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ATTRACTORS OF PARABOLIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS. UNIFORM BOUNDS.

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Resumo

Damos condições suficientes para as quais pode-se obter existência de atratores para problemas parabólicos semilineares de segunda ordem com condições de fronteira não linear. Nós também obtemos estimativas para estes atratores em L^∞ e em C^η. Estes limitantes são uniformes relativamente ao coeficiente de difusão e ao domínio. Também incluímos um exemplo onde uma família a um parâmetro de atratores é semicontínua superiormente como uma consequência simples das estimativas uniformes obtidas.

Attractors of Parabolic Problems with Nonlinear Boundary Conditions. Uniform Bounds.

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Abstract

We give conditions under which we obtain the existence of attractors for parabolic problems with nonlinear boundary conditions. We also obtain bounds for this attractor in L^∞ and C^η. These bounds are uniform relatively to the diffusion coefficient and to the domain. We also include an example where a one parameter family of attractors is uppersemicontinuous as a simple consequence of the uniform bounds obtained.

1 Introduction

In this paper we are concerned with the asymptotic behavior of semilinear parabolic problems with nonlinear boundary conditions. For these type of problems our aim is to determine sufficient conditions under which the correspondig initial value problem is globally well posed in a suitable function space and, in particular, to find conditions under which the dynamical system has a compact global attractor.

In the case of linear boundary coditions, these problems have been extensively studied, see [11, 12, 13, 17, 19, 30] and references therein. For the case of nonlinear boundary conditions, some results in these direction can be found in [14], for example.

Moreover, we are also interested in obtaining bounds for the attractors which are uniform relatively to some parameters of the equation. Of particular interest in this direction is the case where the diffusion coefficient and the spatial domains are used as varying parameters since they appear as natural perturbations of singular type. Obtaning uniform (with respect to the parameters) bounds on attractors can be regarded as a preliminary step towards proving continuity (with respect to the parameters) properties for the asymptotic dynamics.

To be more precise, let $\Omega \subset I\!R^N$ be an open bounded domain with smooth boundary Γ. Assume $\Gamma = \Gamma_0 \cup \Gamma_1$ is a regular partition of the boundary, that is $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Note that either Γ_0 or Γ_1 could be empty. Let a, b, c be $C^1(\Omega)$ functions such that $a(x) \geq m > 0$ for $x \in \Omega$. Consider the parabolic problem

$$\begin{aligned} & u_t - \operatorname{div}(a(x)\nabla u) + c(x)u = f(x,u), \quad \text{in} \quad \Omega \\ & u = 0, \quad \text{in} \quad \Gamma_0, \\ & a(x)\frac{\partial u}{\partial n} + b(x)u = g(x,u), \quad \text{in} \quad \Gamma_1 \\ & u(0) = u_0, \end{aligned} \tag{1.1}$$

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where $f(x,\cdot), g(x,\cdot) : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functions uniformly in $x \in \bar{\Omega}$.

The well possedness of (1.1) has been considered in [7], where the functional setting developed in [4, 5] is used. The results there obtained imply that, under suitable growth assumptions on the nonlinear terms, (1.1) can be solved, locally in time, with initial data in $L^q(\Omega)$, in $W^{1,q}(\Omega)$ or even in spaces of measures; see condition $(G)_X$ in the next Section. The solutions obtained are classical solutions for positive times. We observe that the growth conditions considered in [7] include the so called *critical growth* cases.

Under some sign conditions on the nonlinearities, these solutions are shown to be globally defined in time; see condition (S) in the next Section. These sign conditions combined with the smoothing effect of the equation, will be used to obtain estimates on the solution in the sup norm.

We also show that, under suitable dissipativeness assumptions on the nonlinear terms, (1.1) possesses a global attractor that captures the asymptotic dynamics of the system; see condition (D) in the next Section. This dissipativeness condition, expressed in terms of a suitable eigenvalue problem, reflects some competition balance between both nonlinear terms f and g and was used, although in a different way, in [14]. The key argument in obtaining the existence of the attractor, will be showing a strong absorbing property for the dynamical system in $L^\infty(\Omega)$. Moreover, pointwise estimates of each function in the attractor will also be obtained. These attractors are shown to attract in stronger norms than that of the phase space and are in fact shown to be independent of the choosen phase space.

After obtaining these results, we will consider the situation in which the diffusion coefficient depends on a parameter, that is $a(x) = a_\epsilon(x)$. The final goal in this direction would be comparing the asymptotic dynamics of systems with different "parameter values" by comparing their attractors and the flow on them. In this work we will treat an example in which the diffussion coefficient satisfies a bound of the type: $m \le a_\epsilon(x) \le M$ (that includes cases related to homogeneization theory). In the continuation of this paper, see [8], we will treat a case where the diffussion is singularly perturbed in the sense that $a_\epsilon(x) \to \infty$ in a region $\Omega_1 \subset \Omega$ (see [26] for this type of perturbation).

It is clear that in order to treat both situations we will need first to obtain uniform bounds on the family of attractors. These bounds will be in L^∞ and Hölder norms. The main difficulty to obtain the uniformity on these bounds comes from the fact that the paramenter we have choosen to vary, the diffusion coefficient, enters into the metric properties of the function spaces in which the problems are set, and therefore, they affect the embeddings constants and semigroup estimates in a nontrivial way. Thus, instead of obtaining the L^∞ and Hölder uniform estimates through Sobolev type embeddings we develop in the two Appendices at the end of the paper the main tools that we are using to treat this problems: In Appendix A, we develop in an abstract framework comparison and monotonicity techniques (that actually can be applied to problems beyond the ones in this paper) and in Appendix B, we obtain an elliptic lemma that will give us L^∞ and Hölder uniform estimates for solutions of elliptic equations.

Throughout this paper we have paid special attention on how all the estimates obtained depend on the domain Ω. In certain instances (depending on the boundary conditions) the L^∞ bound on the attractors that we obtain does not depend on the shape of Ω but rather on $|\Omega|$. This again is the first step to obtain certain continuity properties of the attractors when the domain is perturbed. This problem will be treated in a future work.

Most of the analysis and estimates described above can be carried out assuming only that a, b, c lie in L^∞ and therefore they lack regularity. Also, with C^1 coefficients a, b, c, the results obtained here hold for (1.1) with initial data in spaces of measures.

This paper is organized as follows. Section 2 contains some notations and technical details on the functional setting used for (1.1) and state some of the main results of the paper. For simplicity of presentation, these results may have been written in a less general form than they are actually proven in forthcoming sections. In Section 3 we show how the sign condition on the nonlinearities gives global solutions and how the dissipativeness condition implies the existence and attractivity properties of the attractors. In Section 4 we derive the uniform bounds for the attractors and the velocity of trajectories lying on them and in Section 5 we give an example of upper semicontinuity of attractors. In Section 6, we discuss the case of measures as initial datas and the case of nonsmooth coefficients a, b, c, indicating how the previous arguments must be adapted. Finally, the first Appendix is devoted to obtain the abstract comparison and monotonicity arguments that can be used even in the case of critially growing nonlinearities, while in the second one we include some result on $L^\infty(\Omega)$ and Hölder estimates of solutions of elliptic PDEs.

2 Notation and Main Results

Concerning the functional setting, we consider (1.1) in the spaces $L^q(\Omega)$ or $W^{1,q}_{\Gamma_0}(\Omega)$, for $1 < q < \infty$, where $W^{1,q}_{\Gamma_0}(\Omega) = \{u \in W^{1,q}(\Omega) : \ u = 0 \text{ in } \Gamma_0\}$. Therefore, in order to simplify the notations, let us define the family of spaces

$$\mathcal{E} = \{L^q(\Omega),\ W^{1,q}_{\Gamma_0}(\Omega),\ 1 < q < \infty\}.$$

Then we consider (1.1) as a semilinear problem written in the abstract form as

$$\dot{u} = Au + h(u)$$

where the nonlinearity is given by $h := f_\Omega + g_\Gamma$, that is,

$$\langle h(u), \phi\rangle = \int_\Omega f(x,u(x))\phi(x) + \int_{\Gamma_1} g(x,u(x))\phi(x),$$

for all suitably regular test functions.

The unbounded linear operator $A : D(A) \subset L^q(\Omega) \to L^q(\Omega)$ with $D(A) = E^1_q := W^{2,q}_{\Gamma_0,\mathcal{N}}(\Omega) := \{u \in W^{2,q}(\Omega),\ u = 0 \text{ in } \Gamma_0,\ a(x)\frac{\partial u}{\partial n} + b(x)u = 0 \text{ in } \Gamma_1\}$ is given by $Au = -div(a\nabla u) + cu$. Notice that by taking $[\cdot,\cdot]_\theta$ the complex interpolation functor of exponent θ, A has an associated scale of Banach spaces, [4, Thm 5.2 and Section 7], [7], that for $0 \le \alpha \le 1$ verify $E^\alpha_q := [E^1_q, E^0_q]_\alpha$. For $\alpha > \beta$ the inclusion $E^\alpha_q \subset E^\beta_q$ is compact. In particular, $E^{\frac{1}{2}}_q = W^{1,q}_{\Gamma_0}(\Omega)$ and

$$E^\alpha_q := [W^{2,q}_{\Gamma_0,\mathcal{N}}(\Omega), L^q(\Omega)]_\alpha \hookrightarrow H^{2\alpha}_q(\Omega), \quad 0 \le \alpha \le 1, \tag{2.1}$$

with continuous embeddings, where $H^\alpha_q(\Omega)$ are the Bessel potential spaces.

This scale can also be extended to spaces of negative exponents by taking $E^{-\alpha}_q = (E^\alpha_{q'})'$. Also, the operator A can be extended to act on these spaces in such a way that its realization, denoted $A_{\alpha-1}$, verifies $A_{\alpha-1} : E^\alpha_q \to E^{\alpha-1}_q$ is sectorial, for $0 \le \alpha \le 1$. In particular, $-A_{\alpha-1}$ generates an anlytic semigroup $e^{-A_{\alpha-1}t}$ in $E^{\alpha-1}_q$. When refering to these semigroups, and abusing of the notations, we will often drop the subscript in the operator unless some confusion may arise.

We refer the reader to [4] for further details and to [7] for applications to nonlinear boundary conditions. Notice that with this notation, the family $\mathcal{E}$ can be expressed as $\mathcal{E} = \{E^\alpha_q,\ \alpha = 0, 1/2,\ 1 < q < \infty\}$.

Concerning the nonlinear terms f and g, it was shown in [7] that associated to any X in the class $\mathcal{E}$ there exist suitable growth restrictions on the nonlinearities f and g, that we will call $(G)_X$, such that problem (1.1) is locally well posed in X. These growth restrictions are expressed as follows:

(G)$_X$: *Let $f(x,\cdot), g(x,\cdot) : I\!R \to I\!R$ be locally Lipschitz functions, uniformly on $x \in \Omega$ and $x \in \Gamma$ respectively. Assume the following,*

1. *If $X = L^q(\Omega)$, assume that f and g satisfy a relation of the form*

$$|j(x,u) - j(x,v)| \le c|u - v|(|u|^{\rho-1} + |v|^{\rho-1} + 1), \tag{2.2}$$

with exponents ρ_1 and ρ_2 respectively, such that, with $N \ge 2$ (respectively $N = 1$)

$$\rho_1 \le \rho_f := 1 + \frac{2q}{N}, \quad \text{and} \quad \rho_2 \le \rho_g := 1 + \frac{q}{N}, \ (\text{respectively, } \rho_2 < \rho_g := 1 + q),$$

2. If $X = W^{1,q}_{\Gamma_0}(\Omega)$, assume either

i) $q > N$,

ii) $q = N$ and f, g satisfy that for every $\eta > 0$, there exists $c_\eta > 0$ such that

$$|j(x,u) - j(x,v)| \leq c_\eta (e^{\eta |u|^{\frac{N}{N-1}}} + e^{\eta |v|^{\frac{N}{N-1}}})|u - v| \tag{2.3}$$

iii) $1 < q < N$ and f, g satisfy (2.2), with exponents ρ_1 and ρ_2, respectively, such that

$$\rho_1 \leq \rho_f := 1 + \frac{2q}{N-q} \quad \text{and} \quad \rho_2 \leq \rho_g := 1 + \frac{q}{N-q}.$$

The results from [7] can be summarized as follows,

Theorem 2.1 *If X is any space in the class $\mathcal{E}$ and f, g satisfy the growth restriction $(G)_X$, then for any $u_0 \in X$ there exists locally a unique (in certain sense) mild solution $u(\cdot, u_0) \in C([0,\tau), X)$, of problem (1.1) satisfying $u(0,u_0) = u_0$ in X. This solution depends continuously on the initial data $u_0 \in X$ and it is a classical solution for $t > 0$. Also, the following regularizing effect takes place: if $u_0 \in X$ then $u(t,u_0) \in Y$ for any other space Y in the class $\mathcal{E}$, and $t \in (0,\tau)$.*

In order to obtain that all solutions of (1.1) are globally defined, we will assume some sign conditions on the nonlinear terms. These sign conditions are independent of the space X and can be expressed in the form:

(S): *Assume there exist B_0, $C_0 \in \mathbb{R}$ and $B_1, C_1 \geq 0$ such that the following holds: for $u \in \mathbb{R}$ and $x \in \bar{\Omega}$*

$$\begin{aligned} u\, f(x,u) &\leq -C_0 u^2 + C_1 |u|, \\ u\, g(x,u) &\leq -B_0 u^2 + B_1 |u|. \end{aligned} \tag{2.4}$$

Then we have the following result on global existence:

Theorem 2.2 *Let X be an space in the class $\mathcal{E}$. Assume that the growth condition $(G)_X$ and the sign condition (S) hold. Then, for any $u_0 \in X$ the solution $u(t,u_0)$ of (1.1) starting at u_0 exists for all $t \geq 0$. Therefore, we can define in X the semigroup $\{S(t) : \ t \geq 0\}$ associated to (1.1) by $S(t)u_0 = u(t,u_0)$, $t \geq 0$. Moreover, from the regularizing effect of Theorem 2.1, $S(t)u_0 \in Y$ for any other space Y in the class $\mathcal{E}$ and for any $t > 0$.*

To prove the existence of a global attractor for the problem (1.1) we will impose, besides (S), some dissipativeness condition for the vector field in (1.1). This condition is expressed as:

(D) *Assume (S) holds and that with C_0 and B_0 from (S), the first eigenvalue, λ_1, of the following problem is positive*

$$\begin{aligned} &-div(a(x)\nabla u) + (c(x) + C_0)u = \lambda u, \quad \text{in } \Omega \\ &u = 0, \quad \text{in } \Gamma_0 \\ &a(x)\frac{\partial u}{\partial n} + (b(x) + B_0)u = 0, \quad \text{in } \Gamma_1 \end{aligned} \tag{2.5}$$

With all these, it is possible to show the following result

Theorem 2.3 *Let X be an space in the class $\mathcal{E}$. Assume that the growth condition $(G)_X$ and the dissipativeness condition (D) hold. Then the semigroup $\{S(t),\ t \geq 0\}$ associated to (1.1) has a global attractor, $\mathcal{A}_X$, in X. Moreover $\mathcal{A}_X \subset Y$ for any other space Y in the class $\mathcal{E}$ and it lies in a compact set of Y.*

With respect to the dependence of the attractor $\mathcal{A}_X$ on the phase space X, it is important to observe that if X and Y are spaces in the class $\mathcal{E}$, with $Y \hookrightarrow X$ (this is usually acomplished through a Sobolev type embedding) and if $(G)_X$ holds then $(G)_Y$ also holds. That is, the better the space the weaker the growth restriction on the nonlinearities. With this fact in mind and the regularizing effect of the semigroup $S(t)$ we will prove

Theorem 2.4 *Let X be an space in the class $\mathcal{E}$. Assume that the growth condition $(G)_X$ and the dissipative condition (D) hold. Then, for any space Y in the class $\mathcal{E}$ with $Y \hookrightarrow X$ we have the existence of the attractor $\mathcal{A}_Y$. Moreover $\mathcal{A}_X = \mathcal{A}_Y$ and it attracts bounded sets of X in the topology of Y.*

As mentioned in the Introduction, we will also obtain bounds on the attractors which are uniform with respect to variations on the diffusion coefficient a and on the domain Ω. In this direction, a very important first result is a pointwise bound on the attractor $\mathcal{A}_X$ given by the following result

Proposition 2.5 *Let X be an space in the class $\mathcal{E}$. Assume that the growth restriction $(G)_X$ and the dissipative condition (D) hold. Denote by ϕ the solution of*

$$\begin{aligned}
&-div(a(x)\nabla\phi) + (c(x) + C_0)\phi = C_1 \quad in\ \Omega, \\
&u = 0 \quad in \quad \Gamma_0, \\
&a(x)\frac{\partial\phi}{\partial n} + (b(x) + B_0)\phi = B_1 \quad in \quad \Gamma_1.
\end{aligned}$$

Then, $0 \le \phi \in L^\infty(\Omega)$, $\lim_{t\to\infty} |u(t,x,u_0)| \le \phi(x)$, uniformly in $x \in \bar{\Omega}$ and for u_0 in bounded subsets of X. In particular, we have that for any $v \in \mathcal{A}_X$ we have $|v(x)| \le \phi(x)$.

In fact we will prove

Theorem 2.6 *Let X be an space in the class $\mathcal{E}$. Assume that the growth restriction $(G)_X$ and the dissipative condition (D) hold. Then,*

i) $\sup_{v\in\mathcal{A}_X}\{\|v\|_{L^\infty(\Omega)}\} \le K_0$, *where K_0 depends on the domain Ω, on the first eigenvalue of (2.5), λ_1, on $m = \inf\{a(x) : x \in \Omega\} > 0$, on $\sup\{c(x);\ x \in \Omega\}$, $\sup\{b(x);\ x \in \Gamma\}$ and on C_0, C_1, B_0 and B_1.*
Furthermore, if one of the following two conditions hold:

- *there exists a $b_0 > 0$ with $b(x) + B_0 \ge b_0$*
- *$b(x) + B_0 \ge 0$, $B_1 \le 0$ and there exists a $c_0 > 0$ such that $c(x) + C_0 \ge c_0 > 0$,*

then K_0 will depend on $|\Omega|$ rather than on Ω.

ii) $\sup_{v\in\mathcal{A}_X}\{\|v\|_{C^\nu(\bar{\Omega})}\} \le K_\nu$, *where K_ν depends on the same as K_0 plus on ν, on $M = \sup\{a(x);\ x \in \Omega\}$, on $\sup\{\|f'(\psi)\|_{L^\infty(\Omega)} : \psi \in \mathcal{A}_X\}$, and on $\sup\{\|g'(\psi)\|_{L^\infty(\Gamma_1)} : \psi \in \mathcal{A}_X\}$.*

3 Existence of attractors and attractivity properties

In this Section we prove that under assumption (S) problem (1.1) has global solutions, moreover if (D) is assumed then (1.1) has a global attractor that attracts the solutions in strong norms. Moreover we will obtain bounds on the size of the attractor in $L^\infty(\Omega)$ and in $C^\eta(\bar{\Omega})$.

We start with the following lemma

Lemma 3.1 *Let A be the elliptic operator from the previous section and denote by λ_1 its first eigenvalue, which we assume to be positive. Let $q \in (1,\infty)$. There exists a constant $M = M(A, \lambda_1, N, q) \ge 1$ such that for every $\alpha \ge \beta$ we have*

$$\|e^{-At}u_0\|_{E_q^\alpha} \le Me^{-\lambda_1 t}t^{-(\alpha-\beta)}\|u_0\|_{E_q^\beta}, \qquad t > 0, \quad u_0 \in E_q^\beta. \tag{3.1}$$

In particular, there exists a $\delta_0 = \delta_0(q, N) > 0$ *and a constant* $\tilde{M} = \tilde{M}(A, \lambda_1, N, q)$ *such that*

$$\|e^{-At}u_0\|_{L^\infty(\Omega)} \le \tilde{M}t^{-\delta_0}e^{-(\lambda_1-\epsilon)t}\|u_0\|_{L^q(\Omega)}, \qquad t > 0, \quad u_0 \in L^q(\Omega). \tag{3.2}$$

Moreover if $q = 2$ *the constant* M *is independent of everything.*

Proof: The first inequality, including the result for the case $q = 2$ are well known, [4, 19, 25]. Inequality (3.2) comes from an elementary bootstrap argument using (3.1), embeddings (2.1) and Sobolev embeddings $H^\alpha_q \hookrightarrow L^\infty$ for $\alpha > N/q$.$\square$

Let us consider now,

Proposition 3.2 *Let* X *be an space in the class* $\mathcal{E}$, *that is* $X = L^q(\Omega)$ *or* $X = W^{1,q}_{\Gamma_0}(\Omega)$, *and assume* f, g *satisfy* $(G)_X$. *We have,*
i) *If (S) holds, then the solutions of (1.1) are globally defined.*
ii) *If (D) holds, then there exist contants* K_∞ *and* K^α_r, *for* $2\alpha < 1 + \frac{1}{r}$, $r \ge q$, *such that*

$$\limsup_{t\to\infty} \|u(t, u_0)\|_{L^\infty(\Omega)} \le K_\infty, \tag{3.3}$$

$$\limsup_{t\to\infty} \|u(t, u_0)\|_{E^\alpha_r} \le K^\alpha_r, \tag{3.4}$$

and the above limits are uniform relatively to u_0 *in bounded subsets of* X.
iii) *Even more, if* (D) *holds and if* ϕ *is the solution of*

$$\begin{aligned} &-div(a(x)\nabla\phi) + (c(x) + C_0)\phi = C_1 \quad in\ \Omega, \\ &u = 0 \quad on\ \Gamma_0, \\ &a(x)\frac{\partial\phi}{\partial n} + (b(x) + B_0)\phi = B_1 \quad on\ \Gamma_1, \end{aligned} \tag{3.5}$$

then $0 \le \phi \in L^\infty(\Omega)$, $\lim_{t\to\infty} |u(t, x, u_0)| \le \phi(x)$, *uniformly in* $x \in \bar{\Omega}$ *and for* u_0 *in bounded subsets of* X. *Indeed, if* $|u_0(x)| \le \phi(x)$ *for all* $x \in \bar{\Omega}$, *then* $|u(t, x, u_0)| \le \phi(x)$ *for all* $x \in \bar{\Omega}$ *and* $t \ge 0$.

Proof: Denote by B a bounded set in X, and assume that $u_0 \in B$.
i) Let $w^+(t, |u_0|)$ be the solution of

$$\begin{aligned} &w_t - div(a(x)\nabla w) + c(x)w = -C_0w + C_1 \\ &w = 0 \quad on\ \Gamma_0 \\ &a(x)\frac{\partial w}{\partial n} + b(x)w = -B_0w + B_1 \\ &w(0, |u_0|) = |u_0|. \end{aligned} \tag{3.6}$$

Then, since $C_1, B_1 \ge 0$, from Corollary A.14 we have that $w^+(t, |u_0|) \ge 0$. Since $f(x, s) \le -C_0s + C_1$ and $g(x, s) \le -B_0s + B_1$ for all $s \ge 0$ and $x \in \bar{\Omega}$ then, by Corollary A.14 we get $u(t, u_0) \le w^+(t, |u_0|)$, for as long as $u(t, u_0)$ exists. Note that alternatively, we can use that $w^+(t, |u_0|)$ is also a supersolution to (1.1) and then use Theorem A.12 to get the same result. Proceeding similarly we obtain that $u(t, u_0) \ge w^-(t, -|u_0|) := -w^+(t, |u_0|)$ for as long as $u(t, u_0)$ exists. Consequently

$$|u(t, u_0)| \le w^+(t, |u_0|), \tag{3.7}$$

for as long as $u(t, u_0)$ exists.

Let $\phi \in L^\infty(\Omega)$ be the solution of (3.5) and define now $v = w^+ - \phi$, which verifies a linear homogeneous equation. Therefore by Lemma 3.1 we get, in a finite time interval,

$$\|v(t)\|_{L^\infty(\Omega)} \le M_1t^{-\delta_0}\||u_0| - \phi\|_{L^q(\Omega)}$$

Consequently

$$\|w^+(t)\|_{L^\infty(\Omega)} \le \|v(t)\|_{L^\infty(\Omega)} + \|\phi\|_{L^\infty(\Omega)}$$

and therefore the $L^\infty(\Omega)$ norm of $u(t)$ remains bounded in finite time intervals.

Now, using the variations of constants formula for the solution, see [7], for an arbitrary $t_0 > 0$ and $t > t_0 > 0$ we have that

$$u(t, u_0) = e^{-A(t-t_0)}u(t_0, u_0) + \int_{t_0}^{t} e^{-A(t-s)}h(u(s, u_0))ds$$

where h is defined by $\langle h(u), \psi\rangle = \int_\Omega f(u)\psi + \int_{\Gamma_1} g(u)\psi$. Note that if $u \in L^\infty(\Omega)$ and $\psi \in E^r_{q'}$, with $2r > 1/q'$, so that ψ has trace on Γ_1, then we get that $h(u) \in E^{-r}_q$ and $\|h(u)\|_{E^{-r}_q} \leq C(\|u\|_{L^\infty(\Omega)})$. Thus, we have that

$$\|u(t, u_0)\|_{E^\alpha_q} \leq M(t - t_0)^{-\alpha}\|u(t_0)\|_{L^q(\Omega)} + MC\int_{t_0}^{t}(t - s)^{-(\alpha+r)}ds$$

It follows that the E^α_q norm of $u(t)$ remains bounded in finite time intervals provided $\alpha+r < 1$. Since $2r > 1/q'$ is arbitrary, we get the bound for any $2\alpha < 1 + \frac{1}{q}$. Therefore, the solution is global.

ii) When the dissipativeness assumption (D) holds true then, in (3.5) the elliptic operator is positive and then since $C_1, B_1 \geq 0$ we get that $\phi \geq 0$. Now, by Lemma 3.1, we have that $v = w^+ - \phi$ verifies for all $t > 0$

$$\|v(t)\|_{L^\infty(\Omega)} \leq \tilde{M}e^{-t\lambda_1}t^{-\delta_0}\||u_0| - \phi\|_{L^q(\Omega)}$$

Consequently

$$\|w^+(t)\|_{L^\infty(\Omega)} \leq \|v(t)\|_{L^\infty(\Omega)} + \|\phi\|_{L^\infty(\Omega)} \to \|\phi\|_{L^\infty(\Omega)}$$

as $t \to \infty$, uniformly for $u_0 \in B$ and we get (3.3).

For the proof of (3.4) we proceed as follows. As above, we use the variations of constants formula, for $t \geq t_0 > 0$ where now t_0 is large enough such that $\|u(s, u_0)\|_{L^\infty(\Omega)} \leq K_\infty + \eta$ for $s \geq t_0$. Thus, we have, for $2p > 1/q'$

$$\|u(t, u_0)\|_{E^\alpha_q} \leq Me^{-\lambda_1(t-t_0)}(t - t_0)^{-\alpha}\|u(t_0, u_0)\|_{L^\infty(\Omega)} + MC(K_\infty)\int_{t_0}^{t} e^{-\lambda_1(t-s)}(t - s)^{-(\alpha+p)}ds$$

and it follows that $\limsup_{t\to\infty} \|u(t, u_0)\|_{E^\alpha_q} \leq M_0C(K_\infty)\int_0^\infty e^{-\frac{\lambda_1}{2}\theta}\theta^{-(\alpha+p)}d\theta$.

Using that E^α_q is embedded in some $L^s(\Omega)$ for some $s = s(q, \alpha) > q$, the fact that now f and g are subcritical in $L^s(\Omega)$ and repeating the argument above, we get that for any $r \in [q, s]$ and $2\alpha < 1 + 1/r$ there exists a constant K^α_r such that (3.4) holds. A bootstrap argument shows the validity of (3.4) for any $r \geq q$ and $2\alpha < 1 + 1/r$.

iii) Finally, using (3.7) and $w^+ = v + \phi$ and since v goes to zero uniformly in $x \in \bar{\Omega}$ and for $u_0 \in B$, we get

$$\limsup_{t\to\infty} |u(t, x)| \leq \phi(x), \quad \text{for all } x \in \Omega.$$

On the other hand, if $|u_0| \leq \phi$, then $v(0) \leq 0$ on Ω and then $v(x, t) \leq 0$. So, we get that $|u(t, x)| \leq \phi(x)$, for all $x \in \Omega$ for all positive times.□

Remark 3.3 *This proposition, in particular, proves Theorem 2.2. and part of Proposition 2.5.*

Note the optimality of the estimates in the spaces E^α_r for $r \geq q$ and $2\alpha < 1 + \frac{1}{r}$, since for $2\alpha > 1 + \frac{1}{r}$ the space E^α_r incorporates the boundary condition $a(x)\frac{\partial u}{\partial n} + b(x)u = 0$ on Γ_1.

Therefore, we get the following.

Theorem 3.4 *Let X be an space in the class $\mathcal{E}$, that is $X = L^q(\Omega)$ or $X = W^{1,q}_{\Gamma_0}(\Omega)$, assume f, g satisfy the growth restriction $G)_X$ and that the dissipation condition (D) is also satisfied.*

Then, problem (1.1) has a global compact attractor, $\mathcal{A}_X$, in X such that

Finally, note that for the case $q = 2$, the Bessel potential spaces E_2^α that we have considered before coincide with the standard fractional power spaces of the operator A.

For the C^η uniform bounds, we will need to obtain first some L^∞ bounds of $u_t(t, u_0)$ for any u_0 in the attractor and any t. Then, viewing equation (1.1) as an elliptic equation of the form $-\mathrm{div}(a(x)\nabla u) + c(x)u = f(x,u) - u_t$ in Ω with boundary conditions given by (1.1), and applying Lemma B.1 vi), we obtain the C^η-bounds which will depend on $\inf\{a(x)\}$ and $\sup\{a(x)\}$.

Before going to the proof of Theorem 2.6 we need to consider some important lemmas. We start with some uniform estimates on the norm of some inclusions for the spaces E_2^α.

Lemma 4.1 *Consider the operator A as an unbounded operator in $L^2(\Omega)$ and let E_2^α the scale of spaces associated to this operator.*

Then, the norm of the following inclusion depend on $m = \inf\{a(x)\}$, Ω, $\|b\|_{L^\infty(\Gamma_1)}$, $\|c\|_{L^\infty(\Omega)}$:
i) $E_2^\alpha \hookrightarrow H^{2\alpha}(\Omega)$, *for $\alpha \in [0, 1/2]$.*
ii) $L^\infty(\Gamma) \hookrightarrow E_2^{-\beta}$ *for every $\beta \in (1/4, 1/2)$.*
iii) $E_2^\alpha \hookrightarrow E_2^\beta$, *for any $\alpha \geq \beta$.*

Proof: i) Let $\lambda = \lambda(\Omega, m, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}) > 0$ be a large enough number so that there exists a constant $\mu = \mu(\Omega, m, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)})$ with

$$\mu\|u\|^2_{H^1_{\Gamma_0}(\Omega)} \leq \int_\Omega a(x)|\nabla u|^2 + \int_\Omega (c(x) + \lambda)u^2 + \int_{\Gamma_1} b(x)u^2 \tag{4.1}$$

This implies that i) holds for $\alpha = 1/2$. Since it is also trivially true for $\alpha = 0$, we have by interpolation that it is true for any $\alpha \in [0, 1/2]$.
ii) Using i), $g \in L^\infty(\Gamma)$ and $\phi \in E_2^\beta$, for $\beta > 1/4$, we have

$$|\int_\Gamma g\phi| \leq \|g\|_{L^\infty(\Gamma)}\|\phi\|_{L^1(\Gamma)} \leq C\|g\|_{L^\infty(\Gamma)}\|\phi\|_{H^{2\beta}(\Omega)} \leq C\|g\|_{L^\infty(\Gamma)}\|\phi\|_{E_2^\beta}$$

iii) Using the spectral characterization of the fractional power spaces one gets that the norm of the inclusion $E_2^\alpha \subset E_2^\beta$, for any $\alpha \geq \beta$, is given by ${\lambda_1}^{\beta-\alpha}$ where λ_1 is the first eigenvalue of the elliptic operator A, which is assumed to be positive. But from (4.1) we have uniform lower estimates on λ_1 and we get the result.$\square$

As a consequence of Lemma 4.1 and Lemma B.1 we have

Lemma 4.2 *There exists a $\delta_0 = \delta_0(N) > 0$ such that $E_2^{\delta_0} \hookrightarrow L^\infty(\Omega)$ and the norm of the inclusion depends on Ω, $m = \inf\{a(x)\}$, $\|b\|_{L^\infty(\Gamma_1)}$, $\|c\|_{L^\infty(\Omega)}$, δ_0 and N.*

Proof: For $N = 1$, from $E_2^{1/2} = H^1_{\Gamma_0}(\Omega) \subset L^\infty(\Omega)$, and Lemma 4.1 we conclude.

For $N = 2$ and $N = 3$ it follows from Lemma B.1 ii) and $p = 2$ that $E_2^1 = D(A) \hookrightarrow L^\infty(\Omega)$ and the norm of the inclusion depends on Ω, m, $\|b\|_{L^\infty(\Gamma_1)}$ and $\|c\|_{L^\infty(\Omega)}$.

For dimensions $N > 3$, from Lemma B.1 i) and $r(p) = \frac{pN}{N-2p}$ we get that $E_2^1 = D(A) \subset L^{r(2)}(\Omega)$ and the inclusion norm depends on Ω, m, $\|b\|_{L^\infty(\Gamma_1)}$ and $\|c\|_{L^\infty(\Omega)}$. Let $p_1 = \frac{2N}{N-4} = r(2)$, again Lemma B.1 i) implies $E_2^2 = D(A^2) \subset L^{r(p_1)}(\Omega)$ and recursively, $E_2^n = D(A^n) \subset L^{p_n}(\Omega)$, with inclusion norm depending on Ω, m, $\|b\|_{L^\infty(\Gamma_1)}$, $\|c\|_{L^\infty(\Omega)}$ and $p_n = r(p_{n-1})$. Therefore, p_n eventually becomes larger that $N/2$ and then Lemma B.1 ii) gives the result.$\square$

Remark 4.3 *Note that the function $r(p)$ has the property that $r(p) > N/n$ iff $p > N/(n+2)$ which implies that if $p > N/n$ then $r^j(p) > N/(n-2j)$. Starting with $p = 2$ gives that δ_0 can be taken equal to $1/2[N/2]$ if $[N/2]$ is even or equal to $1/2[N/2] + 1/2$ otherwise. In any case we can always choose, $\delta_0 = N$.*

Consider now the following,

Lemma 4.4 *If the first eigenvalue of the operator A is $\lambda_1 > 0$, then there exists a constant $M = M(\Omega, \lambda_1, m, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, N)$ such that*

$$\|e^{-At}u_0\|_{L^\infty(\Omega)} \leq Mt^{-N}\|u_0\|_{L^2(\Omega)}, \qquad t > 0, \quad u_0 \in L^2(\Omega) \tag{4.2}$$

Proof: From Lemma 4.2, we have the existence of a constant $C = C(\Omega, m, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)})$, such that $\|e^{-At}u_0\|_{L^\infty(\Omega)} \leq C\|e^{-At}u_0\|_{E_2^N}$. Moreover, from Lemma 3.1 for the case $q = 2$ we obtain that $\|e^{-At}u_0\|_{E_2^N} \leq Mt^{-N}\|u_0\|_{L^2(\Omega)}$. Combining both inequalities we obtain the result.$\square$

Now we can prove

Theorem 4.5 *Let X be an space in the class $\mathcal{E}$, that is $X = L^q(\Omega)$ or $X = W^{1,q}_{\Gamma_0}(\Omega)$. Assume that the growth restriction $(G)_X$ and the dissipative condition (D) hold.*
i) *Then,*

$$\sup_{v \in \mathcal{A}_X} \|v\|_{L^\infty(\Omega)} \leq K_0$$

where K_0 depends on the domain Ω, on the first eigenvalue of (2.5), λ_1, on $m = \inf\{a(x) : x \in \Omega\} > 0$, on $\sup\{c(x);\ x \in \Omega\}$, $\sup\{b(x);\ x \in \Gamma\}$ and on C_0, C_1, B_0 and B_1.
Furthermore, if one of the following two conditions hold:

- *there exists a $b_0 > 0$ with $b(x) + B_0 \geq b_0$*
- *$b(x) + B_0 \geq 0$, $B_1 \leq 0$ and there exists a $c_0 > 0$ such that $c(x) + C_0 \geq c_0 > 0$,*

then K_0 will depend on $|\Omega|$ rather than on Ω.
ii) *For every $r \geq q$ and $2\alpha < 1 + \frac{1}{r}$ we have that*

$$\sup_{v \in \mathcal{A}_X} \|v\|_{E_r^\alpha} \leq K$$

for some constant $K = K(A, \alpha, r, f, g)$.
If $r = 2$ and $\alpha = 1/2$ or $\alpha = 0$, then $K = K(\Omega, m, \lambda_1, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, f, g)$.

Proof:
i) Notice that from Lemma B.1 ii), we have that

$$\|\phi\|_{L^\infty(\Omega)} \leq C(|B_1| + |C_1| + \|\phi\|_{L^2(\Omega)}),$$

for some constant $C = C(\Omega, m, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, B_0, C_0)$. Moreover, since, from condition (D), the first eigenvalue of A is positive, it is not difficult to show that we must have that $\|\phi\|_{L^2(\Omega)} \leq \tilde{C}(|B_1| + |C_1|)$ for some constant $\tilde{C} = \tilde{C}(\Omega, m, \lambda_1)$.

For the second part, we apply Lemma B.1 iii) or iv) to the function ϕ to obtain the desired result.
ii) The proofs runs as in the proof of point ii) in Proposition 3.2. When $r = 2$ and $\alpha = 1/2$ or $\alpha = 0$ the result comes from the lemmas above.$\square$

The next result, that complements the estimates above, give bounds for the flow of the semigroup on the attractor by estimating the time derivative of solutions lying on it. That is, we estimate the speed of trajectories on the attractor.

Theorem 4.6 *Let X be an space in the class $\mathcal{E}$, that is $X = L^q(\Omega)$ or $X = W^{1,q}_{\Gamma_0}(\Omega)$. Assume f, g satisfy the growth restriction $(G)_X$ and that the dissipation condition (D) is also satisfied. Denote by $u(t, u_0)$ the solution to (1.1) starting at $u_0 \in \mathcal{A}_X$.*
i) *Then for every $r \geq q$ and $2\alpha < 1 + \frac{1}{r}$ we have that*

$$\sup_{u_0 \in \mathcal{A}_X} \sup_{t \in \mathbb{R}} \|u_t(t, u_0)\|_{E_r^\alpha} \leq K$$

for some constant $K = K(A, \alpha, r, f, g)$. In particular, for any $\nu < 1$, we obtain $C^\nu(\bar{\Omega})$ bounds of $u_t(t, u_0)$, uniformly in $t \in I\!R$ and $u_0 \in \mathcal{A}_X$.

If $r = 2$ and $\alpha = 1/2$ or $\alpha = 0$, then $K = K(\Omega, m, \lambda_1, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, f, g)$.

ii) *We also have*

$$\sup_{u_0 \in \mathcal{A}_X} \sup_{t \in I\!R} \|u_t(t, u_0)\|_{L^\infty(\Omega)} \leq K$$

for some constant $K = K(\Omega, \lambda_1, m, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, A_0, D_0)$

Proof:

i) Let $r \geq q$ and $2\alpha < 1 + \frac{1}{r}$. As in Proposition 3.2, note that if $u, v \in L^\infty(\Omega)$ and $2\beta > 1/r'$, so that the trace is well defined in $E^\beta_{r'}$, we get the existence of a constant l_0, such that $\|h(u) - h(v)\|_{E_r^{-\beta}} \leq l_0 \|u - v\|_{L^\infty(\Omega)}$ and $\|h(u) - h(v)\|_{E_r^{-\beta}} \leq l_0 \|u - v\|_{E_r^\gamma}$, as soon as $2\gamma > 1/r$. Therefore for any β such that $1/r' < 2\beta < 1 + \frac{1}{r'}$, we can choose γ as above and such that $\beta + \gamma < 1$. Thus, for these values of β and γ, we get that $h : E_r^\gamma \to E_r^{-\beta}$ is globally Lipschitz. In what follows we follow closely the proof of Theorem 3.5.2 in [19].

Denote $j(s) = h(u(s, u_0))$ and $s \in [t_0, t_1] = [t - 1, t]$ for $t \in I\!R$. Then, for any β in the range above, we get, using the bounds on the attractor, for some $\delta > \gamma$

$$\|j(s+h) - j(s)\|_{E_r^{-\beta}} \leq h l_1 \left((s - t_0)^{-1+\delta-\gamma} + (s - t_0)^{-(\gamma+\beta)} \right) + l_2 \int_{t_0}^{s} (s - z)^{-(1/2+\beta)} \|j(z+h) - j(z)\|_{E_r^{-\beta}} \, dz$$

Using the singular Gronwall lemma in [19, page 6], we get $\|j(s+h) - j(s)\|_{E_r^{-\beta}} \leq h l_3 K(s)$ with $K(s) = \left((s - t_0)^{-1+\delta-\gamma} + (s - t_0)^{-(\gamma+\beta)} \right)$. Therefore, Lemma 3.5.1 in [19], gives for any $\alpha < 1 - \beta$

$$\|u_t(s)\|_{E_r^\alpha} \leq l_4 (s - t_0)^{-(\alpha+\beta)} \|j(s)\|_{E_r^{-\beta}} + l_5 \int_{t_0}^{s} (s - z)^{-(\alpha+\beta)} K(z) \, dz + l_6 (s - t_0)^{-(1+\alpha)} \|u(t_0)\|_{L^q(\Omega)}$$

Now using that $\|j(s)\|_{E_r^{-\beta}}$ and $\|u(t_0)\|_{L^q(\Omega)}$ are uniformly bounded and taking $s = t_1 = t$, we get

$$\|u_t(t)\|_{E_r^\alpha} \leq l_7 \left(1 + \int_{t_0}^{t} (t - z)^{-(\alpha+\beta)} K(z) \, dz \right)$$

which is finite and uniformly bounded in t, since $\delta > \gamma$, $\beta + \gamma < 1$, and $\alpha < 1 - \beta$. Since β is arbitrary verifying $1/r' < 2\beta < 1 + \frac{1}{r'}$ we get the bound for any $2\alpha < 1 + \frac{1}{r}$.

The constants $l_0 - l_7$ involved in this proof depend only on A, r, α, f and g. Moreover, for $r = 2$, $\alpha = 1/2$ or $\alpha = 0$, using Lemma 3.1 it can be seen that actually these constants depend on Ω, m, λ_1, $\|b\|_{L^\infty(\Gamma_1)}$, $\|c\|_{L^\infty(\Omega)}$, f, g.

Finally, the $C^\nu(\bar{\Omega})$ bound, for any $\nu < 1$ follows now from embeddings (2.1) and the Sobolev type embeddings for Bessel potential spaces.

ii) Note that $v = u_t(t)$ verifies

$$\begin{cases} v_t - \mathrm{div}(a(x)\nabla v) + c(x)v = f'(u(t))v, & \text{in } \Omega \\ v = 0, \quad \text{in } \Gamma_0 \\ a(x)\dfrac{\partial v}{\partial \vec{n}} + b(x)v = g'(u(t))v, & \text{in } \Gamma_1 \\ v(0) = u_t(0) \end{cases}$$

Let $A_0 = \sup\{\|f'(\psi)\|_{L^\infty(\Omega)};\ \psi \in \mathcal{A}\}$ and $D_0 = \sup\{\|g'(\psi)\|_{L^\infty(\Gamma_1)};\ \psi \in \mathcal{A}\}$. Let w be the solution of

$$\begin{cases} w_t - \mathrm{div}(a(x)\nabla w) + (c(x) - A_0)w = 0, & \text{in } \Omega \\ w = 0, \quad \text{in } \Gamma_0 \\ a(x)\dfrac{\partial w}{\partial \vec{n}} + (b(x) - D_0)w = 0, & \text{in } \Gamma_1 \\ w(0) = |u_t(0)| \end{cases}$$

Then, using Corollary A.14, we get $w(t,x) \geq 0$ and $|v(t,x)| \leq w(t,x)$. Now Lemma 4.4 gives for $t = 1$

$$\|u_t(1)\|_{L^\infty(\Omega)} \leq M \|u_t(0)\|_{L^2(\Omega)}$$

for some constant $M = M(\Omega, \lambda_1, m, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, A_0, D_0)$. Using the uniform L^2-bound for u_t given in point i) and the invariance of the attractor, we get the result.□

Now we are in a position to give a

Proof of Theorem 2.6.

The first part follows from Theorem 4.5, while for the second, as we mentioned at the introduction of this section, looking at equation (1.1) for t fixed as

$$-\mathrm{div}(a(x)\nabla u) + c(x)u = f(x,u) - u_t$$

and applying Lemma B.1 v) we obtain the Hölder uniform bounds.□

5 An example of Upper Semicontinuity of Attractors

In this section we consider an application of the results obtained in the previous sections to a parabolic problem for which the diffusion coefficient is not precisely determined up to small $L^1(\Omega)$ perturbations. Let Ω be as before, $a_\epsilon(x)$ be a family of C^1 functions in Ω, such that $0 < m \leq a_\epsilon(x) \leq M$, a. e. $x \in \Omega$, $0 \leq \epsilon \leq \epsilon_0$. Consider the problems, for $0 \leq \epsilon \leq \epsilon_0$

$$\begin{aligned} &u_t - \mathrm{div}(a_\epsilon(x)\nabla u) = f(u), \quad \text{in } \Omega \\ &u = 0, \quad \text{in } \Gamma_0, \\ &a_\epsilon(x)\frac{\partial u}{\partial n} = g(u), \quad \text{in } \Gamma_1 \\ &u(0) = u_0 \end{aligned} \tag{5.1}$$

where f and g are locally Lipschitz functions verifying the growth assumptions $(G)_X$ for $X = H^1_{\Gamma_0}(\Omega)$. Assume also that f and g satisfy (2.4).

We are interested on studying continuity properties of the attractors of (5.1) with respect to ϵ. With this in mind we define the concept of **upper semicontinuity**.

Definition 5.1 *A family $\{\mathcal{A}_\epsilon, 0 \leq \epsilon \leq \epsilon_0\}$, $\epsilon_0 > 0$, of subsets of a given metric space X is called upper semicontinuous at $\epsilon = 0$ if given $\eta > 0$ there is an $\epsilon_\eta > 0$ such that $\mathcal{A}_\epsilon \subset \mathcal{N}_\eta(\mathcal{A}_0)$ for all $0 \leq \epsilon \leq \epsilon_\eta$, where $\mathcal{N}_\eta(\mathcal{A}_0)$ is an $\eta-$neighborhood of $\mathcal{A}_0$ in X.*

Now we recall some definition taken from the context of homogenization theory, see [10, 27, 23].

Definition 5.2 *We say that the family a_ϵ of C^1 functions $H-$converges to the C^1 function a_0, as $\epsilon \to 0$, if and only if for $0 \leq \epsilon \leq \epsilon_0$, we have $0 < m \leq a_\epsilon(x) \leq M$, a. e. $x \in \Omega$ and for every $f \in L^2(\Omega)$ the solutions of*

$$\begin{aligned} &-div(a_\epsilon(x)\nabla u^\epsilon) + u^\epsilon = f, \quad in\ \Omega \\ &u^\epsilon = 0, \quad in\ \Gamma_0, \\ &a_\epsilon(x)\frac{\partial u^\epsilon}{\partial n} + u^\epsilon = 0, \quad in\ \Gamma_1 \end{aligned}$$

converge in $H^1_{\Gamma_0}(\Omega)$, as $\epsilon \to 0$, to the solution of the problem above with $\epsilon = 0$.

Observe that a suffcient condition for this is that

$$a_\epsilon \to a_0, \quad \text{strongly in } L^1(\Omega),$$

and that, as example, we can set $\Omega = B(0,1) \subset I\!R^N$ and $a_\epsilon(x) = 1 + |x|^{1/\epsilon}$ and $a_0 = 1$.

Consider the eigenvalue problem

$$(EVP)_\epsilon \qquad \begin{cases} -\mathrm{div}(a_\epsilon(x)\nabla\phi) + C_0\phi = \mu\phi, & \text{in } \Omega \\ \phi = 0, & \text{in } \Gamma_0, \\ a_\epsilon(x)\dfrac{\partial\phi}{\partial n} + B_0\phi = 0, & \text{in } \Gamma_1. \end{cases} \tag{5.2}$$

Assume that the first eigenvalue of $(EVP)_0$ is positive. Then from the results in [20, Theorems VIII.1.14 and VIII.1.15], it follows that the first eigenvalue of $(EVP)_\epsilon$ converges to the first eigenvalue of $(EVP)_0$.

With this it follows from the results in the previous sections that if $\epsilon_0 > 0$ is sufficiently small, the problems (5.1) have global attractors $\mathcal{A}_\epsilon$, $0 \le \epsilon \le \epsilon_0$. Also from the results in the previous sections we obtain that

$$\|\nabla u^\epsilon\|_{L^2(\Omega)},\ \|\nabla u_t^\epsilon\|_{L^2(\Omega)},\ \|u^\epsilon\|_{C^\gamma(\bar\Omega)},\ \|u_t^\epsilon\|_{C(\bar\Omega)}$$

for $\gamma < 1$ are uniformly bounded for $0 < \epsilon < \epsilon_0$ and for trajectories lying on the attractors $\mathcal{A}_\epsilon$.

Now we argue that for fixed t, $u^\epsilon(t)$ is relatively compact in $C(\bar\Omega) \cap H^1_{\Gamma_0}(\Omega)$. In the case of $C(\bar\Omega)$, this is an easy consequence of the fact that the attractors are bounded in $C^\gamma(\bar\Omega)$ uniformly with respect to ϵ. The $H^1_{\Gamma_0}(\Omega)$ case is a little more elaborated. From the uniform bound in $H^1(\Omega)$ we have the compactness of $\{u^\epsilon(t)\}_{\epsilon>0}$ in $H^s(\Omega)$ for any $s < 1$. Now, for fixed t, we read the equation for $u^\epsilon(t)$ as

$$\begin{cases} -\mathrm{Div}(a_\epsilon\nabla u^\epsilon) = -u_t^\epsilon + f(u^\epsilon) = F^\epsilon, & \text{in } \Omega \\ a_\epsilon(x)\dfrac{\partial u^\epsilon}{\partial \vec n} = -g(u^\epsilon) = G^\epsilon, & \text{in } \Gamma_1, \end{cases}$$

By compactness, we have $u^\epsilon(t) \to u^0$ and $u_t^\epsilon(t) \to v^0$ in $H^s(\Omega)$ for any $s < 1$. Since we can assume that after truncation f and g are globally Lipschitz, then $f(u^\epsilon(t)) \to f(u^0)$ in $L^2(\Omega)$ and $g(u^\epsilon(t)) \to g(u^0)$ in $L^2(\Gamma_1)$ which in turn gives $F^\epsilon \to F^0$ in $L^2(\Omega)$ and $G^\epsilon \to G^0$ in $L^2(\Gamma_1)$.

From the results in [26], we know that $u^\epsilon(t) \to u^0$ weakly in $H^1(\Omega)$ and the convergence is strong provided $< F^\epsilon, u^\epsilon >_\Omega + < G^\epsilon, u^\epsilon >_{\Gamma_1} \to < F^0, u^0 >_\Omega + < G^0, u^0 >_{\Gamma_1}$ which is true in this case. Moreover, $u^0 \in H^1_{\Omega_0}(\Omega)$.

Hence, by Arzelá-Ascoli Theorem (taking subsequences if necessary), for complete trajectories lying on the attractors $u^\epsilon(\cdot) \in \mathcal{A}_\epsilon$, $0 < \epsilon < \epsilon_0$ we have that

$$u^\epsilon \to u \quad \text{locally uniformly in } C(I\!R, C(\bar\Omega)) \cap C(I\!R, H^1_{\Gamma_0}(\Omega))$$

Taking the limit in the week formulation of (5.1)

$$\int_\Omega u_t^\epsilon\phi + \int_\Omega a_\epsilon(x)\nabla u^\epsilon\nabla\phi = \int_\Omega f(u^\epsilon)\phi + \int_\Gamma g(u^\epsilon)\phi$$

for all smooth bounded test functions ϕ, we obtain that u is a globally defined solution to (5.1) for $\epsilon = 0$, which is bounded in $H^1_{\Gamma_0}(\Omega) \cap C(\bar\Omega)$. Hence $u(t) \in \mathcal{A}_0$ for all $t \in I\!R$.

From this we have the following result

Theorem 5.3 *The family of attractors $\{\mathcal{A}_\epsilon : \ 0 \le \epsilon \le \epsilon_0\}$ is upper semicontinuous at $\epsilon = 0$ in $H^1_{\Gamma_0}(\Omega)$ and in $C(\bar\Omega)$.*

The proof of this theorem now follows in a standard way (see, for example, [18]).

6 Final remarks

6.1 Measures as initial data

Due to the strong smoothing effect and the strong absorbing property that the dissipativeness condition (D) imposes on (1.1), most of the results in previous sections hold when we consider measures as initial data. Following [7], if f and g satisfy (2.2), with exponents ρ_1 and ρ_2 with

$$\rho_1 < \rho_f := 1 + \frac{2}{N} \quad \text{and} \quad \rho_2 < \rho_g := 1 + \frac{1}{N}$$

and if $u_0 \in (C(\bar{\Omega}))'$ then (1.1) has a unique classical solution that verifies, as $t \to 0$

$$\int_\Omega u(t,x)\phi(x)\,dx \to \int_\Omega \phi\,du_0, \quad \text{for all } \phi \in C(\bar{\Omega}) \tag{6.1}$$

and which depends continuously on the initial data.

All the proofs above go through with minor modifications that we now indicate. For example for the proof of Proposition 3.2 just note that when $u_0 \in (C(\bar{\Omega}))'$ then there exist $q > 1$ and $s > N/q'$ such that $u_0 \in H_q^{-s}(\Omega)$, see [7]. As in the proof of Proposition 3.2, where now $|u_0|$ is the total variation of u_0, using $v = w^+ - \phi$, and again by Lemma 3.1 we get

$$\|v(t)\|_{L^\infty(\Omega)} \leq \frac{M}{t^{\delta_0+s/2}}\||u_0| - \phi\|_{H_q^{-s}(\Omega)} \leq \frac{M}{t^{\delta_0+s/2}}\||u_0| - \phi\|_{(C(\bar{\Omega}))'}$$

and the rest follows as before.

Analogoulsy, we can derive, as in Theorem 3.4, the existence of an attractor in $(C(\bar{\Omega}))'$, which moreover coincides with the attractor in $L^q(\Omega)$ for every $q > 1$.

6.2 Nonsmooth coefficients

When the coefficients a, b, c are just in L^∞, there is not a satisfactory theory for the operator A, as defined in Section 2, in $L^q(\Omega)$ for $q \neq 2$. The main difference with respect to the smooth case is that the scale of interpolation spaces E_q^α is not well understood for $q \neq 2$. In particular we cannot use embeddings (2.1) for $q \neq 2$. Nevertheless, for $q = 2$, the operator A, defined by $Au = -div(a(x)\nabla u) + c(x)u$ with domain $D(A) = \{u \in H^1_{\Gamma_0}(\Omega),\ -Div(a(x)\nabla u) + c(x)u \in L^2(\Omega),\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = 0 \text{ in } \Gamma_1\}$ is selfadjoint and has compact resolvent. With this, it is well known that A generates an analytic semigroup, e^{-At}, in $L^2(\Omega)$. Moreover, it can be seen that $E_2^{\frac{1}{2}} = H^1_{\Gamma_0}(\Omega)$ and therefore, by interpolation and duality the scale of spaces is well understood for $-1/2 \leq \alpha \leq 1/2$. In particular $E_2^\alpha \hookrightarrow H^{2\alpha}$ and $H^{-2\alpha} \hookrightarrow E_2^{-\alpha}$ for any $0 \leq \alpha \leq 1/2$. The fact that only a part of the scale E_2^α is known, imposse restrictions on the allowed growth of the nonlinearities to obtain a locally well posed problem. Following the ideas of [6, 7] we can see that if $X = L^2$ then the growth restrictions for f and g are exactly the same as the ones given in condition $(G)_X$. If $X = H^1_{\Gamma_0}(\Omega)$ the growth restrictions for f and g are the ones given by $(G)_X$ although with strict inequalities for the exponents (that is, subcritical nonlinearities).

One of the main tools of the paper are the monotonicity and comparison results from the Appendix in Section A. We see that with the following result, the operator A with nonsmooth coefficients is still resolvent positive and therefore the theory from the appendix also apply to this case.

Theorem 6.1 *There exists some λ_0 such that for every $\lambda > \lambda_0$, $f \in L^2(\Omega)$ with $f \geq 0$, and if $u \in D(A)$ verifies $Au + \lambda u = f$, then $u \geq 0$. That is, the operator A is resolvent positive in $L^2(\Omega)$.*

Proof: Since for every $u \in H^1_{\Gamma_0}(\Omega)$ the bilinear form associated to the operator A, verifies $\tau(u,u) = \tau(|u|,|u|)$, the result follows from [15, Theorem 1.3.2, page 12].$\square$

Notice also that Lemma B.1 is valid for the case of nonsmooth coefficients.

With all of the above, we have that for this case we also have the existence of an attractor $\mathcal{A}_X$, ($X = L^2(\Omega)$ or $X = H^1_{\gamma_0}(\Omega)$) for problem (1.1) and that the uniform bounds given by Theorem 2.6 also apply.

A Appendix : Abstract Monotonicity and Comparison Results

In this appendix we give general results that allows us to obtain comparison, positivity and monotonicity results for evolutionary equations. Most of the results for the linear case are already known, see [5]. For the nonlinear case we will exploit the variation of constants formula to obtain these results.

Let us start with several well known concepts and results.

Definition A.1 *An ordered Banach space is a pair* $(X, \leq)$, *where* X *is a Banach space and* $\leq$ *is an ordering relation in* X *such that*

i) $x \leq y$ *implies* $x + z \leq y + z$, $x, y, z \in X$.

ii) $x \leq y$ *implies* $\lambda x \leq \lambda y$, *for* $x, y \in X$, λ *real and non negative.*

iii) *The "positive cone"* $C = \{x \in X,\ x \geq 0\}$ *is closed in* X.

Definition A.2 *Let* $(X, \leq)$, $(Y, \preceq)$ *be a couple of ordered spaces. A function* $T : X \to Y$ *is said* **increasing** *iff* $x \leq y$ *implies* $T(x) \preceq T(y)$ *and it is called* **positive** *iff* $0 \leq x$ *implies* $0 \preceq T(x)$.

Observe that if, in the above definition, T is linear then both concepts coincide.

Lemma A.3 *Let* $f \in L^1((t_0, t_1), X)$, *such that* $f(t) \geq 0$ *a.e.* $t \in (t_0, t_1)$. *Then,* $\int_{t_0}^{t_1} f(s)\,ds \geq 0$. *In other words, the mapping*

$$\int_{t_0}^{t_1} : L^1((t_0, t_1), X) \to X$$

is (linear and) increasing.

Proof: Since the integral is a linear and bounded operator between $L^1((t_0, t_1), X)$ and X and the cone C is closed, it is enough to prove the result for f in a dense subset of $L^1((t_0, t_1), X)$. If $f \in C([t_0, t_1], X)$, by the integral mean value theorem, $\int_{t_0}^{t_1} f(s)\,ds \in (t_1 - t_0)\overline{co}(f([t_0, t_1]))$, that is, the closed convex hull of the range of f. If $f(t) \geq 0$, for every t, then its range is in C and then $\overline{co}(f([t_0, t_1])) \subset C$.$\square$

Let $(X, \leq)$ be an ordered Banach space and let A be a sectorial operator. Notice that from the definition of sectorial operator, see [19, 20], we have that there exists a $\lambda_0 \in \mathbb{R}$ such that $Re(\sigma(A + \lambda_0)) > 0$. Consider the already well-known definition

Definition A.4 *A sectorial operator* A *in an ordered Banach space* X *is said to be "resolvent positive" in* X *if for every* $\lambda > \lambda_0$, $(A + \lambda)^{-1}$ *is increasing.*

The next result establishes the equivalence between the positivity of the resolvent and the exponential.

Proposition A.5 *A sectorial operator* A *is resolvent positive if and only if* e^{-At} *is also positive.*

Proof: A proof of this result can be found in [5]. Nevertheless we give the following straightford proof. Assume first that $\lambda_0 = 0$, i.e. $Re(\sigma(A)) > 0$ and for every $\lambda > 0$, $(A + \lambda)^{-1}$ is increasing. Then, for every $t > 0$ and $u_0 \in X$, we have $\left(I + \frac{t}{n}A\right)^{-n} u_0 \to e^{-At}u_0$, as $n \to \infty$. In particular, if $u_0 \geq 0$, then $(I + \frac{t}{n}A)^{-n}u_0 = \frac{t^n}{n^n}(\frac{n}{t} + A)^{-n}u_0 \geq 0$ for every $n \in \mathbb{N}$ and since the positive cone is closed, we get $e^{-At}u_0 \geq 0$.

For arbitrary λ_0, since $(A + \lambda)^{-1} = (A + \lambda_0 + (\lambda - \lambda_0))^{-1}$, taking $A_0 = A + \lambda_0$, we have $(A_0 + sI)^{-1} \geq 0$ for every $s > 0$ and then $0 \leq e^{-A_0 t} = e^{-(\lambda - \lambda_0)t}e^{-At}$.

The converse follows from $(A + \lambda)^{-1} = \int_0^\infty e^{-At}e^{-\lambda t}\,dt \geq 0$, for every $\lambda > \lambda_0$.$\square$

Remark A.6 *Observe that to say that* A *is resolvent positive is equivalent to a* positivity *result for the "elliptic" problem* $Au + \lambda u = f$: *for every* $\lambda > \lambda_0$ *and for every* $f \geq 0$, *the solution verifies* $u \geq 0$.

We also have

Corollary A.7 *Assume* A *is resolvent positive and let* $u = u(t, u_0, f)$ *be the solution of*

$$\begin{cases} u_t + Au = f(t) \\ u(t_0) = u_0 \in X \end{cases}$$

with $f \in L^1((t_0, t_1), X)$. *Then the mapping*

$$X \times L^1((t_0, t_1), X) \ni (u_0, f) \mapsto u \in C([t_0, t_1], X)$$

is increasing.

Proof: Observe that the solution is given by the variation of constants formula,

$$u_i(t) = e^{-A(t-t_0)}u_i + \int_{t_0}^{t} e^{-A(t-s)} f_i(s)\, ds$$

for $i = 0, 1$, and we know already that $e^{-A(t-t_0)}$ is increasing. On the other hand, for every $t_0 < s < t < t_1$, $e^{-A(t-s)} f_0(s) \geq e^{-A(t-s)} f_1(s)$ and since the integral is increasing, we get the result.$\square$

A.1 The nonlinear equation

Now we turn to nonlinear equations of the form

$$\begin{cases} u_t + Au = f(t,u) \\ u(t_0) = u_0 \end{cases} \tag{A.1}$$

Let $(X, \leq)$ be an ordered Banach space, and let A be a sectorial operator which is resolvent positive. Assume we have constructed as in Section 2 and as in [4, 7] a scale of interpolation spaces X^α for $\alpha \geq 0$, with $X^0 = X$ and $X^1 = D(A)$. In each of these X^α we will consider the order induced by X. Consider the following definition,

Definition A.8 *The scale of spaces X^α is an Scale of Ordered Spaces if the injections $X^\alpha \hookrightarrow X^\beta$, $\alpha \geq \beta \geq 0$, are positive, $A_\alpha : X^{\alpha+1} \to X^\alpha$, $\alpha \geq 0$, is resolvent positive and the positive cone of X^α is dense in the positive cone of X^β for any $\alpha \geq \beta \geq 0$.*

Throughout this subsection we will assume that the scale X^α is an scale of ordered spaces.

Moreover, we will assume that problem (A.1) is locally well posed in X^1, that is, for each $u_0 \in X^1$ we have the existence of a solution $u(t, u_0) \in C([0,\tau], X^1)$ for certain $\tau = \tau(u_0)$ and that the solution $u(t, U_0)$ depends continuously on the initial condition u_0.

We will also assume that the nonlinearity f is subcritical in certain space $X^{1+\epsilon}$, $\epsilon \geq 0$. That is, there exists a $\gamma > 0$ with $0 \leq 1+\epsilon-\gamma < 1$ such that $f : [t_0, t_1] \times X^{1+\epsilon} \to X^\gamma$ is locally Hölder in t, locally Lipschitz in u, and that if $u_0 \in X^{1+\epsilon}$, then the solution $u(t, u_0)$ of (A.1) is given by a fixed point of the variations of constants formula

$$\mathcal{F}(u)(t) = e^{-A(t-t_0)}u_0 + \int_{t_0}^{t} e^{-A(t-s)} f(s, u(s))\, ds \tag{A.2}$$

in the space $V = \{u \in C([t_0, t_0+\tau], X^{1+\epsilon});\ \|u(t)\|_{X^1} \leq \|u_0\|_{X^{1+\epsilon}} + \delta\}$, for certain τ and δ; see [6, 7].

Concerning positivity, we have

Theorem A.9 *Under the assumption above, assume that for every $r > 0$ there exists a constant $\beta = \beta(r) > 0$ such that for every $t \in [t_0, t_1]$, $f(t, \cdot) + \beta I$ is positive on the positive elements in the ball in $X^{1+\epsilon}$ of radious r.*

Then, if $0 \leq u_0 \in X^1$, the solution of (A.1) $u_f(t, u_0)$ is positive while it exists.

Proof: Notice first that we just need to show the result when $u_0 \in X^{1+\epsilon}$. Once this is proved, by density arguments and using the continuous dependence on the initial condition and the fact that the positive cone of $X^{1+\epsilon}$ is dense in the positive cone of X^1 we obtain the result in X^1.

Therefore, assume $u_0 \in X^{1+\epsilon}$. Write the equation as $u_t + (A+\beta I)u = f(t,u) + \beta u$ and denote $\mathcal{F}(u) = e^{-(A+\beta I)(t-t_0)}u_0 + \int_{t_0}^{t} e^{-(A+\beta)(t-s)}(f(s,u(s)) + \beta u(s))\, ds$ which is a contraction in $V = \{u \in C([t_0, t_0 + h], X^{1+\epsilon}),\ \|u(t) - u_0\|_{X^{1+\epsilon}} \leq \delta\}$ if h and δ are sufficiently small.

If we take $\beta = \beta(\|u_0\|_{1+\epsilon} + \delta)$, we have $e^{-(A+\beta I)t} \geq 0$ and $f(t, \cdot) + \beta I \geq 0$ on the positive elements of the ball of radious $\|u_0\|_{X^{1+\epsilon}} + \delta$. In particular, if we denote by $V_+ = \{u \in V :\ u \geq 0\}$, then $\mathcal{F}(V_+) \subset V_+$, which implies that the unique fixed point of $\mathcal{F}$ on V must actually lie on V_+. This shows that $u(t) \geq 0$ for $t \in [t_0, t_0 + \tau]$. Now, with an elementary continuation argument we can actually show that $u(t) \geq 0$ as long as the solution is defined.

Concerning comparison, we have

Theorem A.10 *Let X and A as above. Assume also that the nonlinearities f, g and h satisfy the same as described above for f. Then,*

i) Assume that for every $r > 0$ there exists a constant $\beta = \beta(r) > 0$ such that for every $t \in [t_0, t_1]$, $f(t, \cdot) + \beta I$ is increasing on the ball in $X^{1+\epsilon}$ of radious r. Then, $u_0, u_1 \in X^1$ and $u_0 \geq u_1$ implies $u_f(t, u_0) \geq u_f(t, u_1)$ while they exits.

ii) Assume f, g verify $f(t, \cdot) \geq g(t, \cdot)$ for every t. Then, for every $u_0 \in X^1$, $u_f(t, u_0) \geq u_g(t, u_0)$ while they exist.

iii) Assume f, g are such that for every $r > 0$ there exists a constant $\beta = \beta(r) > 0$ and an increasing function $h(t, \cdot)$ such that for every $t \in [t_0, t_1]$, $f(t, \cdot) + \beta I \geq h(t, \cdot) \geq g(t, \cdot) + \beta I$ on the ball in $X^{1+\epsilon}$ of radious r.

Then, $u_0, u_1 \in X^1$, and $u_0 \geq u_1$ implies $u_f(t, u_0) \geq u_g(t, u_1)$ while they exist.

Proof: As above, it will be enough to show the theorem when the initial data are in $X^{1+\epsilon}$. Therefore, let us assume that $u_0, u_1 \in X^{1+\epsilon}$.

i) We know that $u_i(t) = u_f(t, u_i)$ is the unique fixed point of

$$\mathcal{F}(u) = e^{-(A+\beta)(t-t_0)}u_i + \int_{t_0}^{t} e^{-(A+\beta)(t-s)}(f(s, u(s)) + \beta u(s))\, ds$$

in $V_i = \{u \in C([t_0, t_0 + h], X^{1+\epsilon}),\ \|u(t) - u_i\|_{X^{1+\epsilon}} \leq \delta\}$, provided h and δ are small enough. If we consider the space $\hat{V}_0 = \{u \in V_0;\ u(t) \geq u_1(t)\}$, which is nonempty since $u_1(t) + u_0 - u_1$ lies in $\hat{V}_0$, we have that $\mathcal{F}(\hat{V}_0) \subset \hat{V}_0$. This implies that the unique fixed point of $\mathcal{F}$ in V_0 actually lies in $\hat{V}_0$ and therefore $u_0(t) \geq u_1(t)$ for $t \in [t_0, t_0 + \tau]$. An elementary continuation argument will show that actually this inequality is true as long as both solutions exist.

ii) The argument is very similar as the one from i). Now $u_f(t, u_0)$ and $u_g(t, u_0)$ are the unique fixed points of $\mathcal{F}_f(u) = e^{-A(t-t_0)}u_0 + \int_{t_0}^{t} e^{-A(t-s)} f(s, u(s))\, ds$ and $\mathcal{F}_g(u) = e^{-A(t-t_0)}u_0 + \int_{t_0}^{t} e^{-A(t-s)} g(s, u(s))\, ds$, respectively, in $V = \{u \in C([t_0, t_0 + h], X^{1+\epsilon}),\ \|u(t) - u_0\|_{X^{1+\epsilon}} \leq \delta\}$, provided h and δ are small enough. We may consider the space $\hat{V} = \{u \in V;\ u(t) \geq u_g(t)\}$ and $\mathcal{F}_f(\hat{V}) \subset \hat{V}$. We finish the argument as we did in i).

iii) After adding and substracting the term βu in the equation, by ii) we have $u_f(t, u_0) \geq u_h(t, u_0)$ and by i) $u_h(t, u_0) \geq u_h(t, u_1)$. Again ii) implies $u_h(t, u_1) \geq u_g(t, u_1)$.□

We consider now several results concerning sub and super solutions for equation (A.1). Let us start with the following

Definition A.11 *Let $v \in C([t_0, t_1], X^1) \cap C((t_0, t_1], X^{1+\epsilon})$. Then v is a subsolution of (A.1) if $v_t + Av \leq f(t, v)$, in the sense that for every $t_0 \leq r \leq t \leq t_1$ we have*

$$v(t) \leq e^{-A(t-r)}v(r) + \int_{r}^{t} e^{-A(t-s)} f(s, v(s))\, ds \tag{A.3}$$

Analogously, we define a supersolution if the above holds true with reversed inequalities.

Note that if $v(t)$ is a strong solution of $v_t + Av = g(t)$ and $g(t) \leq f(t, v)$ then, by using the variation of constants formula and the positivity of the semigroup and the integral, we would get that $v(t)$ verifies the inequality above.

Theorem A.12 *Assume X, A and f are as above. Assume that for every $r > 0$ there exists a constant $\beta = \beta(r) > 0$ such that for every $t \in [t_0, t_1)$, $f(t, \cdot) + \beta I$ is increasing on the ball in $X^{1+\epsilon}$ or radious r.*

Let $v(t)$ be a subsolution of (A.1). Then $v(t) \leq u_f(t, v(t_0))$ while they exist. Moreover, if u_0 is such that $v(t_0) \leq u_0$, then $v(t) \leq u_f(t, u_0)$ while they exits.

The same holds true for a supersolution, with reversed inequalities.

Proof: Let $\eta > 0$ be a small number and denote by $t_\eta = t_0 + \eta$. Let $\beta = \beta(\|v(t_\eta)\|_{X^{1+\epsilon}} + 1)$. We know that $u_f(t, t_\eta, v(t_\eta))$ is a fixed point of $\mathcal{F}(u) = e^{-(A+\beta)(t-t_\eta)}v(t_\eta) + \int_{t_\eta}^{t} e^{-(A+\beta)(t-s)}(f(s, u(s)) + \beta u(s))\, ds$ in

$V = \{u \in C([t_\eta, t_\eta + h], X^{1+\epsilon}),\ \|u(t) - v(t_\eta)\|_{X^{1+\epsilon}} \leq \delta\}$, provided h and $\delta \leq 1$ are small enough. Moreover, choosing h smaller if necessary we have that $v \in V$. Also, $\mathcal{F}$ is a strict contraction in V.

Since $f + \beta I$ is increasing in V, if we define $\hat{V} = \{u \in V :\ u(t) \geq v(t) \text{ for } t \in [t_\eta, t_\eta + h]\}$, we will have that $\mathcal{F}(\hat{V}) \subset (\hat{V})$. Since $\hat{V}$ is non empty, $v \in \hat{V}$, we get that the unique fixed point of $\mathcal{F}$ in V lies actually in $\hat{V}$ and therefore $u_f(t, t_\eta, v(t_\eta)) \geq v(t)$ for $t \in [t_\eta, t_\eta + h]$. With a continuation argument we get that the inequality is valid as long as both functions exist.

With a density argument, using the continuous dependence of the solutions of (A.1) and the fact that the positive cone of $X^{1+\epsilon}$ is dense in the positive cone of X^1, we obtain the result.$\square$

A.2 Application to nonhomogeneous boundary conditions

Here, we particularize the abstract results above for the case of parabolic equations with nonlinear boundary conditions.

We will refer to Section V.2.7 of [5] for a general theory of Scales of Ordered Banach Spaces. Applying the results of this reference to the operator A, where $Au = -\mathrm{Div}(a(x)\nabla u) + c(x)u$ in $L^q(\Omega)$, with $1 < q < \infty$, with boundary conditions $u = 0$ on Γ_0, and $a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = 0$ on Γ_1, with $a, b, c \in C^1(\overline{\Omega})$ and $\inf_{x \in \overline{\Omega}} a(x) \geq m > 0$, we have that,

Theorem A.13 *For every $\alpha \in [0,1]$, E_q^α, is an ordered space with the ordering induced by $L^q(\Omega)$. The space, $E_q^{-\alpha}$ is an ordered space with the canonical dual order, that is $u \geq 0$ in $E_q^{-\alpha}$, iff $< u, \phi > \geq 0$ for every $\phi \in E_{q'}^\alpha$.*

For $\gamma \geq \beta$, the inclusion $E_q^\gamma \to E_q^\beta$ is positive and the positive cone in the former space in dense in the positive cone of the latter.

Finally, $A_{\alpha-1} : E_q^\alpha \to E_q^{\alpha-1}$ is sectorial and positve resolvent.

This means that the scale E_q^α is an Scale of Ordered Banach Spaces and, therefore, the abstract monotonicity, comparison, sub and super solutions theory from the previous subsection applies to this case.

In particular we have the following results. Consider the problem

$$\begin{cases} u_t - Div(a(x)\nabla u) + c(x)u = f(t,x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Gamma_0 \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = g(t,x,u) & \text{in } \Gamma_1 \\ u(t_0) = u_0 & \end{cases} \tag{A.4}$$

where $f, g : I\!R \times \overline{\Omega} \times I\!R \to I\!R$ are locally Lipschitz functions in u, uniformly in $x \in \overline{\Omega}$ and for t in compact subsets of $I\!R$.

Assume that for sufficiently smooth data (A.4) is well possed and the solutions depend continuously on the initial data in a suitable topology, for example in $L^q(\Omega)$. Then applying abstract results of the previous subsection, we have

Corollary A.14
i) *Assume $f(t,x,0) \geq 0$ and $g(t,x,0) \geq 0$ for every t and $x \in \bar{\Omega}$. Then $u_0 \geq 0$ implies $u(t) \geq 0$ as long as the solution exists.*
ii) *If $u_1 \geq u_0$ then $u(t, u_1, f, g) \geq u(t, u_0, f, g)$ as long as the solution exist.*
iii) *If $f_1(t,x,u) \geq f_0(t,x,u)$, $g_1(t,x,u) \geq g_0(t,x,u)$ and $u_1 \geq u_0$ then $u(t, u_1, f_1, g_1) \geq u(t, u_0, f_0, g_0)$ as long as the solution exist.*

B Appendix: An Elliptic Lemma

In this appendix we want to give a proof of the following elliptic result.

Lemma B.1 *Let $a, c \in L^\infty(\Omega)$, $b \in L^\infty(\Gamma_1)$ and let $u \in H^1(\Omega)$ be a weak solution of the following elliptic problem*

$$\begin{cases} -div(a\nabla u) + cu = f, & in \quad \Omega \\ u = h, \quad in \quad \Gamma_0, \\ a(x)\dfrac{\partial u}{\partial \vec{n}} + bu = g, \quad in \quad \Gamma_1. \end{cases} \tag{B.1}$$

Then
i) *If $f \in L^p(\Omega)$, $g \in L^{\frac{(N-1)p}{N-p}}(\Gamma_1)$, $h \in L^\infty(\Gamma_0)$ and $\frac{2N}{N+2} \le p < \frac{N}{2}$, then $u \in L^{\frac{pN}{N-2p}}(\Omega)$ and*

$$\|u\|_{L^{\frac{pN}{N-2p}}(\Omega)} \le C(\|f\|_{L^p(\Omega)} + \|g\|_{L^{\frac{(N-1)p}{N-p}}(\Gamma_1)} + \|h\|_{L^\infty(\Gamma_0)} + \|u\|_{L^2(\Omega)})$$

for some constant $C = C(\Omega, \inf_\Omega a, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, N, p)$.
ii) *If $f \in L^p(\Omega)$, $p > N/2$, $g \in L^r(\Gamma_1)$, $r > N-1$, $h \in L^\infty(\Gamma_0)$, then $u \in L^\infty(\Omega)$ and*

$$\|u\|_{L^\infty(\Omega)} \le C\left(\|f\|_{L^p(\Omega)} + \|g\|_{L^r(\Gamma_1)} + \|h\|_{L^\infty(\Gamma_0)} + \|u\|_{L^2(\Omega)}\right)$$

for some constant $C = C(\Omega, \inf_\Omega a, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, N, p, r)$.
iii) *If $f \in L^p(\Omega)$, $p > N$, $g \in L^\infty(\Gamma_1)$, $h \in L^\infty(\Gamma_0)$, the principal eigenvalue associated to the linear problem of (B.1) is $\lambda_1 > 0$, and there exists a $b_0 > 0$ such that $b(x) \ge b_0$ for all $x \in \Gamma_1$, then*

$$\|u\|_{L^\infty(\Omega)} \le C\left(\|f\|_{L^p(\Omega)} + \|g\|_{L^\infty(\Gamma_1)} + \|h\|_{L^\infty(\Gamma_0)}\right)$$

and the constant C depends on $C = C(|\Omega|, \lambda_1, b_0, \inf_\Omega a, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, N, p)$.
iv) *If $f \in L^\infty(\Omega)$, $h \in L^\infty(\Gamma_0)$, $b(x) \ge 0$, $g(x) \le 0$ on Γ_1 and there exists a $c_0 > 0$ such that $c(x) \ge c_0$ then*

$$\|u\|_{L^\infty(\Omega)} \le \max\{c_0^{-1}\|f\|_{L^\infty(\Omega)}, \|h\|_{L^\infty(\Gamma_0)}\}$$

v) *If $f \in L^p(\Omega)$, $p > N/2$, $g \in L^r(\Gamma_1)$, $r > N-1$, $h \in L^\infty(\Gamma_0)$ and $K \subset \bar{\Omega} \setminus \Gamma_0$ is a compact set, then, $u \in C^\alpha(K)$ for some $\alpha \in (0,1)$ and*

$$\|u\|_{C^\alpha(K)} \le C(\|f\|_{L^p(\Omega)} + \|g\|_{L^r(\Gamma_1)} + \|h\|_{L^\infty(\Gamma_0)} + \|u\|_{L^2(\Omega)})$$

for a constant $C = C(\Omega, \rho_0, \inf_\Omega a, \sup_{K+B(0,\rho_0)} a, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, N, p, r)$, where $2\rho_0 =$dist(K, Γ_0).
vi) If $f \in L^p(\Omega)$, $p > N/2$, $g \in L^r(\Gamma_1)$, $r > N-1$, $h \in C^{\alpha_0}(\Gamma_0)$ for some $\alpha_0 \in (0,1)$, then $u \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0, \alpha_0]$ and

$$\|u\|_{C^\alpha(\bar{\Omega})} \le C(\|f\|_{L^p(\Omega)} + \|g\|_{L^r(\Gamma_1)} + \|h\|_{C^{\alpha_0}(\Gamma_0)} + \|u\|_{L^2(\Omega)})$$

for a constant $C = C(\Omega, \inf_\Omega a, \sup_\Omega a, \|b\|_{L^\infty(\Gamma_1)}, \|c\|_{L^\infty(\Omega)}, N, p, r, \alpha_0)$.

Proof: The first result relays in a bootstrap argument. The proof of ii), v) and vi) goes with similar ideas as the ones given in [22] to obtain $L^\infty(\Omega)$ and Hölder estimates for solutions of elliptic problems. Parts iii) and iv) establish conditions under which the L^∞ bounds do not depend on the domain Ω but rather on $|\Omega|$.

Throughout the proof we will denote by $m = \inf_\Omega a$, $b_\infty = \|b\|_{L^\infty(\Gamma_1)}$ and $c_\infty = \|c\|_{L^\infty(\Omega)}$.

i) Note first that for $\frac{2N}{N+2} \le p < \frac{N}{2}$, we have that $r = \frac{(N-2)p}{N-2p} \ge 2$. Let $k \ge \|h\|_{L^\infty(\Gamma_0)}$, $\phi = (u-k)^+$. If we multiply the equation by $|\phi|^{r-2}\phi$ and integrate by parts we obtain

$$\begin{aligned} \tfrac{4(r-1)}{r^2}\int_\Omega a|\nabla|\phi|^{\frac{r}{2}}|^2 + \int_\Omega c(|\phi|^{\frac{r}{2}})^2 + \int_{\Gamma_1} b(|\phi|^{\frac{r}{2}})^2 &= \int_\Omega (f-kc)|\phi|^{r-2}\phi + \int_{\Gamma_1}(g-kb)|\phi|^{r-2}\phi \\ &\le \|\tilde{f}\|_{L^p(\Omega)}\|\,|\phi|^{\frac{r}{2}}\,\|^{\frac{2}{r'}}_{L^{2\frac{p'}{r'}}(\Omega)} + \|\tilde{g}\|_{L^{\frac{(N-1)p}{N-p}}(\Gamma_1)}\|\,|\phi|^{\frac{r}{2}}\,\|^{\frac{2}{r'}}_{L^{2\frac{p'}{r'}\frac{N-1}{N}}(\Gamma_1)}. \end{aligned} \tag{B.2}$$

where we denote $\tilde{f} = f - kc$ and $\tilde{g} = g - kb$.

If we let $v = |\phi|^{\frac{r}{2}}$ and taking into account that $p'/r' = N/(N-2)$, we have that, for any $\lambda \in I\!R$,

$$\frac{4(r-1)}{r^2}\int_\Omega a|\nabla v|^2 + \int_\Omega (c+\lambda)v^2 + \int_{\Gamma_1} bv^2 \leq \|\tilde{f}\|_{L^p(\Omega)}\|v\|^{\frac{2}{r'}}_{L^{\frac{2N}{N-2}}(\Omega)} + \|\tilde{g}\|_{L^{\frac{(N-1)p}{N-p}}(\Gamma_1)}\|v\|^{\frac{2}{r'}}_{L^{\frac{2(N-1)}{N-2}}(\Gamma_1)} + \lambda\int_\Omega v^2$$

$$\leq C(\Omega,N,p)(\|\tilde{f}\|_{L^p(\Omega)} + \|\tilde{g}\|_{L^{\frac{(N-1)p}{N-p}}(\Gamma_1)})\|v\|^{\frac{2}{r'}}_{H^1(\Omega)} + \lambda\int_\Omega v^2 \tag{B.3}$$

Where the constant $C(\Omega,N,p)$ comes from the constants of the embeddings $H^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$ and $H^1(\Omega) \hookrightarrow L^{\frac{2(N-1)}{N-2}}(\Gamma)$. Choose $\lambda = \lambda(\Omega, b_\infty, c_\infty, N)$ so that

$$\frac{4(r-1)}{r^2}\int_\Omega a|\nabla v|^2 + \int_\Omega (c+\lambda)v^2 + \int_{\Gamma_1} bv^2 \geq \frac{r-1}{r^2}m\|v\|^2_{H^1(\Omega)}.$$

Hence, we have that

$$\frac{r-1}{r^2}m\|v\|^2_{H^1(\Omega)} \leq C(\|\tilde{f}\|_{L^p(\Omega)} + \|\tilde{g}\|_{L^{\frac{(N-1)p}{N-p}}(\Gamma_1)})\|v\|^{\frac{2}{r'}}_{H^1(\Omega)} + \lambda\|v\|^{\frac{2}{r}}_{L^2(\Omega)}\|v\|^{\frac{2}{r'}}_{H^1(\Omega)}.$$

From this the following holds

$$\|\phi\|_{L^{\frac{Np}{N-2p}}(\Omega)} = \|v\|^{\frac{2}{r}}_{L^{\frac{2N}{N-2}}(\Omega)} \leq C(\|\tilde{f}\|_{L^p(\Omega)} + \|\tilde{g}\|_{L^{\frac{(N-1)p}{N-p}}(\Gamma_1)} + \|\phi\|_{L^p(\Omega)}).$$

Taking into account the definition of $\tilde{f}$, $\tilde{g}$ and with the standard interpolation inequality in L^q-spaces, $\|\phi\|_{L^p(\Omega)} \leq \epsilon\|\phi\|_{L^{\frac{Np}{N-2p}}(\Omega)} + C_\epsilon\|\phi\|_{L^2(\Omega)}$ we obtain the existence of a constant $C = C(\Omega, m, b_\infty, c_\infty, N, p)$ such that

$$\|\phi\|_{L^{\frac{Np}{N-2p}}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|g\|_{L^{\frac{(N-1)p}{N-p}}(\Gamma_1)} + k + \|\phi\|_{L^2(\Omega)}). \tag{B.4}$$

Now, taking $k = \|h\|_{L^\infty(\Gamma_0)}$ and since $0 \leq u^+ \leq (u-k)^+ + k$, $0 \leq \phi \leq |u|$, we get from the above inequality that

$$\|u^+\|_{L^{\frac{Np}{N-2p}}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|g\|_{L^{\frac{(N-1)p}{N-p}}(\Gamma_1)} + \|h\|_{L^\infty(\Gamma_0)} + \|u\|_{L^2(\Omega)}).$$

for some constant $C = C(\Omega, m, b_\infty, c_\infty, N, p)$. with a similar argument for u^- we obtain the result.

ii) Let $k \geq \|h\|_{L^\infty(\Gamma_0)}$, $\phi = (u-k)^+$ and $A_k = \{x \in \Omega : u(x) > k\}$. Notice first that from (B.4) we can obtain that for any $q \geq 2$, the following inequality holds:

$$\|\phi\|_{L^q(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|g\|_{L^r(\Gamma_1)} + k + \|\phi\|_{L^2(\Omega)}) \leq C(\|f\|_{L^p(\Omega)} + \|g\|_{L^r(\Gamma_1)} + k + \|u\|_{L^2(\Omega)}). \tag{B.5}$$

where $C = C(\Omega, m, b_\infty, c_\infty, N, p, r, q)$.

Let $\lambda = \lambda(\Omega, m, b_\infty, c_\infty, N) > 0$ be such that

$$\int_\Omega a|\nabla u|^2 + \int_{\Gamma_1} bu^2 + \int_\Omega (c+\lambda)u^2 \geq \frac{m}{2}\|u\|^2_{H^1(\Omega)}. \tag{B.6}$$

With Hölder inequality and Sobolev embeddings, we have

$$(\|\phi\|_{L^1(\Omega)}|A_k|^{-\frac{N+2}{2N}})\|\phi\|_{H^1(\Omega)} \leq \|\phi\|_{L^{\frac{2N}{N-2}}(\Omega)}\|\phi\|_{H^1(\Omega)} \leq C\|\phi\|^2_{H^1(\Omega)}.$$

where $C = C(\Omega, N)$. If we now multiply the equation by ϕ and integrate by parts we get that

$$\int_\Omega a|\nabla\phi|^2 + \int_{\Gamma_1} b\phi^2 + \int_\Omega (c+\lambda)\phi^2 = \int_\Omega f\phi + \int_{\Gamma_1} g\phi - k\int_{\Gamma_1} b\phi - k\int_\Omega c\phi + \lambda\int_\Omega \phi^2 \tag{B.7}$$

and from (B.6) and (B.7) we obtain the existence of a constant $C = C(\Omega, m, b_\infty, c_\infty, N)$ such that

$$(\|\phi\|_{L^1(\Omega)}|A_k|^{-\frac{N+2}{2N}})\|\phi\|_{H^1(\Omega)} \le C(|\int_\Omega f\phi| + |\int_{\Gamma_1} g\phi| + |k\int_{\Gamma_1} b\phi| + |k\int_\Omega c\phi| + |\lambda\int_\Omega \phi^2|$$

Next we estimate each term on the right hand side of the above expression. For this, we use extensively Holder's inequality, Sobolev embeddings and trace theorems. We obtain:

$$|\int_\Omega f\phi| \le \|f\|_{L^p(\Omega)}\|\phi\|_{L^{p'}(\Omega)} \le \|f\|_{L^p(\Omega)}\|\phi\|_{L^{\frac{2N}{N-2}}(A_k)}|A_k|^{\frac{1}{2}-\frac{1}{p}+\frac{1}{N}} \le C(\Omega, N)\|f\|_{L^p(\Omega)}\|\phi\|_{H^1(\Omega)}|A_k|^{\frac{1}{2}-\frac{1}{p}+\frac{1}{N}}.$$

$$\begin{aligned} |\int_{\Gamma_1} g\phi| \le \|g\|_{L^r(\Gamma_1)}\|\phi\|_{L^{r'}(\Gamma_1)} &\le C(\Omega, N)\|g\|_{L^r(\Gamma_1)}\|\phi\|_{W^{1,\frac{Nr}{Nr-N+1}}(\Omega)} \\ &\le C(\Omega, N)\|g\|_{L^r(\Gamma_1)}\|\phi\|_{H^1(\Omega)}|A_k|^{\frac{1}{2}-\frac{N-1}{rN}}, \end{aligned}$$

$$|k\int_{\Gamma_1} b\phi| \le C(\Omega, N)kb_\infty\|\phi\|_{W^{1,1}(\Omega)} \le C(\Omega, N)kb_\infty\|\phi\|_{H^1(\Omega)}|A_k|^{\frac{1}{2}}.$$

$$|k\int_\Omega (c+\lambda)\phi| \le C(\Omega, N)k(c_\infty + \lambda)\|\phi\|_{H^1(\Omega)}|A_k|^{\frac{1}{2}+\frac{1}{N}}$$

and finally for $\lambda\int_\Omega \phi^2$ we have that, for any $q \ge 2$,

$$\lambda\|\phi\|^2_{L^2(\Omega)} \le \lambda\|\phi\|_{L^q(\Omega)}|A_k|^{\frac{1}{2}-\frac{1}{q}}\|\phi\|_{L^{\frac{2N}{N-2}}(\Omega)}|A_k|^{\frac{1}{N}} \le \lambda C(\Omega, N)\|\phi\|_{L^q(\Omega)}\|\phi\|_{H^1(\Omega)}|A_k|^{\frac{1}{2}+\frac{1}{N}-\frac{1}{q}}.$$

Choosing $q = 2N$, and applying inequality (B.5) to this value of q, we have:

$$\lambda\|\phi\|^2_{L^2(\Omega)} \le C(\|f\|_{L^p(\Omega)} + \|g\|_{L^r(\Gamma_1)} + k + \|u\|_{L^2(\Omega)})\|\phi\|_{H^1(\Omega)}|A_k|^{\frac{1}{2}+\frac{1}{2N}}.$$

where $C = C(\Omega, m, b_\infty, c_\infty, N, p, r)$.

Thus, for $k \ge \max\{1, \|h\|_{L^\infty(\Gamma_0)}\}$, we have that

$$\|\phi\|_{L^1(\Omega)} \le \gamma \cdot k \cdot |A_k|^{1+\epsilon}$$

where $\epsilon = \min\{\frac{2p-N}{N}, \frac{r-(N-1)}{Nr}, \frac{1}{N}, \frac{2}{N}, \frac{3}{2N}\} > 0$ and $\gamma = C \cdot [\|f\|_{L^p(\Omega)} + \|g\|_{L^r(\Gamma_1)} + \|u\|_{L^2(\Omega)} + 1]$ and $C = C(\Omega, m, b_\infty, c_\infty, N, p, r)$. Now, using the Lemma 5.1 in [22] we have that

$$\max\{u(x);\ x \in \Omega\} \le C(\Omega, m, b_\infty, c_\infty, p, r, \|f\|_{L^p(\Omega)}, \|g\|_{L^r(\Gamma_1)}, \|h\|_{L^\infty(\Gamma_0)}, \|u\|_{L^2(\Omega)}).$$

With a similar argument for $w = -u$, we show the same inequality for $-u$. Combining both inequalities we prove the result.

iii) Let $k \ge k_0 = \max\{b_0^{-1}\|g\|_{L^\infty(\Gamma_1)}, \|h\|_{L^\infty(\Gamma_0)}, 1\}$, $\phi = (u-k)^+$ and $A_k = \{x \in \Omega : u(x) > k\}$. Multiplying the equation by ϕ and integrating by parts we get,

$$\int_\Omega a|\nabla\phi|^2 + \int_{\Gamma_1} b\phi^2 + \int_\Omega c\phi^2 = \int_\Omega f\phi - k\int_\Omega c\phi + \int_{\Gamma_1}(g-kb)\phi \le \int_\Omega (f-kc)\phi \tag{B.8}$$

Since the first eigenvalue of the linear problem is $\lambda_1 > 0$, we have that

$$\lambda_1\int_\Omega \phi^2 \le \int_\Omega a|\nabla\phi|^2 + \int_{\Gamma_1} b\phi^2 + \int_\Omega c\phi^2 \le \int_\Omega (f-kc)\phi$$

and this implies that

$$\min\{m, b_0\}\left(\int_\Omega |\nabla\phi|^2 + \int_{\Gamma_1}\phi^2\right) \le \int_\Omega a|\nabla\phi|^2 + \int_{\Gamma_1} b\phi^2 \le \int_\Omega (f-kc)\phi + c_\infty\int_\Omega c\phi^2 \le (1 + c_\infty/\lambda_1)|\int_\Omega (f-kc)\phi|$$

Denoting by $M = (1 + c_\infty/\lambda_1)/\min\{m, b_0\}$, we have that

$$\int_\Omega |\nabla\phi|^2 + \int_\Gamma \phi^2 \leq M|\int_\Omega (f - kc)\phi|$$

Now, with an inequality due to Mazja, [21, 16], we have that there exists a constant $C = C(|\Omega|, N)$ such that

$$\|\phi\|^2_{L^{\frac{2N}{N-1}}(\Omega)} \leq C\left(\int_\Omega |\nabla\phi|^2 + \int_\Gamma \phi^2\right) \leq CM|\int_\Omega (f - kc)\phi|$$

But,

$$|\int_\Omega (f - kc)\phi| \leq \|f - kc\|_{L^p(\Omega)}\|\phi\|_{L^{p'}(\Omega)} \leq \|f - kc\|_{L^p(\Omega)}\|\phi\|_{L^{\frac{2N}{N-1}}(\Omega)}|A_k|^{\frac{1}{2}-\frac{1}{p}+\frac{1}{2N}}$$

which implies that

$$\|\phi\|_{L^{\frac{2N}{N-1}}(\Omega)} \leq CM\|f - kc\|_{L^p(\Omega)}|A_k|^{\frac{1}{2}-\frac{1}{p}+\frac{1}{2N}}$$

and therefore, for any $k \geq k_0$, we have

$$\|\phi\|_{L^1(\Omega)} \leq \|\phi\|_{L^{\frac{2N}{N-1}}(\Omega)}|A_k|^{\frac{1}{2}+\frac{1}{2N}} \leq CM\|f - kc\|_{L^p(\Omega)}|A_k|^{1-\frac{1}{p}+\frac{1}{N}} \leq \gamma \cdot k \cdot |A_k|^{1+\epsilon},$$

where $\gamma = CM(\|f\|_{L^p(\Omega)} + c_\infty)$ and $\epsilon = \frac{1}{N} - \frac{1}{p} > 0$ as long as $p > N$. Now, as in ii), Lemma 5.1 in [22] takes care of the rest of the proof.

iv) Define $k_0 = \max\{c_0^{-1}\|f\|_{L^\infty(\Omega)}, \|h\|_{L^\infty(\Gamma_0)}\}$ and let $\phi = (u - k_0)^+$. Using ϕ as a test function in the equation and via integration by parts, it is not difficult to obtain that

$$\int_\Omega a|\nabla\phi|^2 + \int_\Omega c\phi^2 \leq 0$$

and from here we get that $\phi \equiv 0$ which implies that $u(x) \leq k_0$. With a similar argument for $-u$ we obtain the desired result.

v) Notice first that from part ii) of this lemma, we have that $u \in L^\infty(\Omega)$. Without loss of generality we can assume that $\|f\|_{L^p(\Omega)} + \|g\|_{L^r(\Gamma_1)} + \|h\|_{L^\infty(\Gamma_0)} + \|u\|_{L^\infty(\Omega)} \leq 1$, since, if this is not the case we can define $v = u/(\|f\|_{L^p(\Omega)} + \|g\|_{L^r(\Gamma_1)} + \|h\|_{L^\infty(\Gamma_0)} + \|u\|_{L^\infty(\Omega)})$ which will satisfy the equation above for certain $\tilde{f}$, $\tilde{g}$, $\tilde{h}$ for which the above inequality holds.

Let $x_0 \in K \subset \bar{\Omega} \setminus \Gamma_0$, $\rho_0 = 1/2\text{dist}(K, \Gamma_0) > 0$, $\rho \in (0, \rho_0)$, $\sigma < 1$ and ξ be a smooth function such that $0 \leq \xi(x) \leq 1$ and

$$\xi(x) = \begin{cases} 1, & if\ |x| \leq (1-\sigma)\rho \\ 0, & if\ |x| \geq \rho \end{cases}$$

with $|\nabla\xi(x)| \leq L/\rho\sigma$, for some constant $L = L(N)$.

Let $v = \xi^2(x - x_0)(u - k)^+$ and $A_{k,\rho} = \{x \in \Omega : u(x) > k\} \cap \{x \in I\!R^N : |x - x_0| \leq \rho\}$. Multiplying the equation by v and integrating by parts we have that

$$\int_\Omega a\nabla u\,\nabla(\xi^2(u-k)^+) + \int_\Gamma g\xi^2(u-k)^+ + c\int_\Omega \xi^2 u(u-k)^+ + \int_\Gamma b\xi^2 u(u-k)^+ = \int_\Omega f\xi^2(u-k)^+$$

But noticing that,

$$\int_{A_{k,\rho}} a\xi^2|\nabla u|^2 = \int_\Omega a\xi^2\nabla u\,\nabla(u-k)^+ \leq |\int_\Omega a\nabla u\,\nabla(\xi^2(u-k)^+| + |\int_\Omega a2\xi\nabla\xi\,\nabla u\,(u-k)^+|$$

$$|\int_\Omega a\nabla u\,\nabla(\xi^2(u-k)^+| + \frac{1}{2}\int_{A_{k,\rho}} a\xi^2|\nabla u|^2 + 2\int_{A_{k,\rho}} a|\nabla\xi|^2((u-k)^+)^2;$$

and denoting by $m_0 = \inf\{a(x),\ x \in K + B(0,\rho_0)\}$ and $M_0 = \sup\{a(x),\ x \in K + B(0,\rho_0)\}$ we have that

$$\begin{aligned}\frac{m_0}{2}\int_{A_{k,\rho}} \xi^2|\nabla u|^2 &\leq 2M_0\int_{A_{k,\rho}} |\nabla\xi|^2((u-k)^+)^2 + |\int_\Gamma g\xi^2(u-k)^+| + |\int_\Omega f\xi^2(u-k)^+| \\ &+|c\int_\Omega \xi^2 u(u-k)^+| + |b\int_\Gamma \xi^2 u(u-k)^+|.\end{aligned} \tag{B.9}$$

In what follows we estimate each term on the right hand side of the above expression. To acomplish this we will extensively use Holder and Young's inequalities, Sobolev embeddings and trace theorems. We will obtain:

$$\int_{A_{k,\rho}} |\nabla\xi|^2((u-k)^+)^2 \leq \frac{L^2}{\rho^2\sigma^2}[\max_\Omega(u-k)^+]^2|A_{k,\rho}|$$

$$|\int_\Omega f\xi^2(u-k)^+| \leq \|f\|_{L^p(\Omega)}\|\xi^2(u-k)^+\|_{L^{p'}(\Omega)} \leq \max_\Omega(u-k)^+|A_{k,\rho}|^{1-\frac{1}{p}}$$

$$|\int_\Omega c\xi^2 u(u-k)^+| \leq c_\infty \max_\Omega(u-k)^+|A_{k,\rho}|$$

$$|\int_\Gamma g\xi^2(u-k)^+| \leq \|g\|_{L^r(\Gamma)}\|\xi(u-k)^+\|_{L^{r'}(\Gamma)} \leq C\|\xi(u-k)^+\|_{H^1(\Omega)}|A_{k,\rho}|^{\frac{1}{2}-\frac{N-1}{rN}}$$

$$\leq \frac{m}{8}\|\xi(u-k)^+\|^2_{H^1(\Omega)} + \frac{2}{m}C^2|A_{k,\rho}|^{1-\frac{2(N-1)}{rN}} \leq (\frac{m}{8}+\frac{mL^2}{4\rho^2\sigma^2})[\max_\Omega(u-k)^+]^2|A_{k,\rho}|+$$

$$\frac{m}{8}\|\xi\nabla(u-k)^+\|^2_{L^2(\Omega)} + \frac{2}{m}C^2|A_{k,\rho}|^{1-\frac{2(N-1)}{rN}}$$

$$|\int_\Gamma b\xi^2 u(u-k)^+| \leq b_\infty\|\xi(u-k)^+\|_{L^1(\Gamma_1} \leq Cb_\infty\|\xi(u-k)^+\|_{H^1(\Omega)}|A_{k,\rho}|^{\frac{1}{2}}$$

$$\leq (\frac{m}{8}+\frac{mL^2}{4\rho^2\sigma^2})[\max_\Omega(u-k)^+]^2|A_{k,\rho}| + \frac{m}{8}\|\xi\nabla(u-k)^+\|^2_{L^2(\Omega)} + \frac{2C^2b_\infty^2}{m}|A_{k,\rho}|$$

If we now use the above estimates in (B.9) we obtain that

$$\begin{aligned}\frac{m_0}{2}\int_{A_{k,\rho}} \xi^2|\nabla u|^2 &\leq \frac{m_0}{4}\int_{A_{k,\rho}} \xi^2|\nabla u|^2 + (2M_0\frac{L^2}{\rho^2\sigma^2}+\frac{m}{8}+\frac{mL^2}{4\rho^2\sigma^2})[\max_\Omega(u-k)^+]^2|A_{k,\rho}| \\ &+(1+c_\infty|\Omega|^{\frac{1}{p}})\max_\Omega(u-k)^+|A_{k,\rho}|^{1-\frac{1}{p}} + (\frac{2}{m}C^2+\frac{2C^2b_\infty^2}{m}|\Omega|^{\frac{2(N-1)}{rN}})|A_{k,\rho}|^{1-\frac{2(N-1)}{rN}}\end{aligned} \tag{B.10}$$

But if we define $\nu > N$ such that $\min\{1-\frac{1}{p}, 1-\frac{2(N-1)}{rN}\} \geq 1-\frac{2}{\nu}$, and noticing that $|A_{k,\rho}| \leq |A_{k,\rho}|^{1-\frac{2}{\nu}}\rho^{\frac{2N}{\nu}}$, then we obtain the existence of a constant $\gamma = \gamma(\Omega, m_0, M_0, \rho_0, N, b_\infty, c_\infty, p, r)$ such that

$$\int_{A_{k,\rho(1-\sigma)}} |\nabla u|^2 \leq \int_{A_{k,\rho}} \xi^2|\nabla u|^2 \leq \gamma\left[\frac{1}{\sigma^2\rho^{2[1-\frac{N}{\nu}]}}[\max_\Omega(u-k)^+]^2+1\right]|A_{k,\rho}|^{1-\frac{2}{\nu}} \tag{B.11}$$

The result now follows from Theorem 7.2 in [22].

vi) Let ρ be small and let $\Omega_\rho = \{x \in \Omega;\ \mathrm{dist}(x,\Gamma_0) < \rho\}$. Then, from v) we have that $u \in C^\alpha(\bar\Omega\setminus\Omega_\rho)$. Denote by $k(x)$ the trace of the function u on $\Gamma_\rho = \partial\Omega_\rho\cap\Omega$. The function u satisfies the equation $-div(a\nabla u)+cu = f$ on Ω_ρ with boundary conditions $u = k$ on Γ_ρ and $u = h$ on Γ_0. Since k and h are Hölder continuos we obtain from standard results, see [22], that u is Hölder continuous on $\bar\Omega_\rho$. Combining both results we prove vi).

This proves the lemma.□

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