, in $I\!R^N$, see also [6, 13] and references therein.

Our results are for f and g critically growing and with no monotonicity assumptions whatsoever. In particular we recover all the critical exponents on f appearing in the literature (like the ones from [9] for measures) allowing simultaneously a critically growing nonlinearity at the boundary.

To initiate the discussion and in order to introduce the main ideas of the paper, we consider the case $f \equiv 0$ and look for solutions of (1.1) with initial data $u_0 \in L^2(\Omega)$.

We first introduce some terminology. It is well known that the Laplace operator with homogenenous Neumann boundary conditions can be regarded as an unbounded operator, $A = \Delta$, in $E^0 = L^2(\Omega)$ with domain $H^2_{\mathcal{N}}(\Omega) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega\}$ and that, since this operator turns out to be sectorial, associated to it there is an scale of Banach spaces (the fractional power spaces) E^α, [12]. It is well known that

$$\left.\begin{array}{l} E^\alpha \hookrightarrow H^{2\alpha}(\Omega), \\ E^{-\alpha} = (E^\alpha)' \end{array}\right\} \alpha \geq 0 \tag{1.2}$$

and $E^{\frac{1}{2}} = H^1(\Omega)$, $E^{-\frac{1}{2}} = (H^1(\Omega))'$, $E^0 = L^2(\Omega)$ and $E^1 = H^2_{\mathcal{N}}(\Omega)$. Notice that the operator A can also be regarded as an unbounded operator $A : E^{\alpha+1} \subset E^\alpha \to E^\alpha$, $\alpha \in I\!R$, (this operator is usually called the realization of A in E^α). If we denote by $X^\alpha = E^{\alpha-1}$ then we have $A : X^1 \subset X^0 \to X^0$ and in this setting, we have that for $u \in X^1 = L^2(\Omega)$, Au is given by the linear form

$$< Au, \phi >= \int_\Omega u \Delta \phi$$

for every $\phi \in H^2_{\mathcal{N}}(\Omega)$. Moreover, A is sectorial and hence (1.1) can be rewritten as

$$\left\{\begin{array}{l} \dot{u} = Au + g_\Gamma(u), \\ u(0) = u_0 \in X^1 \equiv L^2(\Omega) \end{array}\right. \tag{1.3}$$

which means that the solution verifies

$$\frac{d}{dt}\int_\Omega u\phi - \int_\Omega u\Delta\phi = \int_\Gamma g(u)\phi$$

for every $\phi \in H^2_{\mathcal{N}}(\Omega)$.

Note that the map g_Γ is not well defined in $X^1 = L^2(\Omega)$. Actually, it is not well defined in any $X^{1+\alpha}$, $\alpha \leq 1/4$, since none of this spaces have a well defined trace operator. Therefore the standard theory (see [12, 3, 10]) does not apply. In this case, it is suitable to apply the theory developed in [5].

To be more precise, we recall some definitions and results from [5]. Consider the abstract problem,

$$\left\{\begin{array}{l} \dot{x} = Ax + h(x), \ t > 0 \\ x(0) = x_0 \in X^1 \end{array}\right. \tag{1.4}$$

where $A : X^1 \subset X^0 \to X^0$ is a sectorial operator. Denote by X^α the fractional power spaces associated to A. Define,

Definition 1.1 *We say that* $x : [0,\tau] \to X^1$ *is an* $\epsilon-$*regular solution to (1.4) if* $x \in C([0,\tau], X^1) \cap C((0,\tau], X^{1+\epsilon})$*, and* $x(t)$ *satisfies*

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}h(x(s))ds. \tag{1.5}$$

Definition 1.2 *For* $\epsilon \geq 0$*, we will say that a map* h*, is an* $\epsilon-$*regular map relatively to the pair* (X^1, X^0)*, if there exist* $\rho > 1$*,* $\gamma(\epsilon)$ *with* $\rho\epsilon \leq \gamma(\epsilon) < 1$ *and a constant* c*, such that* $h : X^{1+\epsilon} \to X^{\gamma(\epsilon)}$ *and*

$$\|h(x) - h(y)\|_{X^{\gamma(\epsilon)}} \leq c\|x - y\|_{X^{1+\epsilon}}(\|x\|^{\rho-1}_{X^{1+\epsilon}} + \|y\|^{\rho-1}_{X^{1+\epsilon}} + 1) \qquad \forall x, y \in X^{1+\epsilon} \tag{1.6}$$

The main result from [5] basically says the following,

Theorem. *If* h *is an* ϵ*-regular map for some* $\epsilon > 0$*, then (1.4) has a unique* ϵ*-regular solution.*

Note that this result does not require h to be well defined in the space were the initial data is taken, i.e. X^1. Also, this results handles some cases in which $h : X^1 \to X^0$, a situation that can not be handled by the standard semilinear technique for parabolic equations, [12, 3, 4, 10].

With this in mind, for (1.3), we will prove that if g is a locally Lipschitz function verifying

$$|g(u) - g(v)| \leq c|u - v|(|u|^{\rho-1} + |v|^{\rho-1} + 1), \tag{1.7}$$

and $\rho \leq \frac{N+2}{N}$, $N \geq 2$, then g_Γ is an ϵ*-regular map relatively to the pair* (X^1, X^0), for $\frac{1}{4} < \epsilon < \frac{3N}{4(N+2)}$ and $\gamma(\epsilon) = \rho\epsilon$. From the above theorem, we have the existence and uniqueness of ϵ-regular solutions of (1.3). Moreover these solutions will be shown to be classical solutions.

For problem (1.1), with f not necessarily zero, the abstract formulation becomes

$$\left\{ \begin{array}{l} \dot{u} = Au + f_\Omega(u) + g_\Gamma(u), \\ u(0) = u_0 \in X^1 = L^2(\Omega); \end{array} \right. \tag{1.8}$$

that, is

$$\frac{d}{dt}\int_\Omega u\phi - \int_\Omega u\Delta\phi = \int_\Omega f(u)\phi + \int_\Gamma g(u)\phi$$

for every $\phi \in H^2_N(\Omega)$.

Therefore, it is clear that we need to treat problems of type (1.4) with h being a sum of nonlinearities with different $\epsilon-$regularity properties. This is presented in Section 2, where an extension of the results from [5] is presented to include this case.

Moreover, we would like to treat these problems not only in the Hilbert setting, but include also the $L^q(\Omega)$ and $W^{1,q}(\Omega)$ setting, $1 < q < \infty$. For these cases, and as it is explained in Section 2.1 (see also Remark 1 of [5]), it is more appropriate to work with scales of Banach spaces coming from interpolation instead of fractional power spaces.

Among all results proved in this paper we would like to emphasize the following,

Theorem 1.1 *For* $1 < q < \infty$ *and* $N \geq 2$ *(respectively,* $N = 1$*) and* f*,* g *satisfying (1.7) with exponents* ρ_1 *and* ρ_2 *respectively, such that*

$$\rho_1 \leq \rho_f = 1 + \frac{2q}{N}, \quad \textit{and} \quad \rho_2 \leq \rho_g = 1 + \frac{q}{N}, \ (\textit{respectively}, \ \rho_2 < 1 + q),$$

problem (1.1) with $u_0 \in L^q(\Omega)$ *has a unique* ϵ*-regular solution, for some* $\epsilon > 0$*. Furthermore, this solution is a classical solution.*

This paper is organized as follows. In Section 2 we introduce some background results on scales of Banach spaces and we extend the abstract results in [5] to include a sum of nonlinearities with different ϵ−regular properties. In Section 3 we consider the problem (1.1) in $W^{1,q}(\Omega)$, $L^q(\Omega)$ and in spaces of measures.

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2 Abstract Results

The results for parabolic equations with nonlinear boundary conditions presented in the applications at the end of this paper follow from general abstract results for semilinear parabolic equations which allow critically growing nonlinearities. This approach requires the use of scales of Banach spaces, that are briefly described below.

2.1 Scales of Banach Spaces

In what follows we give a description of how to construct an Scale of Banach Spaces associated to a sectorial operator $-A : D(A) \subset E_0 \to E_0$. For that we need the concept of interpolation which we briefly describe in the very restricted context on what it will be used, in this paper. For a more complete treatment of interpolation we refer to [16].

Let $\mathcal{B}$ be the set of all Banach spaces and let E, F be in $\mathcal{B}$, we denote by $\mathcal{L}(E, F)$ the set of bounded linear operators from E into F. A pair (E_1, E_0) of elements of $\mathcal{B}$ is called an *interpolation couple* if $E_1 \overset{d}{\hookrightarrow} E_0$ (continuous and dense embedding). Denote by $\mathcal{C}$ the set of all interpolation couples.

An *interpolation functor* of exponent θ, $(\cdot,\cdot)_\theta$, $0 < \theta < 1$, is a map from $\mathcal{C}$ into $\mathcal{B}$ such that given (E_1, E_0) and (F_1, F_0) interpolation couples we have that $E_\theta := (E_1, E_0)_\theta$ and $F_\theta := (F_1, F_0)_\theta$ are such that $E_1 \overset{d}{\hookrightarrow} E_\theta \overset{d}{\hookrightarrow} E_0$, $F_1 \overset{d}{\hookrightarrow} F_\theta \overset{d}{\hookrightarrow} F_0$, $A \in \mathcal{L}(E_\theta, F_\theta)$ whenever $A \in \mathcal{L}(E_0, F_0) \cap \mathcal{L}(E_1, F_1)$ and

$$\|A\|_{\mathcal{L}(E,F)} \le \|A\|^{1-\theta}_{\mathcal{L}(E_0,F_0)} \|A\|^{\theta}_{\mathcal{L}(E_1,F_1)}.$$

There are many examples of interpolation functors of exponent θ for $0 < \theta < 1$ (see, for example, [4, 16, 14]). The interpolation functors that we will be concerned with are the real, the complex and the continuous interpolation functors (see [4]). We observe that what we call an interpolation functor is called an exact admissible interpolation functor in [4].

We now construct the scale of spaces that will be used in the abstract results that follow.

Let $-A : D(A) \subset E_0 \to E_0$ be a sectorial operator and assume that $\mathrm{Re}(\sigma(A)) < 0$ where $\sigma(A)$ is the spectrum of A. Then A generates an analytic semigroup $\{e^{At} : t \ge 0\}$ in E_0. Let $E_1 = D(A)$ with the graph norm $\|\cdot\|_{E_1} = \|A\cdot\|_{E_0}$ and similarly let $E_2 = D(A^2)$ with the graph norm $\|\cdot\|_{E_2} = \|A\cdot\|_{E_1} = \|A^2\cdot\|_{E^0}$. For each $0 < \theta < 1$ we fix an interpolation functor $(\cdot,\cdot)_\theta$ and let

$$E_\theta := (E_1, E_0)_\theta, \qquad E_{1+\theta} := (E_2, E_1)_\theta.$$

Then, since $A \in \mathcal{L}(E_2, E_1) \cap \mathcal{L}(E_1, E_0)$ we have that $A \in \mathcal{L}(E_{\theta+1}, E_\theta)$ in addition the following theorem holds.

Theorem 2.1 *If A_α denotes the realization of A in E_α then $A_\alpha \in \mathcal{L}(E_{\alpha+1}, E_\alpha)$ and $A_\alpha : E_{\alpha+1} \subset E_\alpha \to E_\alpha$ generates an analytic semigroup $e^{A_\alpha t}$ in E_α which is the restriction of e^{At} to E_α. Furthermore, there is a constant $M \geq 1$ such that the following holds*

$$t^{\alpha-\beta}\|e^{At}x\|_{E_\beta} \leq M\|x\|_{E_\alpha}, \quad 0 \leq \alpha \leq \beta \leq 2.$$

For a proof of this theorem and much more, see [4].

In order that one can see why we may want to consider these scales coming from interpolation instead of the scale of fractional power spaces associated to the operator A we consider a very standard problem for which the scale of fractional power spaces seems not to be suitable in the treatment of critical problems.

Let $\Omega \subset I\!R^N$ be a bounded smooth domain and $a : \bar{\Omega} \to I\!R$ be a C^1 function with $a(x) \geq a_0 > 0$, $\forall x \in \Omega$. Consider the elliptic operator

$$Lu = \operatorname{div}(a\nabla u)$$

with homogeneous Neumann boundary conditions. Then L defines a sectorial operator A_q in $L^q(\Omega)$ with domain $D(A) = \{u \in W^{2,q}(\Omega) : \frac{\partial u}{\partial n} = 0\} =: W^{2,q}_{\mathcal{N}}(\Omega)$ (see, [3]).

Note that if we consider the complex interpolation functor of exponent θ (see [16, 4]), denoted by $[\cdot,\cdot]_\theta$ we have the following

$$E_\alpha = [W^{2,q}_{\mathcal{N}}(\Omega), L^q(\Omega)]_\alpha \hookrightarrow H^{2\alpha}_q(\Omega) \tag{2.1}$$

for $0 \leq \alpha \leq 1$.

We note that if F^α_q denotes the fractional power spaces for the operator A_q we have that

$$F^\alpha_q \hookrightarrow H^{2\alpha^-}_q(\Omega)$$

for any $\alpha^- < \alpha$ (see [12]). In general it is not known that F^α_q is embedded in $H^{2\alpha}_q(\Omega)$ (though this is true if the boundary conditions are Dirichlet or in the case of Neumann boundary conditions if the domain Ω is C^∞ (for references on this, see [3], Remark 7.3)). To obtain existence results for parabolic problems with critically growing nonlinearities one needs to work with an scale for which the sharp embedings (2.1) hold. This is one of the reasons why it is convenient to use the scales coming from interpolation theory instead of the scale of fractional power spaces and the reason why the theory of scales of Banach spaces has been developed by H. Amann (see [4]).

2.2 Local existence of solutions

In this section we are interested in existence of solutions for the problems of the form

$$\begin{array}{l} \dot{x} = Ax + \sum_{i=1}^n f_i(x), \ t > 0 \\ x(0) = x_0. \end{array} \tag{2.2}$$

where we restrict ourselves to the case of autonomous nonlinearities for the sake of simplicity in the notation.

The results presented here are natural and simple extensions of the abstract results presented in [5], that are needed to deal with nonlinear boundary conditions, systems and many other applications. With this in mind we will only indicate the differences in the proofs and refer to [5] for details. Also, note that the results in [5] were presented mainly for the scale of fractional power spaces, although the proofs given there hold for any scale of interpolation spaces.

As above, let X^0 be a Banach space, $\|\cdot\|_{X^0}$ its norm and $-A : D(A) \subset X^0 \to X^0$ be a sectorial operator. If $\sigma(A)$ denotes the spectrum of A, we assume that $\mathrm{Re}\sigma(A) < 0$. Let $\{X^\alpha,\ 2 \geq \alpha \geq 0\}$, be a scale of Banach spaces associated to A constructed as before.

Consider the following class of nonlinearities: for $0 \leq \epsilon < 1$, $\rho > 1$, $\rho\epsilon \leq \gamma(\epsilon) < 1$, and c a positive constant, define $\mathcal{F} := \mathcal{F}(\epsilon, \rho, \gamma(\epsilon), c)$ as the family of functions f such that $f(\cdot)$ is an ϵ-regular map relative to the pair (X^1, X^0), that is $f : X^{1+\epsilon} \to X^{\gamma(\epsilon)}$ and

$$\|f(x) - f(y)\|_{X^{\gamma(\epsilon)}} \leq c\|x - y\|_{X^{1+\epsilon}}(\|x\|^{\rho-1}_{X^{1+\epsilon}} + \|y\|^{\rho-1}_{X^{1+\epsilon}} + 1) \tag{2.3}$$

$$\|f(x)\|_{X^{\gamma(\epsilon)}} \leq c(\|x\|^{\rho}_{X^{1+\epsilon}} + 1). \tag{2.4}$$

for all $x, y \in X^{1+\epsilon}$. Note that the first condition implies the second.

Then, we look for functions that verify

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} \sum_{i=1}^{n} f_i(x(s))ds$$

and we have the following result.

Theorem 2.2 *Assume that for $1 \leq i \leq n$, $f_i \in \mathcal{F}(\epsilon_i, \rho_i, \gamma_i(\epsilon_i), c)$, for some $\epsilon_i > 0$. Assume also that*

$$\min\{\gamma_i(\epsilon_i);\ 1 \leq i \leq n\} =: \underline{\gamma} > \max\{\epsilon_i;\ 1 \leq i \leq n\} =: \overline{\epsilon}. \tag{2.5}$$

Then, if $y_0 \in X^1$, there exist $r = r(y_0) > 0$ and $\tau_0 = \tau_0(y_0) > 0$ such that for $x_0 \in X^1$ with $\|x_0 - y_0\|_{X^1} < r$ there exists a continuous function $x : [0, \tau_0] \to X^1$, with $x(0) = x_0$, which is the unique $\overline{\epsilon}-$regular mild solution to (2.2) starting at x_0. In addition, this solution satisfies

$$x \in C((0, \tau_0], X^{1+\theta}), \qquad t^\theta \|x(t, x_0)\|_{X^{1+\theta}} \overset{t\to 0^+}{\longrightarrow} 0, \quad 0 \leq \theta < \underline{\gamma}.$$

Moreover, if $x_0, z_0 \in B_{X^1}(y_0, r)$ the following holds

$$t^\theta \|x(t, x_0) - x(t, z_0)\|_{X^{1+\theta}} \leq C(\theta_0)\|x_0 - z_0\|_{X^1}, \ \forall t \in [0, \tau_0], \qquad 0 \leq \theta \leq \theta_0 < \underline{\gamma}.$$

Also, if $\gamma_i(\epsilon_i) > \rho_i\epsilon_i$, for all $1 \leq i \leq n$, then r can be chosen arbitrarily large. That is, the time of existence can be chosen uniform on bounded sets of X^1.

Furthermore, $x \in C((0, \tau_0], X^{1+\underline{\gamma}})$ and $x_t \in C((0, \tau_0], X^{1+\theta})$, for any $\theta < \underline{\gamma}$, and (2.2)is satisfied, that is, $x(\cdot, x_0)$ is an strict solution.

The constants above depend on the following: $\tau_0 = \tau_0(y_0, A, \epsilon_i, \rho_i, \gamma_i(\epsilon_i), c, M)$, $r = r(y_0, \epsilon_i, \rho_i, \gamma_i(\epsilon_i), c, M)$, $C = C(\theta_0, \epsilon_i, \rho_i, \gamma_i(\epsilon_i), M)$.

Proof. The proof of this theorem basically follows line by line the proof of Theorem 1 from [5]. We will just outline the proof pointing out the few differences.

Let $\mathbf{B}(\cdot,\cdot) : (0,\infty) \times (0,\infty) \to (0,\infty)$ denote the beta function and define

$$\mathbf{B}^{\theta}_{\epsilon_i} = \max_{0\le\xi\le\theta} \{\mathbf{B}(\gamma_i(\epsilon_i), 1-\gamma_i(\epsilon_i)), \mathbf{B}(\gamma_i(\epsilon_i), 1-\rho_i\epsilon_i), \mathbf{B}(\gamma_i(\epsilon_i)-\xi, 1-\gamma_i(\epsilon_i)), \mathbf{B}(\gamma_i(\epsilon_i)-\xi, 1-\rho_i\epsilon_i)\}.$$

For the proof of existence, define μ as the unique positive solution of $\sup_{i,j} cM\mathbf{B}^{\epsilon_j}_{\epsilon_i}\mu^{\rho_i-1} = \frac{1}{4n}$. Let us choose now $r = r(\mu, M) > 0$ such that

$$r = \frac{\mu}{4M}. \tag{2.6}$$

Also, for y_0 fixed, choose $\delta > 0$ and $\tau_0 = \tau_0(y_0, A, \mu, \epsilon_i, \rho_i, \gamma_i(\epsilon_i), c, M) \in (0,1]$ such that

$$\sup_{i,j} cM\delta\mathbf{B}^{\epsilon_j}_{\epsilon_i} = \frac{\mu}{4n}, \quad \|t^{\epsilon_i}e^{-At}y_0\|_{X^{1+\epsilon_i}} \le \frac{\mu}{4}, \ 0 \le t \le \tau_0, \quad \sup_{i,j}\sup_{0\le t\le\tau_0} \{t^{\gamma_i(\epsilon_i)-\epsilon_j}\} \le \delta. \tag{2.7}$$

Now, for $x_0 \in X^1$ with $\|x_0 - y_0\|_{X^1} < r$, we look for fixed points of the map

$$(Tx)(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} \sum_{i=1}^n f_i(s, x(s))ds.$$

in the space, $K(\tau_0) = \{x \in C((0,\tau_0], X^{1+\bar{\epsilon}}) : \sup_{1\le i\le n}\sup_{t\in(0,\tau_0]} t^{\epsilon_i}\|x(t)\|_{X^{1+\epsilon_i}} \le \mu\}$, with norm $\|x\|_{K(\tau_0)} = \sup_{1\le i\le n}\sup_{t\in(0,\tau_0]} t^{\epsilon_i}\|x(t)\|_{X^{1+\epsilon_i}}$.

With the above choice of parameters we have that, $T(K(\tau_0)) \subset K(\tau_0)$,

$$\text{if } x \in K(\tau_0) \quad \text{then} \quad Tx \in C((0,\tau_0], X^{1+\theta}), \qquad \forall\theta \in [0, \underline{\gamma}) \tag{2.8}$$

and that T is a strict contraction in $K(\tau_0)$. In proving all these, condition (2.5) is used thoroughly. By the Banach contraction principle we have that T has a unique fixed point in $K(\tau_0)$, $x(t, x_0)$ which is defined for $\|x_0 - y_0\|_{X^1} < r$, $0 \le t \le \tau_0$. Note that, from (2.8), $x(\cdot, x_0) \in C((0,\tau_0], X^{1+\theta})$, for all $0 \le \theta < \underline{\gamma}$. Also $t^{\theta}\|x(t,x_0)\|_{X^{1+\theta}} \to 0$ as $t \to 0$ for all $0 < \theta < \underline{\gamma}$ and

$$\lim_{t\to 0+} \|x(t,x_0) - x_0\|_{X^1} = 0.$$

With all these we have that $x(t, x_0)$ is an $\bar{\epsilon}-$regular solution starting at x_0 and it is the unique $\bar{\epsilon}$-regular solution starting at x_0, in the set $K(\tau_0)$.

Moreover, if $x_0,\ z_0 \in B_{X^1}(y_0, r)$ and for $0 \le \theta \le \theta_0 < \underline{\gamma}$, we have that

$$t^{\theta}\|x(t,x_0) - x(t,z_0)\|_{X^{1+\theta}} \le C(\theta_0)\|x_0 - z_0\|_{X^1}$$

for some constant $C(\theta_0)$.

The proof of uniqueness of $\bar{\epsilon}$-regular solutions follows exactly as in Theorem 1 of [5]. The same applies for the estimate of the existence time when $\gamma_i(\epsilon_i) > \rho_i\epsilon_i$, for all $1 \le i \le n$. We omit the proof of these facts.

For the extra regularity in the statement, observe that, since $h : X^{1+\bar{\epsilon}} \to X^{\underline{\gamma}}$ is locally Lipschitz on bounded sets, and since the solution smoothes for positive time, we can apply the technique in Theorem 3.5.2 in [12], to obtain that $x(t, x_0)$ satisfies (2.2) and $x_t \in C((0,\tau_0], X^{1+\theta})$, for any $\theta < \underline{\gamma}$. From here, we also get $x \in C((0,\tau_0], X^{1+\underline{\gamma}})$.□

Like in [5], we may also prove a result on the maximal time of existence of an $\bar{\epsilon}$-regular solution.

Proposition 2.1 *If f_i are as in Theorem 2.2 and $x(t, x_0)$ is an $\bar{\epsilon}$−regular solution starting at x_0 with a maximal time of existence τ_m, then either $\tau_m = \infty$ or $\liminf_{t\to\tau_m^-} \|x(t, x_0)\|_{X^{1+\delta}} = \infty$, for any $\delta > 0$.*

Proof. It is easy to check that of f_i is ϵ_i–regular relatively to (X^1, X^0) with $\gamma_i(\epsilon_i) \geq \rho_i\epsilon_i$, then for any $\delta > 0$, f_i is ϵ_i^*–regular relatively to $(X^{1+\delta}, X^\delta)$, for $\epsilon_i^* = \epsilon_i - \delta$, with $\gamma_i(\epsilon_i^*) > \rho_i\epsilon_i^*$. Assume the solution remains bounded in $X^{1+\delta}$ along a sequence that converges to τ_m. Then using Theorem 2.2 for the pair $(X^{1+\delta}, X^\delta)$ and using now the uniform existence time on bounded sets, we get that the solution can be extended beyond τ_m, which is a contradiction.$\square$

If f is an ϵ−regular map relatively to the pair (X^1, X^0), for ϵ in an interval I, as in [5], we classify the map in the following way:

- If $I = [0, \epsilon_1]$ for some $\epsilon_1 > 0$ and $\gamma(0) > 0$. We say that f is a *subcritical* map relative to (X^1, X^0).

  In this case, $f : X^1 \to X^{\gamma(0)}$ and then the standard technique for solving $u_t = Au + f(u)$ can be applied.

- If $I = [0, \epsilon_1]$ for some $\epsilon_1 > 0$ with $\gamma(\epsilon) = \rho\epsilon$, $\epsilon \in I$ and if f is not subcritical; then, we say that f is a *critical* map relative to (X^1, X^0).

  In this case, $f : X^1 \to X^0$ and then the standard technique for solving $u_t = Au + f(u)$ can not be applied.

- If $I = (0, \epsilon_1]$ for some $\epsilon_1 > 0$ with $\gamma(\epsilon) = \rho\epsilon$, $\epsilon \in I$, and f is not subcritical or critical; then, we say that f is a *double-critical* map relative to (X^1, X^0).

- If $I = [\epsilon_0, \epsilon_1]$ for some $\epsilon_1 > \epsilon_0 > 0$ with $\gamma(\epsilon_0) > \rho\epsilon_0$ and f is not subcritical, critical or double critical; then, we say that f is an *ultra-subcritical* map relative to (X^1, X^0).

- If $I = [\epsilon_0, \epsilon_1]$ for some $\epsilon_1 > \epsilon_0 > 0$ with $\gamma(\epsilon) = \rho\epsilon$, $\epsilon \in I$, and if f is not subcritical, critical, double critical or ultra-subcritical; then, we say that f is an *ultra-critical* map relative to (X^1, X^0).

Note that in the last three cases f is not even defined on X^1. However, in all cases, the results above allow to construct solutions of (2.2).

3 Heat Equations with Nonlinear Boundary Conditions

It is well known that for semilinear problems like (1.1) it is always desirable to work not on a fixed space (say $W^{1,q}(\Omega)$, for example), but to work on families of spaces in which the initial data can lie and in which the nonlinear term is well behaved. This amounts to constructing scales of spaces associated to the operator $A = \Delta$ in $L^q(\Omega)$, with homogeneous Neumann boundary conditions, as described above. Among several reasons for this, we point out that, with this approach, one can give a unified treatment to (1.1) which allows one to solve it in the following two senses:

Definition 3.1 *For $u_0 \in W^{1,q}(\Omega)$, a variational weak solution of (1.1) is a function $u \in C([0, \tau), W^{1,q}(\Omega))$ such that*

$$\frac{d}{dt}\int_\Omega u\phi + \int_\Omega \nabla u \nabla \phi = \int_\Omega f(u)\phi + \int_\Gamma g(u)\phi \tag{3.1}$$

for every $\phi \in W^{1,q'}(\Omega)$.

Definition 3.2 *For* $u_0 \in L^q(\Omega)$, *a variational very weak solution of (1.1) is a function* $u \in C([0,\tau), L^q(\Omega))$ *such that*

$$\frac{d}{dt}\int_\Omega u\phi - \int_\Omega u\Delta\phi = \int_\Omega f(u)\phi + \int_\Gamma g(u)\phi \tag{3.2}$$

for every $\phi \in W^{2,q'}_N(\Omega)$, *where* $W^{2,q'}_N(\Omega)$ *consist of the functions in* $W^{2,q'}(\Omega)$ *with zero normal derivative on* Γ, *and* q' *the conjugate exponent to* q.

This formulations come from multiplying the equation by a test function and integrating by parts. See [3, (2.7), (2,9), page 13] and [3, pages 61 and 62].

In the Hilbert case, that is, when $q = 2$ all methods of construction of scales give the same spaces, while if $q \neq 2$ the result depends on the choice of the interpolation method. One of the scales of spaces that have been widely used is the scale of fractional power spaces, see [12], but some other scales have been shown to be very well suited to that purpose, [3, 4, 14]. In fact, we will show below that this scales are naturally better adapted for the problem of critical exponents, since some more precise information is available than for the scale of fractional power spaces.

It is clear from the results in the previous section that for a given problem $\dot{x} = Ax + \sum_{i=1}^n f_i(x)$, where A is a sectorial operator with associated scale X^θ, $\theta \in [0, 2]$, we need to identify these fractional order spaces and to study the $\epsilon-$regularity properties of the maps f_i. This last part is usually a consequence of the previous knowledge of the integer order spaces, interpolation theory and standard techniques involving Sobolev type embeddings. In this way the local existence of solutions for this problem is reduced to a good knowledge of scale of spaces associated to A.

With this in mind we will breafly describe a way to identify these fractional order spaces following closely [4]. Assume that E_0 is a reflexive Banach space and let $-A_0 : D(A_0) \subset E_0 \to E_0$ be a sectorial operator on E_0 which for simplicity is assumed to have positive spectrum. Then one can define the spaces $E_1 := D(A_0)$ with the "graph" norm $\|\cdot\|_{E_1} := \|A_0 \cdot\|_{E_0}$. If $A_1 : D(A_1) \subset E_1 \to E_1$ denotes the realization of A_0 in E_1 we can define $E_2 := D(A_1)$ with the "graph" norm $\|\cdot\|_{E_2} := \|A_1 \cdot\|_{E_1}$. This procedure allow us to define a discrete scale of Banach Space $\{E_j,\ j = 0, 1, 2, \cdots\}$ and as in the Section 2 we fix an interpolation functor $(\cdot,\cdot)_\theta$ for each $0 < \theta < 1$ and obtain a continuous one-sided scale $\{E_\theta;\ \theta \in [0, \infty)\}$ associated to A. This is enough for the abstract results but in the applications it may be also needed to consider an extension of this scale to "negative" values of θ. This is accomplished in the foll

$$E_{-1} := \left(E_0, \|A_0^{-1} \cdot\|_{E_0}\right)^\sim \tag{3.3}$$

that is, a completion of E_0 relatively the norm $\|A^{-1}\cdot\|_{E_0}$. That gives a Banach Space, $E_0 \overset{d}{\hookrightarrow} E_{-1}$, and there is a unique closed extension of A_0 to E_{-1} which we denote by $A_{-1} : E_0 \subset E_{-1} \to E_{-1}$. Clearly E_0 with the norm $\|A_{-1}\cdot\|_{E_{-1}}$ is isomorphic to E_0. With this proceedure one can then construct E_{-2} and A_{-2} and E_{-1} with the norm $\|A_{-2}\cdot\|_{E_{-2}}$ is isomorphic to E_{-1}. This proceedure can be carried out to obtain the discrete scale of $\{E_j,\ j = -m, -m+1, -m+2, \cdots\}$ for any natural number m. Once the discrete scale is constructed we fix the operator $A = A_{-m+1} : E_{-m+1} \subset E_m \to E_m$ and construct the one sided continuous scale associated to A $\{E_\theta;\ \theta \in [-m, \infty)\}$ as above. Of course this proceedure will only render practical if one can identify the spaces in the negative scale. With this in mind we describe briefly the construction of the dual scale.

Let $E_0^{\#}$ denote the topological dual of E_0 and $A_0^{\#} : D(A^{\#}) \subset E_0^{\#} \to E_0^{\#}$ be the dual operator of A_0. Then $A_0^{\#}$ is a sectorial operator in $E_0^{\#}$ and proceeding as before one may construct the scale of Banach spaces associated to $A_0^{\#}$ denoted by $\{E_\theta^{\#};\ \theta \in [-m, \infty)\}$. Depending on the way one constructs this scale it is possible to identify the fractional spaces $E_{-\theta}$ through the spaces $E_\theta^{\#}$, $\theta \geq 0$. This can be done in the following way:

For $0 < \theta < 1$ let $[\cdot,\cdot]_\theta$, $(\cdot,\cdot)_{\theta,q}$, $1 \leq q \leq \infty$ and $(\cdot,\cdot)_{\theta,\infty}^0$ denote respectively the complex, the real and the continuous interpolation functors of exponent θ. Then, if $(\cdot,\cdot)_\theta \in \{[\cdot,\cdot]_\theta, (\cdot,\cdot)_{\theta,q}, (\cdot,\cdot)_{\theta,\infty}^0, 1 < q < \infty\}$ we define the dual functor $(\cdot,\cdot)_\theta^{\#}$ of $(\cdot,\cdot)_\theta$ as

$$(\cdot,\cdot)_\theta^{\#} := \begin{cases} [\cdot,\cdot]_\theta & \text{if} \quad (\cdot,\cdot)_\theta = [\cdot,\cdot]_\theta \\ (\cdot,\cdot)_{\theta,q'} & \text{if} \quad (\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,q},\ 1 < q < \infty \\ (\cdot,\cdot)_{\theta,1} & \text{if} \quad (\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,\infty}^0. \end{cases}$$

If for each $0 < \theta < 1$ we choose $(\cdot,\cdot)_\theta \in \{[\cdot,\cdot]_\theta, (\cdot,\cdot)_{\theta,q}, (\cdot,\cdot)_{\theta,\infty}^0\}$ to obtain the scale E_θ and its dual functor $(\cdot,\cdot)_\theta^{\#}$ to obtain the scale $E_\theta^{\#}$ the following holds

$$E_\theta = (E_\theta^{\#})'.$$

See [4] for the proof.

Now we overview the description of the scales of spaces constructed above for problem (1.1). Again we follow [3]. Let Ω be a bounded smooth domain and $H_q^l(\Omega)$, be the Bessel potential spaces and $H_q^{-l}(\Omega) := (H_{q'}^l(\Omega))'$, $1 < q < \infty$, $l \geq 0$. In this section we will constantly use certain well known embeddings that we summarize as:

$$\begin{aligned} &H_q^l(\Omega) \subset L^r(\Omega), \quad \text{if } \tfrac{l}{N} - \tfrac{1}{q} \geq -\tfrac{1}{r},\ 1 < q \leq r < \infty \\ &H_q^l(\Omega) \subset C^\eta(\bar{\Omega}), \quad \text{if } l - \tfrac{N}{q} > \eta > 0 \end{aligned} \tag{3.4}$$

with continuous embeddings (see, [1]). This embeddings are known to be optimal.

For later use, we state here some trace results for Bessel potential spaces, [1]. If T denotes the trace operator, then for $l > \frac{1}{q}$, T is well defined on $H_q^l(\Omega)$ and

$$H_q^l(\Omega) \xrightarrow{T} L^r(\Gamma), \quad \text{for } r \begin{cases} \leq \infty & \text{if } lq > N \\ < \infty & \text{if } lq = N \\ \leq \frac{(N-1)q}{N-lq} & \text{if } lq < N \end{cases} \tag{3.5}$$

Following the results and notation of the example of Subsection 2.1, the operator $L = \Delta$ (or in general an elliptic operator of the type $L = \text{div}(a(x)\nabla)$) with homogeneous Neumann boundary condition, can be seen as an unbounded operator in $E_q^0 := L^q(\Omega)$ for $1 < q < \infty$ with domain $E_q^1 := W_N^{2,q}(\Omega) := \{u \in W^{2,q}(\Omega),\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. It has an associated scale of interpolation spaces E_q^α that, thanks to (2.1), (3.4) and the fact that $E_q^{-\alpha} = (E_{q'}^\alpha)'$, satisfy

$$\left.\begin{array}{ll} E_q^\alpha \hookrightarrow L^r(\Omega), \text{ for } r \leq \frac{Nq}{N-2\alpha q}, & 0 \leq \alpha < \frac{N}{2q} \\ E_q^0 = L^q(\Omega) & \\ E_q^\alpha \hookleftarrow L^s(\Omega), \text{ for } s \geq \frac{Nq}{N-2\alpha q}, & -\frac{N}{2q'} < \alpha \leq 0 \end{array}\right\} 1 < q < \infty. \tag{3.6}$$

with continuous embeddings. Moreover, we have $E_q^{-\frac{1}{2}} = (W^{1,q'}(\Omega))' =: W^{-1,q}(\Omega)$.

If we now consider the realizations of L on the scale above, we get the following, [3, pages 39–43]. The operator $L_{-1/2} \in \mathcal{L}(E_q^{\frac{1}{2}}, E_q^{-\frac{1}{2}}) = \mathcal{L}(W^{1,q}(\Omega), (W^{1,q'}(\Omega))') = \mathcal{L}(W^{1,q}(\Omega), W^{-1,q}(\Omega))$ is given by

$$< L_{-1/2}u, \phi >= -\int_\Omega \nabla u \nabla \phi \tag{3.7}$$

for every $\phi \in W^{1,q'}(\Omega)$. On the other hand, the operator $L_{-1} \in \mathcal{L}(E_q^0, E_q^{-1}) = \mathcal{L}(L^q(\Omega), (W_N^{2,q'}(\Omega))')$ is given by

$$< L_{-1}u, \phi >= \int_\Omega u \Delta \phi \tag{3.8}$$

for every $\phi \in W_N^{2,q'}(\Omega)$.

With all these, (1.1) can now be written as

$$u_t = L_{-1/2}u + h(u), \quad u(0) = u_0 \in W^{1,q}(\Omega)$$

which corresponds to the weak formulation (3.1). Alternatively, (1.1) can be written as

$$u_t = L_{-1}u + h(u), \quad u(0) = u_0 \in L^q(\Omega)$$

which corresponds to the very weak formulation (3.2), see (1.3) and (1.8). In both cases h is defined as $h(u) = f_\Omega(u) + g_\Gamma(u)$, in the sense that

$$\langle f_\Omega(u) + g_\Gamma(u), \phi \rangle = \int_\Omega f(u)\phi + \int_\Gamma g(u)\phi \tag{3.9}$$

for all suitable choosen test function ϕ. As for (1.4), we look for solutions of the integral equation

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}[f_\Omega(u(s)) + g_\Gamma(u(s))]ds \tag{3.10}$$

where A denotes either $L_{-1/2}$ or L_{-1}.

Therefore, we are in a position to apply the abstract results of Subsection 2.2. Then, we will assume some growth assumptions on f and g and study the $\epsilon-$regularity properties of the corresponding mappings f_Ω and g_Γ in order to apply Theorem 2.2.

3.1 Critical exponents for initial data in $L^q(\Omega)$

In this section we prove Theorem 1.1. Therefore, we assume that in (1.1), f and g verify (1.7) with exponents ρ_1 and ρ_2 respectively. We will see that as long as $\rho_1 \leq \rho_f = \frac{N+2q}{N}$ and $\rho_2 \leq \rho_g = \frac{N+q}{N}$ we can give an existence and uniqueness theorem for the problem above in $L^q(\Omega)$.

With the notations of Section 2.1 and more precisely with (2.1), denote by $X_q^\alpha := E_q^{\alpha-1}$, $\alpha \in I\!R$ and by $A_q : X_q^1 \subset X_q^0 \to X_q^0$ the realization of Δ in E_q^{-1} given by (3.8). Then, from (3.6), we have

$$\left.\begin{array}{ll} X_q^\alpha \hookrightarrow L^r(\Omega), \text{ for } r \le \frac{Nq}{N+2q-2\alpha q}, & 1 \le \alpha < \frac{N}{2q} \\ X_q^1 = L^q(\Omega) & \\ X_q^\alpha \hookleftarrow L^s(\Omega), \text{ for } s \ge \frac{Nq}{N+2q-2\alpha q}, & -\frac{N}{2q'} < \alpha \le 1 \end{array}\right\} 1 < q < \infty. \tag{3.11}$$

with continuous embeddings. Note that in this setting, when applying Theorem 2.2, we will obtain a solution on (1.1) in the sense of (3.2). Later on we will show that this solution is even classical.

The ϵ-regularity properties of the map f_Ω are given by the following lemma that is taken from [5]. The main argument relies in the fact that if f grows like in (1.7), then f_Ω is a well defined mapping between $L^{\rho_1 r}(\Omega)$ and $L^r(\Omega)$, for any $r \ge 1$, which is moreover Lipschitz on bounded sets. Therefore, we use the sharp embedings in (3.11) for $X_q^{1+\epsilon} \hookrightarrow L^{\rho_1 r(\epsilon)}(\Omega)$ and then the sharp embeddings in (3.11) for $L^{r(\epsilon)}(\Omega) \hookrightarrow X_q^{\gamma(\epsilon)}$. Whith this, (2.3) and (2.4) come out easily. In fact, it is not difficult to check that $\gamma(\epsilon) = 1 + \frac{N}{2q}(1-\rho_1) + \epsilon\rho_1$. When checking that the parameters in the exponents, verify $0 \le \epsilon < 1$, $\rho_1\epsilon \le \gamma(\epsilon) < 1$, and according to the classification of nonlinearities stated in the previous section, we get the results below. Notice that $\rho_1\epsilon \le \gamma(\epsilon)$ iff $\rho_1 \le \frac{N+2q}{N}$ and the critical case corresponds to the equal sign.

Lemma 3.1 (Critical Case) *If $1 < q < \infty$ and if $\rho_1 = \rho_f = \frac{N+2q}{N}$, then*

- *If $N \ge 3$, $q > \frac{N}{N-2}$, then f_Ω is an ϵ-regular map relative to (X_q^1, X_q^0) for $0 \le \epsilon < \frac{1}{\rho_1}$ and $\gamma(\epsilon) = \rho_1\epsilon$. Therefore f_Ω is a critical map.*
- *If $N \ge 3$, $q = \frac{N}{N-2}$, then f_Ω is an ϵ-regular map relative to (X_q^1, X_q^0) for $0 < \epsilon < \frac{1}{\rho_1}$ and $\gamma(\epsilon) = \rho_1\epsilon$. Therefore, f_Ω is a double-critical map.*
- *If $N = 1, 2$ and $1 < q < \infty$ or if $N \ge 3$ and $1 < q < \frac{N}{N-2}$, then f_Ω is an ϵ-regular map relative to (X_q^1, X_q^0) for $0 < \frac{1}{\rho_1}\left(1 - \frac{N}{2q'}\right) < \epsilon < \frac{1}{\rho_1}$ and $\gamma(\epsilon) = \rho_1\epsilon$. Therefore, f_Ω is an ultra-critical map.*

Lemma 3.2 (Subcritical Case) *If $N \ge 1$, $1 < q < \infty$ and if $\rho_1 < \rho_f = \frac{N+2q}{N}$, then*

- *If $\frac{N}{N-2} \le q$ then f_Ω is a subcritical map relative to (X_q^1, X_q^0).*
- *If $1 < q < \frac{N}{N-2}$ and $\rho_1 \le q$ then f_Ω is a subcritical map relative to (X_q^1, X_q^0).*
- *If $1 < q < \frac{N}{N-2}$ and $\frac{N}{2}(\rho_1 - 1) < q < \rho_1$ then f_Ω is an ultra-subcritical map relative to (X_q^1, X_q^0).*

For the nonlinearity g_Γ we have the following analysis. If g grows like in (1.7), with exponent ρ_2, then g_Γ is a well defined mapping between $L^{\rho_2 r}(\Gamma)$ and $L^r(\Gamma)$, for any $r \ge 1$, which is moreover Lipschitz on bounded sets. Therefore, we use first the sharp embeddings (2.1) for $X_q^{1+\epsilon} \subset H_q^{2\epsilon}(\Omega)$ together with the properties for the trace operator, T, $H_q^{2\epsilon}(\Omega) \xrightarrow{T} L^{\rho_2 r(\epsilon)}(\Gamma)$, see (3.5). Then, in a similar way, we use sharp embeddings of the form $L^{r(\epsilon)}(\Gamma) \hookrightarrow X_q^{\gamma(\epsilon)}$. Again, (2.3) and (2.4) come out easily.

Indeed, in this case, it is easy to check that $\gamma(\epsilon) = \frac{1}{2} + \frac{N}{2q}(1-\rho_2) + \epsilon\rho_2$. Again, when checking that the parameters in the exponents, verify $0 \le \epsilon < 1$, $\rho_2\epsilon \le \gamma(\epsilon) < 1$, and according to the classification of nonlinearities stated in the previous section, we get the results below. Notice that now $\rho_2\epsilon \le \gamma(\epsilon)$ iff $\rho_2 \le \frac{N+q}{N}$ and the critical case corresponds to the equal sign.

Also, observe that since X_q^1 is a space with no trace, ϵ has to be taken large enough such that the functions in $X_q^{1+\epsilon}$ have trace. This implies that the mapping g_Γ will be always ultra-subcritical or ultra-critical, according to the notations in the previous section.

Lemma 3.3 (Critical Case) *If $N > 1$, $1 < q < \infty$ and if $\rho_2 = \rho_g = \frac{N+q}{N}$, then*

- *If $1 < q < \frac{N}{N-1}$ then, g_Γ is an ϵ-regular map relative to (X_q^1, X_q^0) for $\frac{1}{\rho_2}(1 - \frac{N}{2q'}) < \epsilon < \frac{1}{2\rho_2}\left(\frac{q+1}{q}\right)$ and $\gamma(\epsilon) = \rho_2\epsilon$.*
- *If $q \geq \frac{N}{N-1}$ then g_Γ is an ϵ-regular map relative to (X_q^1, X_q^0) for $\frac{1}{2q} < \epsilon < \frac{1}{2\rho_2}\left(\frac{q+1}{q}\right)$ and $\gamma(\epsilon) = \rho_2\epsilon$.*

In both cases the nonlinearity g_Γ is an ultra-critical map.

Proof: We have to prove that g_Γ maps $X_q^{1+\epsilon}$ into $X_q^{\gamma(\epsilon)}$ with $\gamma(\epsilon) = \rho_2\epsilon$ and satisfies

$$\begin{aligned}
&\|g_\Gamma(u)\|_{X_q^{\gamma(\epsilon)}} \leq c\|u\|^{\rho_2}_{X_q^{1+\epsilon}} + 1),\\
&\|g_\Gamma(u) - g_\Gamma(v)\|_{X_q^{\gamma(\epsilon)}} \leq c(\|u\|^{\rho_2-1}_{X_q^{1+\epsilon}} + \|v\|^{\rho_2-1}_{X_q^{1+\epsilon}} + 1)\|u-v\|_{X_q^{1+\epsilon}}.
\end{aligned}$$

Since the proofs of both inequalities are similar we restrict our attention to the first one. Note that if T denotes the trace operator then, with the notations in (2.1), we have

$$X_q^{1+\epsilon} \hookrightarrow H_q^{2\epsilon}(\Omega) \xrightarrow{T} L^{\rho_2 r(\epsilon)}(\Gamma),$$

for $\rho_2 r(\epsilon) = \frac{Nq-q}{N-2q\epsilon}$, provided that $\frac{Nq-q}{N-2q\epsilon} > q$ and, again using (3.5),

$$(X_q^{\gamma(\epsilon)})' \hookrightarrow H_{q'}^{2-2\rho_2\epsilon}(\Omega) \xrightarrow{T} L^{r(\epsilon)'}(\Gamma),$$

provided that $q > \frac{Nq-q}{(N-2q\epsilon)\rho_2} > 1$. Now, for $\phi \in (X_q^{\gamma(\epsilon)})'$ we have that

$$\begin{aligned}
|\langle g_\Gamma(u), \phi\rangle| &= \left|\int_\Gamma g(u)\phi\right| \leq \|g(u)\|_{L^{r(\epsilon)}(\Gamma)}\|\phi\|_{L^{r(\epsilon)'}(\Gamma)}\\
&\leq c(1 + \|u\|^{\rho_2}_{L^{\rho_2 r(\epsilon)}(\Gamma)})\|\phi\|_{L^{r(\epsilon)'}(\Gamma)} \leq c(1 + \|u\|^{\rho_2}_{X_q^{1+\epsilon}})\|\phi\|_{(X_q^{\gamma(\epsilon)})'}
\end{aligned}$$

which implies $\|g_\Gamma(u)\|_{X_q^{\gamma(\epsilon)}} \leq c(1 + \|u\|^{\rho_2}_{X_q^{1+\epsilon}})$.

The intervals of ϵ−regularity will come from the trace restrictions above, namely: $\rho_2\, q > \frac{Nq-q}{N-2q\epsilon} > \max\{\rho_2, q\}$. If $\rho_2 \geq q$, that is, if $1 < q \leq \frac{N}{N-1}$ then we have $\frac{1}{\rho_2}(1 - \frac{N}{2q'}) < \epsilon < \frac{1}{2\rho_2}\left(\frac{q+1}{q}\right)$. On the other hand, if $q \leq \rho_2$, that is, if $q \geq \frac{N}{N-1}$ we have $\frac{1}{2q} < \epsilon < \frac{1}{2\rho_2}\left(\frac{q+1}{q}\right)$. This concludes the proof.□

With respect to subcritical cases we have the following results,

Lemma 3.4 (Subcritical Case) *If $N \geq 1$, $1 < q < \infty$ and if $\rho_2 < \rho_g = \frac{N+q}{N}$, then g_Γ is an ultra-subcritical map.*

Note that in either case for f_Ω or g_Γ, since the worst situation correspond to the critial cases, the intervals of ϵ–regularity for subcritical cases contain those for the critical ones.

The results above are summarized in the following figures.

Insert here Figures 1 and 2

Insert here Figures 3 and 4

Remark 3.1 *Notice that for q close to 1 the intervals where f_Ω and g_Γ are $\epsilon-$regular are disjoint, see Figure 4. Hence, in this case, $h = f_\Omega + g_\Gamma$ is not ϵ-regular for any value of ϵ. Therefore it is essential that in the abstract results we may deal with a sum of functions with different $\epsilon-$regularity properties and for different values of ϵ.*

We can now give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let $1 < q < \infty$ be fixed, $N \geq 2$ (the case $N = 1$ is similar) and let f and g satisfy (1.7) with $\rho_1 \leq (N + 2q)/N$ and $\rho_2 \leq (N + q)/N$, respectively. From the lemmas above we obtain that f_Ω is an ϵ_1-regular map and g_Γ is an ϵ_2-regular map for some $\epsilon_1, \epsilon_2 > 0$.

In order to check condition (2.5) in Theorem 2.2, we proceed as follows. As noted above the intervals of ϵ–regularity of subcritical cases are larger than those for the critical ones, so it is enough to check (2.5) for some ϵ_1 and ϵ_2 in the intervals given by Lemma 3.1 and Lemma 3.3 and for $\rho_1 = \frac{N+2q}{N}$ and $\rho_2 = \frac{N+q}{N}$.

Now, observe that if $q \leq \frac{N}{N-2}$, taking ϵ_1 and ϵ_2 as large as possible in these intervals makes $\underline{\gamma}$ to be close to $\frac{q+1}{2q}$ while $\bar{\epsilon}$ is close to $\frac{N}{N+q}$ and (2.5) is verified.

On the other hand, if $q > \frac{N}{N-2}$, since $\rho_1 > \rho_2$, the interval for f contains that for g and we can choose $\epsilon_1 = \epsilon_2 = \bar{\epsilon}$ close to $\frac{q+1}{\rho_2 2q}$ which makes $\underline{\gamma}$ to be close to $\frac{q+1}{2q}$ and again (2.5) is verified. Note also that in any case, $\underline{\gamma}$ can be choosen larger than $\frac{1}{2}$.

Consequently, Theorem 2.2 gives the existence of an $\bar{\epsilon}-$regular solution for $\bar{\epsilon} = \max\{\epsilon_1, \epsilon_2\}$. In particular, the solution satisfies (1.1) in the sense of (3.2). Also, Theorem 2.2 gives $u \in C((0, \tau], X_q^{1+\underline{\gamma}})$, with $\underline{\gamma} = \min\{\gamma_1(\epsilon_1), \gamma_2(\epsilon_2)\}$ and also $u_t \in C((0, \tau], X_q^{1+\underline{\gamma}^-})$, for any $\underline{\gamma}^- < \underline{\gamma}$.

Since, as observed above, ϵ_1 and ϵ_2 can be choosen so that $\underline{\gamma} > \frac{1}{2}$, then $X_q^{1+\underline{\gamma}^-} \hookrightarrow H_q^1(\Omega)$. Therefore, the solution found satisfies $u, u_t \in C((0, \tau], H_q^1(\Omega))$, that is, $u \in C^1((0, \tau], H_q^1(\Omega))$.

If $q > N$, we can use the embedding $H_q^1(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ for some $\alpha > 0$. Therefore $u \in C^1((0, \tau], C^\alpha(\bar{\Omega}))$. Moreover, viewing equation (1.1) for $0 < t \leq \tau$ as an elliptic equation $-\Delta u = f(u) - u_t$, in Ω, $\partial u/\partial n = g(u)$, on Γ, and usign standard elliptic regularity theory we get that $u(\cdot, t) \in C^{2,\alpha}(\Omega)$. This shows that the solution is classical.

If $1 < q < N$ we apply a bootstrap argument. Since $H_q^1(\Omega) \hookrightarrow L^r(\Omega)$ for $r = Nq/(N - q) > q$, and since $\rho_1 \leq (N + 2q)/N < (N + 2r)/N$ and $\rho_2 \leq (N + q)/N < (N + r)/N$ we can apply the argument above in $L^r(\Omega)$ which now gives $u \in C^1((0, \tau], H_r^1(\Omega))$. Repeating the above, in a finite number of steps we can show that $u \in C^1((0, \tau], H_p^1(\Omega))$, for some $p > N$. This, as before, implies that the solution is classical.$\square$

Note that in either case for f_Ω or g_Γ, since the worst situation correspond to the critial cases, the intervals of ϵ-regularity for subcritical cases contain those for the critical ones.

The results above are summarized in the following figures.

Insert here Figures 1 and 2

Insert here Figures 3 and 4

Remark 3.1 *Notice that for q close to 1 the intervals where f_Ω and g_Γ are $\epsilon-$regular are disjoint, see Figure 4. Hence, in this case, $h = f_\Omega + g_\Gamma$ is not ϵ-regular for any value of ϵ. Therefore it is essential that in the abstract results we may deal with a sum of functions with different $\epsilon-$regularity properties and for different values of ϵ.*

We can now give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let $1 < q < \infty$ be fixed, $N \geq 2$ (the case $N = 1$ is similar) and let f and g satisfy (1.7) with $\rho_1 \leq (N + 2q)/N$ and $\rho_2 \leq (N + q)/N$, respectively. From the lemmas above we obtain that f_Ω is an ϵ_1-regular map and g_Γ is an ϵ_2-regular map for some $\epsilon_1, \epsilon_2 > 0$.

In order to check condition (2.5) in Theorem 2.2, we proceed as follows. As noted above the intervals of ϵ-regularity of subcritical cases are larger than those for the critical ones, so it is enough to check (2.5) for some ϵ_1 and ϵ_2 in the intervals given by Lemma 3.1 and Lemma 3.3 and for $\rho_1 = \frac{N+2q}{N}$ and $\rho_2 = \frac{N+q}{N}$.

Now, observe that if $q \leq \frac{N}{N-2}$, taking ϵ_1 and ϵ_2 as large as possible in these intervals makes $\underline{\gamma}$ to be close to $\frac{q+1}{2q}$ while $\bar{\epsilon}$ is close to $\frac{N}{N+q}$ and (2.5) is verified.

On the other hand, if $q > \frac{N}{N-2}$, since $\rho_1 > \rho_2$, the interval for f contains that for g and we can choose $\epsilon_1 = \epsilon_2 = \bar{\epsilon}$ close to $\frac{q+1}{\rho_2 2q}$ which makes $\underline{\gamma}$ to be close to $\frac{q+1}{2q}$ and again (2.5) is verified. Note also that in any case, $\underline{\gamma}$ can be choosen larger than $\frac{1}{2}$.

Consequently, Theorem 2.2 gives the existence of an $\bar{\epsilon}-$regular solution for $\bar{\epsilon} = \max\{\epsilon_1, \epsilon_2\}$. In particular, the solution satisfies (1.1) in the sense of (3.2). Also, Theorem 2.2 gives $u \in C((0,\tau], X_q^{1+\underline{\gamma}})$, with $\underline{\gamma} = \min\{\gamma_1(\epsilon_1), \gamma_2(\epsilon_2)\}$ and also $u_t \in C((0,\tau], X_q^{1+\underline{\gamma}^-})$, for any $\underline{\gamma}^- < \underline{\gamma}$.

Since, as observed above, ϵ_1 and ϵ_2 can be choosen so that $\underline{\gamma} > \frac{1}{2}$, then $X_q^{1+\underline{\gamma}^-} \hookrightarrow H_q^1(\Omega)$. Therefore, the solution found satisfies $u, u_t \in C((0,\tau], H_q^1(\Omega))$, that is, $u \in C^1((0,\tau], H_q^1(\Omega))$.

If $q > N$, we can use the embedding $H_q^1(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ for some $\alpha > 0$. Therefore $u \in C^1((0,\tau], C^\alpha(\bar{\Omega}))$. Moreover, viewing equation (1.1) for $0 < t \leq \tau$ as an elliptic equation $-\Delta u = f(u) - u_t$, in Ω, $\partial u/\partial n = g(u)$, on Γ, and usign standard elliptic regularity theory we get that $u(\cdot, t) \in C^{2,\alpha}(\Omega)$. This shows that the solution is classical.

If $1 < q < N$ we apply a bootstrap argument. Since $H_q^1(\Omega) \hookrightarrow L^r(\Omega)$ for $r = Nq/(N-q) > q$, and since $\rho_1 \leq (N+2q)/N < (N+2r)/N$ and $\rho_2 \leq (N+q)/N < (N+r)/N$ we can apply the argument above in $L^r(\Omega)$ which now gives $u \in C^1((0,\tau], H_r^1(\Omega))$. Repeating the above, in a finite number of steps we can show that $u \in C^1((0,\tau], H_p^1(\Omega))$, for some $p > N$. This, as before, implies that the solution is classical.$\square$

3.2 Critical exponents for initial data in $W^{1,q}(\Omega)$

Now, besides (1.7), f and g are allowed in some cases to belong to the class of locally Lipschitz mapping in $I\!R$ such that for every $\eta > 0$, there exists $c_\eta > 0$ such that

$$|f(u) - f(v)| \leq c_\eta (e^{\eta |u|^{\frac{N}{N-1}}} + e^{\eta |v|^{\frac{N}{N-1}}})|u - v| \tag{3.12}$$

This growth will be allowed for the case $q = N$, in $W^{1,N}(\Omega)$, because of Trudinger's inequality, [17].

In this case, and again with the notations in (2.1), denote by $X_q^\alpha := E_q^{\alpha - \frac{1}{2}}$, $\alpha \in I\!R$ and by $A_q : X_q^1 \subset X_q^0 \to X_q^0$ the realization of Δ in $E_q^{-\frac{1}{2}}$, given by (3.7). Moreover, from (3.6), we have

$$\left.\begin{array}{ll} X_q^\alpha \hookrightarrow L^r(\Omega), \text{ for } r \leq \frac{Nq}{N+q-2\alpha q}, & \frac{1}{2} \leq \alpha < \frac{N}{2q} \\ X_q^0 = W^{-1,q}(\Omega) & \\ X_q^\alpha \hookleftarrow L^s(\Omega), \text{ for } s \geq \frac{Nq}{N+q-2\alpha q}, & -\frac{N}{2q'} < \alpha \leq \frac{1}{2} \end{array}\right\} 1 < q < \infty. \tag{3.13}$$

with continuous embeddings. Note that in this setting, when applying Theorem 2.2, below, we will obtain a solution on (1.1) in the sense of (3.1). Later on we will show that this solution is even classical. More precisely, in this section we will prove the following result.

Theorem 3.1 *If $N \geq 1$, $u_0 \in W^{1,q}(\Omega)$ and f, g are locally Lipschitz functions, then*

- *if $q > N$, or*
- *if $q = N$ and f, g satisfy (3.12), or*
- *if $1 < q < N$ and f, g satisfy (1.7), with $\rho_1 \leq \frac{N+q}{N-q}$ and $\rho_2 \leq \frac{N}{N-q}$ respectively,*

problem (1.1) has a unique $\bar{\epsilon}-$regular solution for some $\bar{\epsilon} > 0$. This solution is a classical solution.

The $\epsilon-$regularity properties of the map f_Ω are given by the following lemma that is taken from [5].

Lemma 3.5 *Assume that f is a locally Lipschitz function. The nonlinearity f_Ω can be classified as follows:*

- *If $q > N$ then f_Ω is a subcritical map relatively to (X_q^1, X_q^0). Note that, for $N = 1$ f_Ω, is always subcritical.*
- *If $N \geq 2$, $q = N$ and f satisfies (3.12). Then f_Ω is a subcritical map relatively to (X_q^1, X_q^0).*
- *If $N \geq 2$, $\frac{N}{N-1} < q < N$ and f satisfies (1.7) with exponent ρ_1*
    1. *If $\rho_1 = \frac{N+q}{N-q}$, f_Ω is an $\epsilon-$regular map relative (X_q^1, X_q^0) for $0 \leq \epsilon < \frac{1}{2\rho_1}$. Therefore f_Ω is a critical map.*
    2. *If $\rho_1 < \frac{N+q}{N-q}$, f_Ω is a subcritical map relatively to (X_q^1, X_q^0).*
- *If $N \geq 3$, $\frac{N}{N-1} = q$ and f satisfies (1.7) with exponent ρ_1 (here the case $N = 2$ falls into the case $N = q$)*

1. *If* $\rho_1 = \frac{N+q}{N-q} = \frac{N}{N-2}$, f_Ω *is an* $\epsilon-$*regular map relative* (X_q^1, X_q^0) *for* $0 < \epsilon < \frac{1}{2\rho_1}$. *Therefore* f_Ω *is a double-critical map.*
2. *If* $\rho_1 < \frac{N}{N-2}$, f_Ω *is a subcritical map relatively to* (X_q^1, X_q^0).

- *If* $N \geq 2$, $1 < q < \frac{N}{N-1}$ *and* f *satisfies (1.7) with exponent* ρ_1

1. *If* $\rho_1 = \frac{N+q}{N-q}$, f_Ω *is an* $\epsilon-$*regular map relative* (X_q^1, X_q^0) *for* $0 < \frac{N-q}{2q} - \frac{N}{2\rho_1} < \epsilon < \frac{N-q}{2q} - \frac{N}{2q\rho_1}$, *with* $\gamma(\epsilon) = \rho_1\epsilon$. *Therefore* f_Ω *is an ultra-critical map.*
2. *If* $\frac{Nq}{N-q} < \rho_1 < \frac{N+q}{N-q}$, f_Ω *is an* $\epsilon-$*regular map relative* (X_q^1, X_q^0) *for* $0 < \frac{N-q}{2q} - \frac{N}{2\rho_1} < \epsilon < \frac{N-q}{2q} - \frac{N}{2q\rho_1}$, *with* $\gamma(\epsilon) > \rho_1\epsilon$. *Therefore* f_Ω *is an ultra-subcritical map.*
3. *If* $1 < \rho_1 \leq \frac{Nq}{N-q}$, f_Ω *is a subcritical map relatively to* (X_q^1, X_q^0).

The proof runs as in Lemma 3.1 and Lemma 3.2. For example, when (1.7) is assumed with exponent ρ_1 and $1 < q < N$, we have $f_\Omega : X_q^{1+\epsilon} \to X_q^{\gamma(\epsilon)}$, for $\gamma(\epsilon) = \frac{N+q}{2q} - \frac{N-q}{2q}\rho_1 + \rho_1\epsilon$.

The $\epsilon-$regularity properties of the map g are given by the following lemma.

Lemma 3.6 *Assume that* g *is a locally Lipschitz function. The nonlinearity* g_Γ *can be classified as follows:*

- *If* $q > N$ *then* g_Γ *is a subcritical map relatively to* (X_q^1, X_q^0). *Note that, for* $N = 1$, g_Γ *is always subcritical.*
- *If* $q = N$ *and* g *satisfies (3.12); then,* g_Γ *is a subcritical map relatively to* (X_q^1, X_q^0).
- *If* $N \geq 2$, $\frac{N}{N-1} < q < N$ *and* g *satisfies (1.7) with exponent* ρ_2

1. *If* $\rho_2 = \frac{N}{N-q}$, g_Γ *is an* $\epsilon-$*regular map relative* (X_q^1, X_q^0) *for* $0 \leq \epsilon < \frac{1}{2q\rho_2}$. *Therefore* g_Γ *is a critical map.*
2. *If* $\rho_2 < \frac{N}{N-q}$, g_Γ *is a subcritical map relatively to* (X_q^1, X_q^0).

- *If* $N \geq 3$, $\frac{N}{N-1} = q$ *and* g *satisfies (1.7) with exponent* ρ_2 *(the case* $N = 2$ *falls into the case* $N = q$*)*

1. *If* $\rho_2 = \frac{N}{N-q} = \frac{N-1}{N-2}$, g_Γ *is an* $\epsilon-$*regular map relative* (X_q^1, X_q^0) *for* $0 < \epsilon < \frac{1}{2q\rho_2}$. *Therefore* g_Γ *is a double-critical map.*
2. *If* $\rho_2 < \frac{N-1}{N-2}$, g_Γ *is a subcritical map relatively to* (X_q^1, X_q^0).

- *If* $N \geq 2$, $1 < q < \frac{N}{N-1}$ *and* g *satisfies (1.7) with exponent* ρ_2

1. *If* $\rho_2 = \frac{N}{N-q}$, g_Γ *is an* $\epsilon-$*regular map relative* (X_q^1, X_q^0) *for* $0 < \frac{N-q}{2q} - \frac{N-1}{2\rho_2} < \epsilon < \frac{1}{2q\rho_2}$, *with* $\gamma(\epsilon) = \rho_2\epsilon$. *Therefore* g_Γ *is an ultra-critical map.*
2. *If* $\frac{(N-1)q}{N-q} < \rho_2 < \frac{N}{N-q}$, g_Γ *is an* $\epsilon-$*regular map relative* (X_q^1, X_q^0) *for* $0 < \frac{N-q}{2q} - \frac{N-1}{2\rho_2} < \epsilon < \frac{1}{2q\rho_2}$, *with* $\gamma(\epsilon) > \rho_2\epsilon$. *Therefore* g_Γ *is an ultra-subcritical map.*

3. If $1 < \rho_2 \le \frac{(N-1)q}{N-q}$, g_Γ is a subcritical map relatively to (X_q^1, X_q^0).

Again, the proof runs as in Lemma 3.3 and Lemma 3.4. For instance, when (1.7) is assumed with exponent ρ_2 and $1 < q < N$, we have $g_\Gamma : X_q^{1+\epsilon} \to X_q^{\gamma(\epsilon)}$, for $\gamma(\epsilon) = \frac{N}{2q} - \frac{N-q}{2q}\rho_2 + \rho_2\epsilon$.

The results above are summarized in the following figures.

Insert here Figures 5 and 6

Proof of Theorem 3.1 Again it can be seen that for some ϵ_1 and ϵ_2, (2.5) is satisfied. Applying Theorem 2.2, we get that (1.1) has a unique locally defined ϵ-regular solution, for some $\bar{\epsilon} > 0$ and $u_0 \in W^{1,q}(\Omega)$. This solution verifies (1.1) in the sense of (3.1). An entirely similar bootstrap argument as in the proof of Theorem 1.1 proves that this solution is classical.$\square$

3.3 Critical exponents for measures as initial data

In this section we want to solve problem (1.1) with an initial condition $u_0 \in (C^0(\bar{\Omega}))'$, that is, a bounded measure.

Notice first that if $s > N/q'$, then $H^s_{q'}(\Omega) \overset{d}{\hookrightarrow} C^0(\bar{\Omega})$, and therefore, $(C^0(\bar{\Omega}))' \overset{d}{\hookrightarrow} (H^s_{q'}(\Omega))' = H^{-s}_q(\Omega)$. Therefore, we will try to obtain solutions of (1.1) with initial data in the latter space.

We can have an heuristical indication of which will be the growth allowed in the nonlinear terms. If we let $q \to 1$ and $s \to 0$, in the region $s > N/q'$, we will have $H^{-s}_q(\Omega) \sim L^1(\Omega)$. Hence, if $\rho_1 < \frac{N+2}{N}$, $\rho_2 < \frac{N+1}{N}$, (which should be the critical growth for $q = 1$), we may expect that f_Ω and g_Γ are ϵ_1- and ϵ_2-regular with respect to (X_q^1, X_q^0) where $X_q^1 = H_q^{-s}(\Omega)$, for some $\epsilon_1 > 0$, $\epsilon_2 > 0$. In fact, we prove

Lemma 3.7 *If f and g satisfy (1.7) with exponents $\rho_1 < \frac{N+2}{N}$ and $\rho_2 < \frac{N+1}{N}$, then there exists $q > 1$ and $s > N/q'$ such that if $X_q^1 = H_q^{-s}(\Omega)$; then f_Ω is an ϵ_1-regular map relative to (X_q^1, X_q^0) and g_Γ is an ϵ_2-regular map relative to (X_q^1, X_q^0), for some $\epsilon_1 > 0$, $\epsilon_2 > 0$ with $\gamma_1(\epsilon_1) > \rho_1\epsilon_1$, $\gamma_2(\epsilon_2) > \rho_2\epsilon_2$ and $\min\{\gamma_i(\epsilon_i)\} > \max\{\epsilon_i\}$.*

Proof. Notice that if $1 < q < \min\{\rho_1, \rho_2\}$, $s > N/q'$ and we denote by T the trace operator, then

$$H_q^{\frac{N}{q} - \frac{N}{\rho_1}}(\Omega) \hookrightarrow L^{\rho_1}(\Omega) \xrightarrow{f_\Omega} L^1(\Omega) \hookrightarrow H_q^{-s}(\Omega)$$

$$H_q^{\frac{N}{q} - \frac{N-1}{\rho_2}}(\Omega) \xrightarrow{T} L^{\rho_2}(\Gamma) \xrightarrow{g_\Gamma} L^1(\Gamma) \hookrightarrow H_q^{-s}(\Omega)$$

which implies

$$\|f(u) - f(v)\|_{H_q^{-s}(\Omega)} \le c\|u - v\|_{H_q^{\frac{N}{q} - \frac{N}{\rho_1}}(\Omega)} (1 + \|u\|^{\rho_1 - 1}_{H_q^{\frac{N}{q} - \frac{N}{\rho_1}}(\Omega)} + \|v\|^{\rho_1 - 1}_{H_q^{\frac{N}{q} - \frac{N-1}{\rho_1}}(\Omega)}) \tag{3.14}$$

$$\|g_\Gamma(u) - g_\Gamma(v)\|_{H_q^{-s}(\Omega)} \le c\|u - v\|_{H_q^{\frac{N}{q} - \frac{N-1}{\rho_2}}(\Omega)} (1 + \|u\|^{\rho_2 - 1}_{H_q^{\frac{N}{q} - \frac{N-1}{\rho_2}}(\Omega)} + \|v\|^{\rho_2 - 1}_{H_q^{\frac{N}{q} - \frac{N-1}{\rho_2}}(\Omega)}) \tag{3.15}$$

From the assumption on ρ_1 and ρ_2 we can take $\delta > 0$ small enough so that

$$\epsilon_1 = \frac{1}{2}(N - \frac{N}{\rho_1}) + \delta < \rho_1\epsilon_1 = \frac{1}{2}(N(\rho_1 - 1) + \delta\rho_1) < 1 - 2\delta,$$

$$\epsilon_2 = \frac{1}{2}(N - \frac{N-1}{\rho_2}) + \delta < \rho_2\epsilon_2 = \frac{1}{2}(N(\rho_2 - 1) + 1 + \delta\rho_2) < 1 - 2\delta.$$

Let $s_0 = N/q' + \delta$, and $X_q^1 = H_q^{-s_0}(\Omega)$. Then $H_q^{\frac{N}{q}-\frac{N}{\rho_1}}(\Omega) = X_q^{1+\epsilon_1}$, $H_q^{\frac{N}{q}-\frac{N-1}{\rho_2}}(\Omega) = X_q^{1+\epsilon_2}$ and $H_q^{-s_0-2\delta}(\Omega) = X_q^{1-\delta}$. Hence, $f_\Omega : X_q^{1+\epsilon_1} \to X_q^{\gamma_1(\epsilon_1)}$, $g_\Gamma : X_q^{1+\epsilon_2} \to X_q^{\gamma_2(\epsilon_2)}$ with $\gamma_1(\epsilon_1) = \gamma_2(\epsilon_2) = 1 - \delta > \max\{\rho_1\epsilon_1, \rho_2\epsilon_2\}$ and from (3.14), (3.15) we obtain (2.3) for f_Ω and g_Γ. This proves that f_Ω is an ϵ_1−regular map and that g_Γ is an ϵ_2−regular map.□

Again by Theorem 2.2 and a bootstrap argument, we get

Theorem 3.2 *If f and g satisfy (1.7) with $\rho_1 < \frac{N+2}{N}$ and $\rho_2 < \frac{N+1}{N}$, and if $X_q^1 = H_q^{-s}(\Omega)$ with q and s as in Lemma 3.7, then problem (1.1) with $u_0 \in X_q^1$ has a unique $\bar{\epsilon}$-regular solution, for some $\bar{\epsilon} > 0$. Furthermore, this solution is a classical solution.*

We also have

Proposition 3.1 *With the assumption above, if $u_0 \in (C(\bar{\Omega}))'$ then $u_0 \in X_q^1$ and the solution $u(t, u_0)$ in Theorem 3.2 satisfies*

$$\langle u(t, u_0), \phi\rangle = \int_\Omega u(t, x)\phi(x)\, dx \overset{t\to 0}{\longrightarrow} \langle u_0, \phi\rangle, \quad \forall \phi \in C(\bar{\Omega})$$

Proof. Consider δ as in the proof of Lemma 3.7. Since we have a lot of freedom choosing q for Lemma 3.7, we choose it with the condition that $N/q' < \delta$. Then $u_0 \in X_q^1$ and since $u(\cdot)$ is an $\bar{\epsilon}$−regular solution, we have $u(t) = e^{A\, t}u_0 + \int_0^t e^{A(t-s)}h(u(s))\, ds$.

From this, observe that

$$\| \int_0^t e^{A(t-s)}h(u(s))\, ds\|_{L^q(\Omega)} \leq \int_0^t \|e^{A(t-s)}h(u(s))\|_{L^q(\Omega)}\, ds$$

Notice that, if $2l = N/q' + \delta$, then $X_q^{1+l} = L^q(\Omega)$ and then using $h = f_\Omega + g_\Gamma$, the ϵ_1 and ϵ_2 regularity of each term and the estimate for the linear semigroup given in Theorem 2.1, with $\alpha = \gamma_i(\epsilon_i)$ and $\beta = 1 + l$, we can bound the expression above by

$$M \sum_{i=1}^{2} \int_0^t (t-s)^{-1+\gamma_i(\epsilon_i)-l} c(1 + \|u(s)\|^{\rho_i}_{X^{1+\epsilon_i}})\, ds$$

Now multiplying and dividing by $s^{\epsilon_i\rho_i}$ inside each integral, taking the sup and rescaling the remaining integrals, we get

$$\sum_{i=1}^{2} (t^{\gamma_i(\epsilon_i)-l} + t^{\gamma_i(\epsilon_i)-\rho_i\epsilon_i-l} \sup_{0\leq s\leq t} (s^{\epsilon_i}\|u(s)\|_{X^{1+\epsilon_i}})^{\rho_i}) \overset{t\to 0}{\longrightarrow} 0.$$

since, as u is an $\bar{\epsilon}$-regular solution, the sup above are finite and the exponents for t are positive by construction of ϵ_i, $\gamma_i(\epsilon_i)$, ρ_i and l.

With this,

$$\lim_{t\to 0}\langle u(t), \phi\rangle = \lim_{t\to 0}\langle e^{A\, t}u_0, \phi\rangle = \lim_{t\to 0}\langle u_0, e^{A\, t}\phi\rangle = \langle u_0, \phi\rangle$$

and we get the result.□

Finally, we can obtain the following

Theorem 3.3 *If $u_0 \in (C(\bar{\Omega}))'$ then there exist a unique classical solution to (1.1) which satisfies*

$$\int_\Omega u(t,x)\phi(x)\,dx \stackrel{t\to 0}{\longrightarrow} \langle u_0, \phi\rangle, \quad \forall \phi \in C(\bar{\Omega}) \tag{3.16}$$

Proof. The existence is guaranteed by Theorem 3.2 and Proposition 3.1.

For the uniqueness observe that by the uniform boundedness principle it follows that any classical solution $v(t,x)$ satisfying (3.16) is bounded, as $t \to 0$ in $(C(\bar{\Omega}))'$. Since the inclusion $(C(\bar{\Omega}))' \hookrightarrow X_q^1$, with X_q^1 as in Theorem 3.2, is compact, we obtain that that $v(t) \stackrel{t\to 0}{\longrightarrow} u_0$ in X_q^1.

Moreover, it is clear that if $v(t)$ is a classical solution then $v \in C((0,\tau], X_q^{1+\bar{\epsilon}})$ and then it is an $\bar{\epsilon}$-regular solution. Then, the uniqueness follows from the uniqueness of Theorem 3.2.□

Remark 3.2 *Similar ideas can be used to obtain existence and uniqueness of classical solutions to (1.1) when the initial data is even more singular. Actually with the same techniques as above, it can be shown that if $1 < \rho_1 < \frac{N+2}{N+\nu}$, $1 < \rho_2 < \frac{N+1}{N+\nu}$, $j \geq 0$, and $u_0 \in (C^\nu(\bar{\Omega}))'$ then problem (1.1) has a unique classical solution.*

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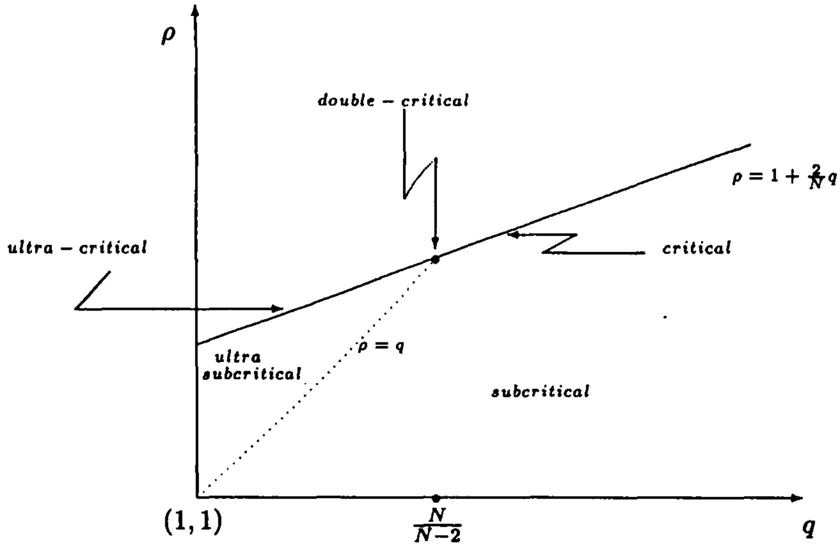

Figure 1: Criticallity of f_Ω in $L^q(\Omega)$ for $N \geq 3$

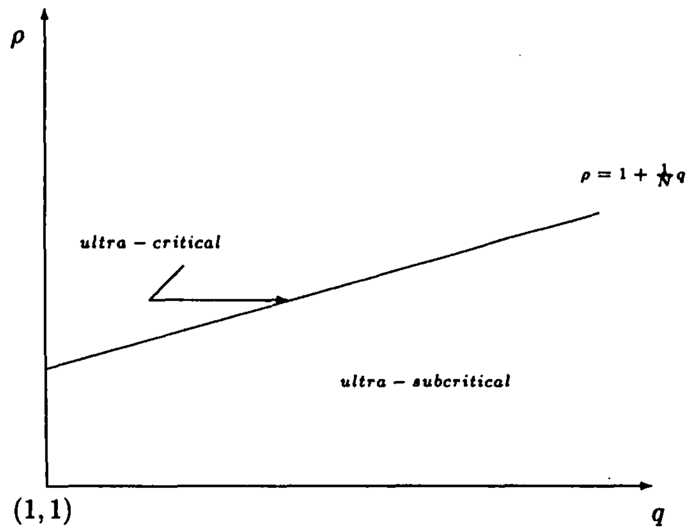

Figure 2: Criticallity of g_Γ in $L^q(\Omega)$ for $N \geq 3$

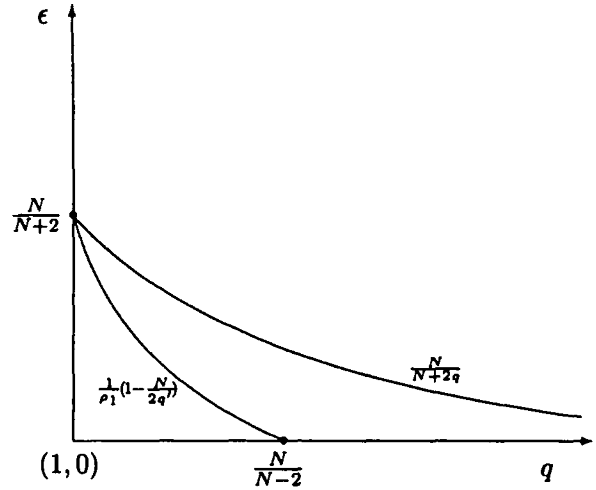

Figure 3: Intervals of ϵ−regularity of f_Ω.

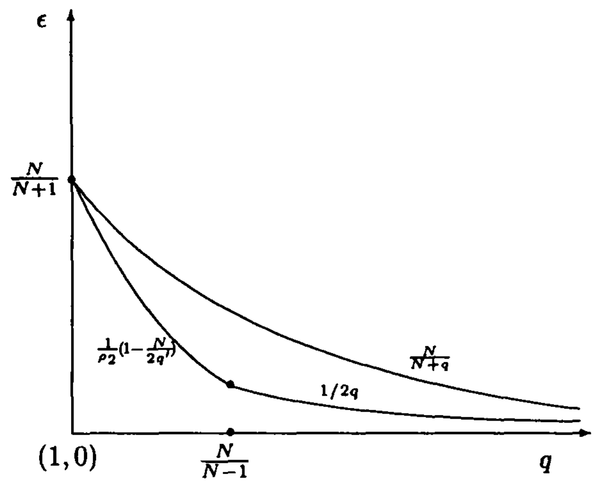

Figure 4: Intervals of ϵ−regularity of g_Γ.

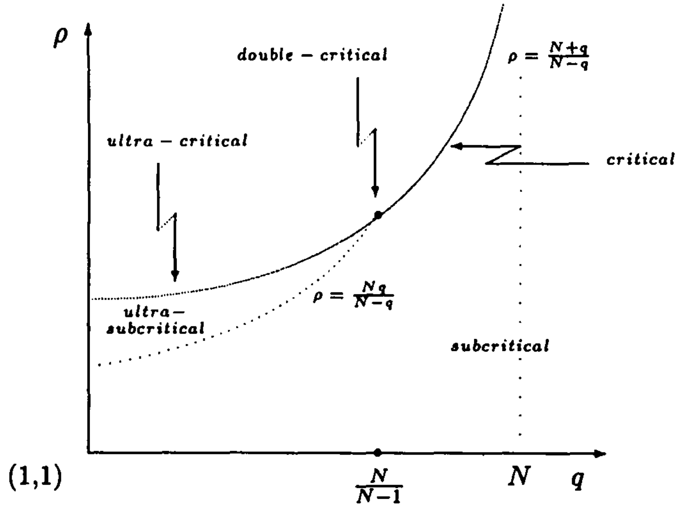

Figure 5: Criticallity of f_Ω in $W^{1,q}(\Omega)$ for $N \geq 3$

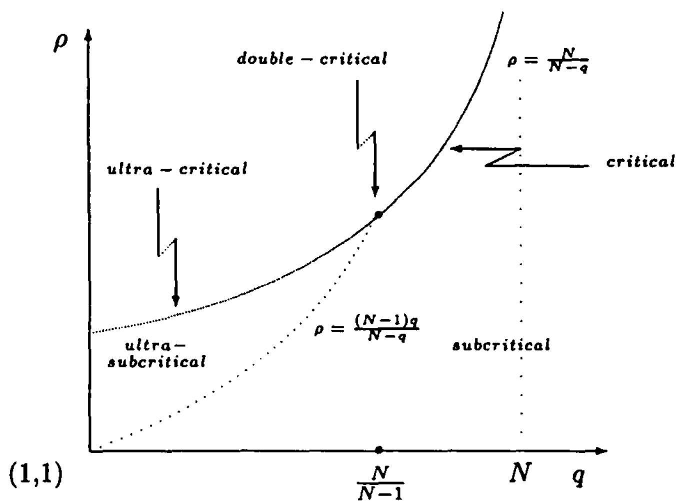

Figure 6: Criticallity of g_Γ in $W^{1,q}(\Omega)$ for $N \geq 3$

NOTAS DO ICMSC

SÉRIE MATEMÁTICA

058/97 BIASI, C.; GONÇALVES, D.L.; LIBARDI, A.K.M. - Metastable immersion with the normal bordism approach.

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