Instituto de Ciências Matemáticas e de Computação

ISSN - 0103-2577

BOUNDARY SYNCHRONIZATION IN PARABOLIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

ALEXANDRE N. CARVALHO
MARCOS R.T. PRIMO

Nº 66

NOTAS DO ICMC
Série Matemática

São Carlos
Nov./1998

Resumo

Neste trabalho provamos a sincronização de dois problemas parabólicos com condição de fronteira não linear e acoplamento dissipativo na fronteira; isto é, mostramos que as coordenadas do sistema, formado pelas duas equações parabólicas com condição de fronteira não linear (acopladas linearmente na fronteira), estão próximas uma da outra para tempos grandes. Isto é obtido se os termos não lineares são aproximadamente os mesmos.

Boundary Synchronization in Parabolic Problems with Nonlinear Boundary Conditions

Alexandre N. Carvalho*

Departamento de Matemática

Instituto de Ciências Matemáticas e de Computação

Universidade de São Paulo

13560-970 - São Carlos - SP - Brazil

Marcos R. T. Primo†

Departamento de Matemática

Fundação Universidade Estadual de Maringá

87020-900 - Maringá - Pr - Brazil

Abstract

In this work we prove the synchronization of two parabolic problems with nonlinear boundary condition and dissipative boundary coupling; that is, we show that the coordinates of the system, made of the two parabolic equations with nonlinear boundary condition (linearly coupled in the boundary), are close for large times. This is obtained if the nonlinear terms are approximately the same.

1 Introduction and Notations

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain with $\Gamma := \partial\Omega$. Consider the parabolic problem

$$\begin{cases} u_t(t,x) = \triangle u(t,x) + f_1(u(t,x)) & t>0, \;\; x \in \Omega, \\ v_t(t,x) = \triangle v(t,x) + f_2(v(t,x)) & t>0, \;\; x \in \Omega, \\ \frac{\partial u}{\partial \vec{n}} = g_1(u) - k(u-v) & t>0, \;\; x \in \Gamma, \\ \frac{\partial v}{\partial \vec{n}} = g_2(v) + k(u-v) & t>0, \;\; x \in \Gamma, \end{cases} \tag{1.1}$$

*Partially supported by CNPq grant # 300.889/92-5 and FAPESP grant # 97/11323-0 - Brazil
†Partially supported by CNPq and PICDT-CAPES - Brazil

where k is a positive parameter, $f_i : \mathbb{R} \to \mathbb{R}$, and $g_i : \mathbb{R} \to \mathbb{R}$,, $i = 1, 2$, are locally Lipschitz functions satifying

$$\sup_{s \in \mathbb{R}} |f_1(s) - f_2(s)| \leq C_f \quad \text{and} \quad \sup_{s \in \mathbb{R}} |g_1(s) - g_2(s)| \leq C_g. \tag{1.2}$$

for some C_f and C_g. We remark that, if (1.1) is dissipative (1.2) can be assumed without loss of generality (as we will see later).

The aim of this work is to show that there exists $k_0 > 0$ such that, for $k > k_0$, any globally defined bounded solution $(u(t, \cdot), v(t, \cdot))$ of (1.1), in an appropriate Banach space Y, will satisfy

$$\lim_{t \to \infty} \|u(t, \cdot) - v(t, \cdot)\|_Y \leq C(|C_f| + |C_g|),$$

with C_f, C_g as before, and C independent of k. Notice that if $f_1 = f_2$ and $g_1 = g_2$, an immediate consequence of this result is that

$$\lim_{t \to \infty} \|u(t, \cdot) - v(t, \cdot)\|_Y = 0.$$

Such property is called *synchronization.*

In order to obtain that the problem (1.1) is locally well posed, we need to introduce some notation and results. To solve problems like (1.1), we always work with a family of Banach spaces, in which the initial data can lie and the nonlinear terms are well behaved. For this we introduce an appropriate scale of Banach spaces. The results presented now can be found, with details, in [3, 4].

Consider the linear operator $A = \text{diag}(\mathbf{A}, \mathbf{A})$, where $\mathbf{A} : D(\mathbf{A}) \subset L^q(\Omega) \to L^q(\Omega)$, is given by

$$\begin{aligned} &D(\mathbf{A}) = \{\phi \in W^{2,q}(\Omega); \tfrac{\partial \phi}{\partial \vec{n}} = 0, \quad \text{in } \Gamma\} := W^{2,q}_N(\Omega); \\ &\mathbf{A}\phi = -\Delta \phi. \end{aligned} \tag{1.3}$$

Let $\mathbf{E}^1_q := D(\mathbf{A})$ and $\mathbf{E}^0_q := L^q(\Omega)$. Taking $[\cdot, \cdot]_\theta$ the complex interpolation functor of exponent θ, $\mathbf{A}$ has an associated scale of Banach spaces, see [6], that for $0 \leq \alpha \leq 1$ is given by $\mathbf{E}^\alpha_q = [\mathbf{E}^1_q, \mathbf{E}^0_q]_\alpha$. Let $E^\alpha_q := \mathbf{E}^\alpha_q \times \mathbf{E}^\alpha_q$.

We have that for every $\alpha > \beta$ the inclusion $E^\alpha_q \subset E^\beta_q$ is compact, $E^{\frac{1}{2}}_q = W^{1,q}(\Omega, \mathbb{R}^2)$ and

$$E^\alpha_q := [W^{2,q}_N(\Omega, \mathbb{R}^2), L^q(\Omega, \mathbb{R}^2)]_\alpha \hookrightarrow H^{2\alpha}_q(\Omega, \mathbb{R}^2), \tag{1.4}$$

where $\alpha \in (0, 1)$, and $H^{2\alpha}_q(\Omega, \mathbb{R}^2)$ are the Bessel potential spaces, see [6]. We can extend this scale to Banach spaces of negative exponents, by taking

$$E^{-\alpha}_q := (E^\alpha_{q'})', \quad \alpha > 0. \tag{1.5}$$

Also, the operator A can be extended to act on these spaces in such way that its realization, $A_{\alpha-1}$, verifies $A_{\alpha-1} : E^\alpha_q \to E^{\alpha-1}_q$ and is a sectorial operator, for $0 \leq \alpha \leq 1$.

Therefore, we are considering problems of the form

$$\begin{array}{rcl} \dot{x}(t) & = & -Ax(t) + \sum_{i=1}^{n} F_i(x(t)), \ \ t > 0; \\ x(0) & = & x_0, \end{array} \tag{1.6}$$

where A is the operator defined previously, and F_i, $i = 1, 2$ are suitable functions, which we will define below. Let $\{X^\alpha,\ \alpha \geq 0\}$ be the scale of Banach Spaces constructed above (shifted if needed). We are looking for solutions of (1.1) defined as follows.

Definition 1.1 *We say that a function $x : [0, \tau] \to X^1$ is an ε-regular solution to* (1.1) *if $x \in C([0, \tau], X^1) \cap C((0, \tau], X^{1+\varepsilon})$, and $x(t)$ satisfies*

$$x(t) = e^{-At}x_0 + \int_0^t e^{-A(t-s)} \sum_{i=1}^{n} F_i(x(s))ds.$$

A reason to work with the scale of Banach spaces constructed above is that we can solve the problem (1.1) in the two following senses:

Definition 1.2 *For $u_0 \in W^{1,q}(\Omega, \mathbb{R}^2)$, a variational weak solution of* (1.1) *is a function $u \in C([0, \tau), W^{1,q}(\Omega, \mathbb{R}^2))$ such that*

$$\frac{d}{dt}\int_\Omega u\phi + \int_\Omega \nabla u \nabla \phi = \int_\Omega F(u)\phi + \int_\Gamma G(u)\phi, \tag{1.7}$$

for every $\phi \in W^{1,q'}(\Omega, \mathbb{R}^2)$, where for $v = (v_1, v_2) \in \mathbb{R}^2$,

$$\begin{array}{rcl} F(v) & = & (f_1(v_1), f_2(v_2)) \\ G(v) & = & (g_1(v_1) - k(v_1 - v_2), g_2(v_2) + k(v_1 - v_2)). \end{array} \tag{1.8}$$

Definition 1.3 *For $u_0 \in L^q(\Omega, \mathbb{R}^2)$, a variational very weak solution of* (1.1) *is a function $u \in C([0, \tau), L^q(\Omega, \mathbb{R}^2))$ such that*

$$\frac{d}{dt}\int_\Omega u\phi - \int_\Omega u\Delta\phi = \int_\Omega F(u)\phi + \int_\Gamma G(u)\phi, \tag{1.9}$$

for every $\phi \in W_N^{2,q'}(\Omega, \mathbb{R}^2)$, where $W_N^{2,q'}(\Omega, \mathbb{R}^2)$ consists of the functions in $W^{2,q'}(\Omega, \mathbb{R}^2)$ with zero normal derivative on Γ, F and G are given in (1.8).

However we need to impose some growth restrictions on the nonlinearities f_i, g_i, $i = 1, 2$ of (1.1). These growth restrictions are expressed as follows:

(C1) In order to obtain solutions in the sense of Definition 1.3, assume that f_i, g_i, $i = 1, 2$ satisfy a relation of the form

$$|h(u) - h(v)| \leq c|u - v|(|u|^{\rho-1} + |v|^{\rho-1} + 1), \ \ u, v \in \mathbb{R}, \tag{1.10}$$

with exponents ρ_i and $\overline{\rho_i}$, $i = 1, 2$ respectively, such that, if $N \geq 1$,

$$\rho_i \leq \rho_f := 1 + \frac{2q}{N} \ \text{ and } \ \overline{\rho_i} \leq \rho_g := 1 + \frac{q}{N}, \ \ i = 1, 2,$$

where the second inequality is strict when $N = 1$.

(C2) In order to obtain solutions in the sense of Definition 1.2, assume that

1. $q > N$,
2. $q = N$ and for every $\eta > 0$, there exists a constant $c_\eta > 0$ such that f_i and g_i, $i = 1, 2$, satisfy a relation of the form

$$|h(u) - h(v)| \leq c_\eta |u - v| (e^{\eta |u|^{\frac{N}{N-1}}} + e^{\eta |v|^{\frac{N}{N-1}}}), \quad u, v \in \mathbb{R}, \tag{1.11}$$

3. $1 < q < N$ and f_i, g_i, $i = 1, 2$ satisfy (1.10) with exponents ρ_i and $\overline{\rho_i}$, $i = 1, 2$ respectively, such that,

$$\rho_i \leq \rho_f := 1 + \frac{2q}{N - q} \quad \text{and} \quad \overline{\rho_i} \leq \rho_g := 1 + \frac{q}{N - q}, \quad i = 1, 2.$$

Then we have the following result from [4].

Theorem 1.1 *Assume that f_i, g_i, $i = 1, 2$ satisfy the growth restriction* (**C1**). *Then, for every $u_0 \in L^q(\Omega, \mathbb{R}^2)$, there exists a unique local solution, $u(\,\cdot\,; u_0)$, of problem* (1.1) *in the sense of* Definition 1.3, *such that $u(0; u_0) = u_0$. This solution depends continuously on the initial data $u_0 \in L^q(\Omega, \mathbb{R}^2)$. Furthermore, this solution is a classical solution for every $t > 0$.*

Also, if f_i, g_i, $i = 1, 2$ satisfy the growth restriction (**C2**), *then for every $u_0 \in W^{1,q}(\Omega, \mathbb{R}^2)$, there exists a unique local solution, $u(\,\cdot\,; u_0)$, of problem* (1.1) *in the sense of* Definition 1.2, *such that $u(0; u_0) = u_0$. This solution depends continuously on the initial data $u_0 \in W^{1,q}(\Omega, \mathbb{R}^2)$. Furthermore, this solution is a classical solution for every $t > 0$.*

In order to show global existence of solutions for (1.1), we will impose some sign restrictions on the nonlinear terms of (1.1). Assume that there exist constants $B_0, C_0 \in \mathbb{R}$ and $B_1, C_1 \geq 0$ such that for every $u \in \mathbb{R}$,

$$\begin{aligned} u f_i(u) &\leq -C_0 u^2 + C_1 |u|, \\ u g_i(u) &\leq -B_0 u^2 + B_1 |u|, \end{aligned} \tag{1.12}$$

$i = 1, 2$. In addition, to prove the existence of a global attractors for the problem (1.1), we will need to impose that the first eigenvalue, λ_1, of the problem

$$\begin{cases} -\Delta u + C_0 u = \lambda u & \text{in } \Omega, \\ \frac{\partial u}{\partial \vec{n}} + B_0 u = 0 & \text{in } \Gamma, \end{cases} \tag{1.13}$$

is positive, where B_0, C_0 are given in (1.12). With all these, we can prove the principal result of this work:

Theorem 1.2 *Under the all above assumptions, there exists a constant $k_0 > 0$ such that for every $k > k_0$,*

$$\lim_{t\to\infty} \|u(t,\cdot) - v(t,\cdot)\|_{L^\infty(\Omega)} \le C(|C_f| + |C_g|),$$

where C_f e C_g are given in (1.2). Moreover, if $f_1 = f_2$ and $g_1 = g_2$, then

$$\lim_{t\to\infty} \|u(t,\cdot) - v(t,\cdot)\|_{L^\infty(\Omega)} = 0,$$

for every $k > k_0$.

Synchronization of coupled parabolic problems have been previously considered in [7] for homogeneous Neumann boundary conditions and with the coupling acting inside the region. The synchronization there obtained is not uniform as it happens here.

This paper is organized as follows: In Section 2 the positivity and comparison results introduced in [5] are extended to include problems of the form (1.1); in Section 3 we obtain global existence of solutions to (1.1) and existence of global attractors; in Section 4 the synchronization is proved and finally in Section 5 an example is considered.

2 Positivity and Comparison Results

In this section, we give some positivity and comparison abstract results, then we apply these results to obtain global existence, asymptotic properties and synchronization of solutions of problem (1.1).

Definition 2.1 *An ordered Banach space is a pair $(X, \le)$, where X is a Banach space and $\le$ is an order relation in X such that*

- *$x \le y$ implies $x + z \le y + z$, for every $x, y, z \in X$;*
- *$x \le y$ implies $\lambda x \le \lambda y$ for $\lambda \in \mathbb{R}^+$ and $x, y \in X$;*
- *The "positive cone" $C = \{x \in X; 0 \le x\}$ is closed in X.*

Definition 2.2 *Let $(X, \le)$ be a ordered Banach space. A function $T : X \to X$ is said* **increasing** *if and only if $x \le y$ implies $T(x) \le T(y)$, $x, y \in X$, and its is called* **positive** *if and only if $0 \le x$ implies $0 \le T(x)$, $x \in X$.*

Definition 2.3 *A sectorial operator A in an ordered Banach space X is said to be* **resolvent positive** *in X if for every $\lambda > \lambda_0$, $(A + \lambda)^{-1}$ is increasing in X, $\lambda_0 \in \mathbb{R}$.*

Remark 2.1 *Observe that to say that A is resolvent positive in X, is equivalent to the following comparison result: for every $\lambda > \lambda_0$ and $f \le g$ in X, if $Au + \lambda u = f$ and $Av + \lambda v = g$, then $u \le v$ in X.*

Now, let us consider problems of the form

$$\begin{cases} u_t + Au = f(u) \\ u(0) = u_0, \end{cases} \tag{2.1}$$

in an ordered Banach space X, where A is a sectorial operator which is resolvent positive in X. Assume that we have constructed, as in Section 1, a scale of interpolation spaces X^α, $\alpha \geq 0$, with $X^0 = X$ and $X^1 = D(A)$. In each X^α, we will consider the order induced by X. Throughout this section, the scale X^α will satisfy the following definition

Definition 2.4 *The scales of Banach spaces X^α, $\alpha \geq 0$ is an Scale of Ordered Spaces if the injections $X^\alpha \hookrightarrow X^\beta$, $\alpha \geq \beta$ are positives, $A_\alpha : X^{\alpha+1} \to X^\alpha$, $\alpha \geq 0$, is resolvent positive and the positive cone of X^α is dense in the positive cone of X^β for every $\alpha \geq \beta \geq 0$.*

We will assume that the nonlinear term in (2.1), f, is subcritical in a certain space $X^{1+\varepsilon}$, $\varepsilon \geq 0$, that is, there exists a $\gamma > 0$ with $0 \leq 1+\varepsilon-\gamma < 1$ such that $f : X^{1+\varepsilon} \to X^\gamma$ is locally Lipschitz. Moreover, suppose that the problem (2.1) is locally well posed in X^1. Then we have the following result about positivity (see [5] for a proof)

Theorem 2.1 *Under the all assumption above, assume that for every $r > 0$ there exists a constant $\beta = \beta(r) > 0$ such that $f(\cdot) + \beta I$ is positive on the positive elements in the ball in $X^{1+\varepsilon}$ of radius r. Then, if $0 \leq u_0 \in X^1$ the solution $u_f(\,\cdot\,; u_0)$, of (2.1), is positive while it exists.*

Concerning comparison, we have (see [5] for a proof)

Theorem 2.2 *Let A and X as above. Assume that the nonlinearities f, g and h satisfy the same conditions as describe above for f. Then,*

(i) *Assume that for every $r > 0$ there exists a constant $\beta = \beta(r) > 0$ such that $f(\cdot)+\beta I$ is increasing on the ball in $X^{1+\varepsilon}$ of radius r. If $u_0, u_1 \in X^1$, with $u_0 \leq u_1$, then $u_f(\,\cdot\,; u_0) \leq u_f(\,\cdot\,; u_1)$ while they exist.*

(ii) *Assume that f, g verify $f(\cdot) \leq g(\cdot)$. Then, for any $u_0 \in X^1$, $u_f(\,\cdot\,; u_0) \leq u_g(\,\cdot\,; u_0)$ while they exist.*

(iii) *Assume that f, g are such that for every $r > 0$, there exists a constant $\beta = \beta(r) > 0$ and an increasing function h such that $f + \beta I \leq h \leq g + \beta I$ on the ball in $X^{1+\varepsilon}$ of radius r. If $u_0, u_1 \in X^1$ with $u_0 \leq u_1$, then $u_f(\,\cdot\,; u_0) \leq u_f(\,\cdot\,; u_0)$ while they exist.*

Now we will apply the abstract results above to the problem (1.1). First, we will define an order, $\preceq$, in $L^q(\Omega, \mathbb{R}^2)$. Let $f = (f_1, f_2), g = (g_1, g_2)$ be functions in $L^q(\Omega, \mathbb{R}^2)$. We say that $f \preceq g$ if $f_1 \leq g_1$ and $f_2 \leq g_2$, where $\leq$ is the usual order in $L^q(\Omega)$. It is clear that $(L^q(\Omega, \mathbb{R}^2), \preceq)$ is a ordered Banach space.

Let us consider now the operator B defined by $B : D(B) \subset L^q(\Omega, \mathbb{R}^2) \to L^q(\Omega, \mathbb{R}^2)$ with

$$\begin{aligned} &D(B) = \{\phi \in W^{2,q}(\Omega, \mathbb{R}^2)\ ; -\tfrac{\partial \phi_1}{\partial \vec{n}} - b\phi_1 = \tfrac{\partial \phi_2}{\partial \vec{n}} + b\phi_2 = k(\phi_1 - \phi_2) \text{ in } \Gamma\}, \\ &B\phi = -\Delta\phi + c\phi. \end{aligned} \tag{2.2}$$

We have the following result

Proposition 2.1 *The operator B as given in* (2.2) *is a sectorial operator and resolvent positive in $L^q(\Omega, \mathbb{R}^2)$.*

Proof: We will prove only that the operator B is resolvent positive in $L^q(\Omega, \mathbb{R}^2)$, in the sense of Definition 2.3.

Let $f = (f_1, f_2)$ be functions in $L^q(\Omega, \mathbb{R}^2)$ such that $0 \preceq f$. Then, $0 \leq f_1, f_2$ in $L^q(\Omega)$. Consider now the following elliptic problem

$$\begin{array}{ll} -\Delta u_1 + cu_1 + \lambda u_1 = f_1 & \text{in } \Omega; \\ -\Delta u_2 + cu_2 + \lambda u_2 = f_1 & \text{in } \Omega; \\ \frac{\partial u_1}{\partial n} + bu_1 + k(u_1 - u_2) = 0 & \text{in } \Gamma; \\ \frac{\partial u_2}{\partial n} + bu_2 - k(u_1 - u_2) = 0 & \text{in } \Gamma. \end{array} \tag{2.3}$$

Notice that from Remark 2.1 it is enough to show that there exists λ_0 such that for every $\lambda > \lambda_0$, $0 \preceq u$, that is, $0 \leq u_1, u_2$ in $L^q(\Omega)$.

Now, consider the following change of variables $w = u_1 - u_2$ and $z = u_1 + u_2$, that is, $u_1 = \frac{w+z}{2}$ and $u_2 = \frac{z-w}{2}$. Therefore, from (2.3) we obtain, in the new variables, the following elliptic problem

$$\begin{array}{ll} -\Delta w + cw + \lambda w = f_1 - f_2 & \text{in } \Omega; \\ -\Delta z + cz + \lambda z = f_1 + f_2 & \text{in } \Omega; \\ \frac{\partial w}{\partial n} + (b + 2k)w = 0 & \text{in } \Gamma; \\ \frac{\partial z}{\partial n} + bz = 0 & \text{in } \Gamma. \end{array}$$

Let $\overline{z}$ be a solution of the elliptic problem

$$\begin{array}{ll} -\Delta \overline{z} + c\overline{z} + \lambda \overline{z} = f_1 + f_2 & \text{in } \Omega; \\ \frac{\partial \overline{z}}{\partial n} + (b + 2k)\overline{z} = 0 & \text{in } \Gamma. \end{array}$$

Since $k > 0$, the Maximum Principle (see [14], page 78), for elliptic problems implies that $\overline{z} \leq z$ in $L^q(\Omega)$. Therefore, $\phi := w + \overline{z} \leq w + z$ in $L^q(\Omega)$, and ϕ is a solution of the problem

$$\begin{array}{ll} -\Delta \phi + c\phi + \lambda \phi = 2f_1 & \text{in } \Omega; \\ \frac{\partial \phi}{\partial n} + (b + 2k)\phi = 0 & \text{in } \Gamma. \end{array}$$

But $0 \leq 2f_1$ in $L^q(\Omega)$, then there exists $\lambda_0 \in \mathbb{R}$ such that for every $\lambda > \lambda_0$, $0 \leq \phi$. Hence, we conclude that for every $\lambda > \lambda_0$, $0 \leq \frac{\phi}{2} = u_1$, in $L^q(\Omega)$.

Similarly, we show that $0 \leq u_2$ in $L^q(\Omega)$, for every $\lambda > \lambda_0$, then we get the result. ■

As in Section 1, we can construct a scale of Banach spaces, X_q^α, $\alpha \in [-1,1]$, $1 < q < \infty$, to the operator B as defined in (2.2). Assume that the constants c and b on (2.2) are such that the first eigenvalue of B is positive. Then, with the Proposition 2.1 and the Theorem 2.7.2 from [2], we can show the next result

Theorem 2.3 *For every $1 < q < \infty$ and $\alpha \in [0,1]$, X_q^α is a ordered Banach space, with the order induced by $L^q(\Omega, \mathbb{R}^2)$. The space $X_q^{-\alpha}$ is an ordered Banach space with the dual canonical order, that is, $0 \leq u$ in $X_q^{-\alpha}$ if and only if $0 \leq \langle u, \phi \rangle$ for every $\phi \in X_{q'}^\alpha$.*

For $\gamma \geq \beta$, the inclusion $X_q^\gamma \to X_q^\beta$ is positive and the positive cone of X_q^γ is dense in the positive cone of X_q^β.

Furthermore, $B_{\alpha-1} : X_q^\alpha \to X_q^{\alpha-1}$ is a sectorial operator and resolvent positive.

For smooth initial data, an immediate consequence of the abstract results of this section is the next result concerning positivity and comparison to the solutions of the problem (1.1). Since the solutions of (1.1) depend continuously on initial data in $W^{1,q}(\Omega, \mathbb{R}^2)$ or in $L^q(\Omega, \mathbb{R}^2)$, we obtain the next corollary

Corollary 2.1 *Suppose that for $i = 0, 1, 2$ the functions f_i, g_i satisfy the assumptions of* Theorem 1.1. *Then*

(i) *Assume that $f_i(0), g_i(0) \geq 0$, $i = 1, 2$. If $0 \preceq u_0$, then $0 \preceq u(t; u_0)$, while it exists.*

(ii) *If $u_0 \preceq u_1$, then $u(t; u_0) \preceq u(t; u_1)$, while they exist.*

(iii) *Assume that for every $u \in \mathbb{R}$, $f_0(u) \leq f_i(u)$ and $g_0(u) \leq g_i(u)$, for $i = 1, 2$. If $u_0 \preceq u_1$, then $u(t; u_0, f_0, g_0) \preceq u(t; u_1, f_1, f_2, g_1, g_2)$, while they exist.*

Where "$\preceq$" denotes the order defined in this section to the space in which the initial data u_i, $i = 0, 1$ belongs, that is, $L^q(\Omega, \mathbb{R}^2)$ or $W^{1,q}(\Omega, \mathbb{R}^2)$.

Let $F_i, G_i : \mathbb{R}^2 \to \mathbb{R}$ be functions locally Lipschitz, and consider the problem

$$\begin{cases} w_t = \Delta w + F_1(w,z) & \text{in } \Omega, \\ z_t = \Delta z + F_2(w,z) & \text{in } \Omega, \\ \frac{\partial w}{\partial \vec{n}} = G_1(w,z) & \text{in } \Gamma, \\ \frac{\partial z}{\partial \vec{n}} = G_2(w,z) & \text{in } \Gamma. \end{cases} \tag{2.4}$$

With a similar argument, we obtain the next result

Corollary 2.2 *Suppose that the problem (2.4) is locally well posed in $L^q(\Omega, \mathbb{R}^2)$ or $W^{1,q}(\Omega, \mathbb{R}^2)$ and denote by $u(t; u_0) = (w(t; u_0), z(t; u_0))$, such solution. Then,*

(i) *Assume that $F_i(0,0), G_i(0,0) \geq 0$, $i = 1, 2$. If $0 \preceq u_0$, then $0 \preceq u(t; u_0) = (w(t; u_0), z(t; u_0))$, while it exists.*

(ii) *If $u_0 \preceq u_1$, then $u(t;u_0) \preceq u(t;u_1)$, while they exist.*

(iii) *Assume that for every $w, z \in \mathbb{R}$, $F_0(w,z) \leq F_i(w,z)$ and $G_0(w,z) \leq G_i(w,z)$, for $i = 1,2$. If $u_0 \preceq u_1$, then $u(t;u_0,F_0,G_0) \preceq u(t;u_1,F_1,F_2,G_1,G_1)$, while they exist.*

Where "$\preceq$" denotes the order defined in this section to the space in which the initial data u_i, $i = 0,1$ belongs, that is, $L^q(\Omega,\mathbb{R}^2)$ or $W^{1,q}(\Omega,\mathbb{R}^2)$.

3 Global Existence and Existence Attractors

In this section we will prove that the solutions obtained in Theorem 1.1 to the problem (1.1) are globally defined. We will also show some asymptotic properties of such solutions. Here we follow the ideas contained in [5]. We start with the following lemma

Lemma 3.1 *Suppose that the first eigenvalue, λ_1, of the problem (1.13) is positive. Let $\lambda_1(k)$ be the first eigenvalue of the problem*

$$\begin{cases} -\Delta u + C_0 u = \lambda u & in\ \ \Omega, \\ -\Delta v + C_0 v = \lambda v & in\ \ \Omega, \\ \frac{\partial u}{\partial \vec{n}} + B_0 u + k(u-v) = 0 & in\ \ \Gamma, \\ \frac{\partial v}{\partial \vec{n}} + B_0 v - k(u-v) = 0 & in\ \ \Gamma, \end{cases} \tag{3.1}$$

where B_0, C_0 are given in (1.12). Then, $\lambda_1(k) = \lambda_1 > 0$, for every $k > 0$.

Proof: Let us consider the following change of variables $w = u - v$ e $z = u + v$. From (3.1), we obtain that $\lambda_1(k)$ still is the first eigenvalue of the problem

$$\begin{cases} -\Delta w + C_0 w = \lambda w & \text{in}\ \ \Omega, \\ -\Delta z + C_0 z = \lambda z & \text{in}\ \ \Omega, \\ \frac{\partial w}{\partial \vec{n}} + B_0 w + 2kw = 0 & \text{in}\ \ \Gamma, \\ \frac{\partial z}{\partial \vec{n}} + B_0 z = 0 & \text{in}\ \ \Gamma. \end{cases}$$

Notice that the above problem is decoupled. Denote by $\mu_1(k)$ the first eigenvalue of

$$\begin{cases} -\Delta \phi + C_0 \phi = \lambda \phi & \text{in}\ \ \Omega, \\ \frac{\partial \phi}{\partial \vec{n}} + B_0 \phi + 2k\phi = 0 & \text{in}\ \ \Gamma. \end{cases}$$

The results contained in [9], page 399, imply that $\mu_1(k) \geq \lambda_1 > 0$, for every $k \geq 0$. The result is proved. ■

The next result is a technical lemma, and your proof can be found in [2, 11].

Lemma 3.2 *Consider the operator B as defined in (2.2) with $c = C_0$ and $b = B_0$, where C_0, B_0 as given in (1.12). Assume that the first eigenvalue, λ_1, from (1.13) is positive. Let α_0 be a positive constant and $q \in (1, \infty)$. Then, there exists a constant $M_1 = M_1(B, \lambda_1, N, q, \alpha_0)$ such that for every $\alpha_0 \geq \alpha \geq \beta \geq -\alpha_0$ we have*

$$\|e^{-Bt}u_0\|_{X_q^\alpha} \leq M_1 e^{-\lambda_1 t} t^{-(\alpha-\beta)} \|u_0\|_{X^\alpha{}_q}, \quad t > 0, \quad u_0 \in X^\alpha{}_q.$$

In particular, there exists $\delta_0 = \delta_0(q, N) > 0$ and a constant $\tilde{M_1} = \tilde{M_1}(B, \lambda_1, N, q)$ such that

$$\|e^{-Bt}u_0\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \tilde{M_1} e^{-\lambda_1 t} t^{-\delta_0} \|u_0\|_{L^q(\Omega,\mathbb{R}^2)}, \quad t > 0, \quad u_0 \in L^q(\Omega, \mathbb{R}^2).$$

Theorem 3.1 *Suppose that the functions f_i, g_i $i = 1, 2$ satisfy* **(C1)** *or* **(C2)**, *respectively. We have*

(i) *If for $i = 1, 2$ f_i and g_i verify (1.12), then the locally solutions of (1.1) as given in the* Theorem 1.1 *are globally defined.*

(ii) *Also, assume that the first eigenvalue, λ_1, of (1.13) is positive. Then, there exists a positive constant K_∞ such that*

$$\limsup_{t\to\infty} \|u(t; u_0)\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq K_\infty, \tag{3.2}$$

with the limit above being uniform to u_0 in bounded subsets of $L^q(\Omega, \mathbb{R}^2)$ or $W^{1,q}(\Omega, \mathbb{R}^2)$, respectively.

(iii) *If the assumptions of* (i) *and* (ii) *hold and if $\phi = (\phi_1, \phi_2)$ is the solution of the elliptic problem*

$$\begin{array}{ll} -\Delta\phi_1 + C_0\phi_1 = C_1 & \text{in } \Omega, \\ -\Delta\phi_2 + C_0\phi_2 = C_1 & \text{in } \Omega, \\ \frac{\partial\phi_1}{\partial\vec{n}} + B_0\phi_1 + k(\phi_1 - \phi_2) = B_1 & \text{in } \Gamma, \\ \frac{\partial\phi_2}{\partial\vec{n}} + B_0\phi_2 - k(\phi_1 - \phi_2) = B_1 & \text{in } \Gamma, \end{array} \tag{3.3}$$

then $0 \leq \phi_1, \phi_2 \in L^\infty(\Omega)$. If $u(t, x; u_0) = (u_1(t, x; u_0), u_2(t, x; u_0))$ is the solution of (1.1) with initial data $u_0 = (u_0^1, u_0^2)$, we have for $i = 1, 2$

$$\limsup_{t\to\infty} |u_i(t, x; u_0)| \leq \phi_i(x),$$

uniformly in $x \in \overline{\Omega}$ and for u_0 in bounded subsets of $L^q(\Omega, \mathbb{R}^2)$ or $W^{1,q}(\Omega, \mathbb{R}^2)$, respectively. Indeed, if for $i = 1, 2$, $|u_0^i| \leq \phi_i(x)$ for every $x \in \overline{\Omega}$, then for $i = 1, 2$,

$$|u_i(t, x, u_0)| \leq \phi_i(x), \quad \forall x \in \overline{\Omega}, \quad \forall t \geq 0.$$

Proof: The proof of this result is similar to the proof of Proposition 3.1 from [5]. ■

From Theorem 3.1 and the results of [10], we obtain the next result

Theorem 3.2 *Let us consider* $X = L^q(\Omega, \mathbb{R}^2)$ *or* $X = W^{1,q}(\Omega, \mathbb{R}^2)$, *respectively and suppose that for* $i = 1, 2$ f_i *and* g_i *verify* **(C1)** *or* **(C2)**, *respectively and* (1.12). *Then, the problem* (1.1) *has a global compact attractor* $\mathcal{A}_X$ *such that*

$$\mathcal{A}_X \subset \Sigma(\phi_1, \phi_2) := \{u \in L^\infty(\Omega, \mathbb{R}^2)\,;\, |u_i(x)| \leq \phi_i(x), \forall x \in \Omega, \;\; i = 1, 2\}.$$

Now the main idea is to find estimates, independent of the constant k, to the elements of the global attractor $\mathcal{A}_X$. Before, we need to show the following lemma.

Lemma 3.3 *Let* $\phi = (\phi_1, \phi_2) \in H^1(\Omega, \mathbb{R}^2)$ *a weak solution of the elliptic problem* (3.3). *Assume that the first eigenvalue,* λ_1, *of* (1.13) *is positive. Then,* $\phi \in L^\infty(\Omega, \mathbb{R}^2)$ *and for* $i = 1, 2$

$$\|\phi_i\|_{L^\infty(\Omega)} \leq C(|B_1| + |C_1|), \tag{3.4}$$

where $C = C(|\Omega|, \lambda_1, N)$. *Furthermore, there exists* $k_0 > 0$, *large enough, such that the constant* C *in* (3.4) *can be chosen independent of* k, *for every* $k > k_0$.

Proof: Introducing the change of variables $\psi_1 = \phi_1 - \phi_2$ and $\psi_2 = \phi_1 + \phi_2$, the Lemma 8.1, item (iii) from [5] implies the result. ∎

Finally, from Theorem 3.2 and from Lemma 3.3 we have

Theorem 3.3 *Consider* $X = L^q(\Omega, \mathbb{R}^2)$ *or* $X = W^{1,q}(\Omega, \mathbb{R}^2)$, *respectively. For* $i = 1, 2$ *assume that the functions* f_i, g_i *verify the conditions* **(C1)** *or* **(C2)**, *respectively and* (1.12). *Suppose that the first eigenvalue,* λ_1, *of* (1.13) *is positive. Then,*

$$\sup_{v \in \mathcal{A}_X} \|v\|_{L^\infty(\Omega, \mathbb{R}^2)} \leq K_0,$$

where $K_0 = K_0(|\Gamma|, \lambda_1, C_0, C_1, B_0, B_1)$.

4 Synchronization in Parabolic Problems

In this section, we will show that the problem (1.1), for functions f_i and g_i, $i = 1, 2$, verifying conditions as the ones given in the Sections 1 e 3, "synchronize" in the sense that the coordinates u and v are close to one another if f_1, g_1 are close to f_2, g_2. Let us consider the following problem

$$\begin{cases} u_t = \Delta u + f_1(u) & \text{in } \Omega, \\ v_t = \Delta v + f_2(v) & \text{in } \Omega, \\ \frac{\partial u}{\partial \vec{n}} = g_1(u) - k(u - v) & \text{in } \Gamma, \\ \frac{\partial v}{\partial \vec{n}} = g_2(v) + k(u - v) & \text{in } \Gamma. \end{cases} \tag{4.1}$$

Let $(u(t; u_0), v(t; v_0))$ be the solution of (4.1), with initial data $(u_0, v_0) \in X$, where $X = L^q(\Omega, \mathbb{R}^2)$ or $X = W^{1,q}(\Omega, \mathbb{R}^2)$, as the previous sections.

The results from Section 3 imply that there exists a bounded set $\Sigma_0 \subset \mathbb{R}^2$, such that for $k > 0$,

$$(u(t; u_0)(x), v(t; v_0)(x)) \in \Sigma_0,$$

for any $x \in \Omega$, $t > 0$ and for every $(u_0, v_0) \in X$, where $X = L^q(\Omega, \mathbb{R}^2)$ or $X = W^{1,q}(\Omega, \mathbb{R}^2)$. Therefore, let L_f and L_g be positive constants such that, for every $(u, v) \in \Sigma_0$ and $i = 1, 2$,

$$\begin{aligned} |f_1(u) - f_1(v)| &\le L_f|u - v|, \\ |g_1(u) - g_1(v)| &\le L_g|u - v|. \end{aligned} \tag{4.2}$$

We can assume, without loss of generality, that (1.2) and (4.2) hold for every $u, v \in \mathbb{R}$.

Let us consider the change of variables, $w = u - v$ and $z = u + v$, in (4.1). Then we obtain that $(w(t; w_0), z(t; z_0))$ is a solution of the problem

$$\begin{cases} w_t = \Delta w + f_1(\frac{w+z}{2}) - f_2(\frac{z-w}{2}) & \text{in } \Omega, \\ z_t = \Delta z + f_1(\frac{w+z}{2}) + f_2(\frac{z-w}{2}) & \text{in } \Omega, \\ \frac{\partial w}{\partial \vec{n}} = g_1(\frac{w+z}{2}) - g_2(\frac{z-w}{2}) - 2kw & \text{in } \Gamma, \\ \frac{\partial z}{\partial \vec{n}} = g_1(\frac{w+z}{2}) + g_2(\frac{z-w}{2}) & \text{in } \Gamma, \end{cases} \tag{4.3}$$

with initial data $w_0 = u_0 - v_0$ and $z_0 = u_0 + v_0$.

The next two results will be essential to the development of this section.

Lemma 4.1 *Let $(w(t; w_0), z(t; z_0))$ be a solution of (4.3), with initial data (w_0, z_0) as defined above. Then,*

$$w(t, w_0)(x) \le \overline{w}(t; |w_0|)(x) \quad \text{and} \quad z(t, z_0)(x) \le \overline{z}(t; |z_0|)(x), \tag{4.4}$$

for every $t \ge 0$ and $x \in \Omega$, where $(\overline{w}(t; |w_0|), \overline{z}(t; |z_0|))$ is the solution of the problem

$$\begin{cases} \overline{w}_t = \Delta \overline{w} + L_f|\overline{w}| + C_f & \text{in } \Omega, \\ \overline{z}_t = \Delta \overline{z} + f_1(\frac{\overline{w}+\overline{z}}{2}) + f_2(\frac{\overline{z}-\overline{w}}{2}) & \text{in } \Omega, \\ \frac{\partial \overline{w}}{\partial \vec{n}} = L_g|\overline{w}| + C_g - 2k\overline{w} & \text{in } \Gamma, \\ \frac{\partial \overline{z}}{\partial \vec{n}} = g_1(\frac{\overline{w}+\overline{z}}{2}) + g_2(\frac{\overline{w}-\overline{z}}{2}) & \text{in } \Gamma, \end{cases} \tag{4.5}$$

with initial data $(\overline{w}_0, \overline{z}_0) = (|w_0|, |z_0|)$.

Proof: For $z, w \in \mathbb{R}$, let us consider

$$\begin{aligned} F_1(z, w) &= f_1(\tfrac{z+w}{2}) - f_1(\tfrac{z-w}{2}) + f_1(\tfrac{z-w}{2}) - f_2(\tfrac{z-w}{2}), \\ F_2(z, w) &= f_1(\tfrac{z+w}{2}) + f_2(\tfrac{z-w}{2}), \\ G_1(z, w) &= g_1(\tfrac{z+w}{2}) - g_1(\tfrac{z-w}{2}) + g_1(\tfrac{z-w}{2}) - g_2(\tfrac{z-w}{2}) - 2kw, \\ G_2(z, w) &= g_1(\tfrac{z+w}{2}) + g_2(\tfrac{z-w}{2}). \end{aligned}$$

Then, from (1.2) and (4.2) we obtain that

$$F_1(z, w) \le L_f|w| + C_f \quad \text{and} \quad G_1(z, w) \le L_g|w| + C_g - 2kw.$$

Since $w_0 \le |w_0|$ and $z_0 \le |z_0|$, the Corollary 2.2 shows the lemma. ■

With obvious modifications, we can prove the next result

Lemma 4.2 *Let $(w(t;w_0), z(t;z_0))$ be a solution of (4.3), with initial data (w_0, z_0) as defined above. Then,*

$$w(t,w_0)(x) \geq \underline{w}(t;-|w_0|)(x) \quad and \quad z(t,z_0)(x) \geq \underline{z}(t;-|z_0|)(x), \tag{4.6}$$

for every $t \geq 0$ and $x \in \Omega$, where $(\underline{w}(t;-|w_0|), \underline{z}(t;-|z_0|))$ is the solution of the problem

$$\begin{cases} \underline{w}_t = \Delta \underline{w} - L_f|\underline{w}| - C_f & in \ \ \Omega, \\ \underline{z}_t = \Delta \underline{z} + f_1(\frac{\underline{w}+\underline{z}}{2}) + f_2(\frac{\underline{z}-\underline{w}}{2}) & in \ \ \Omega, \\ \frac{\partial \underline{w}}{\partial \vec{n}} = -L_g|\underline{w}| - C_g - 2k\underline{w} & in \ \ \Gamma, \\ \frac{\partial \underline{z}}{\partial \vec{n}} = g_1(\frac{\underline{w}+\underline{z}}{2}) + g_2(\frac{\underline{w}-\underline{z}}{2}) & in \ \ \Gamma, \end{cases} \tag{4.7}$$

with initial data $(\underline{w}_0, \underline{z}_0) = (-|w_0|, -|z_0|)$.

We are interested in estimating the difference $u(t;u_0)(x) - v(t;v_0)(x)$, for $t \geq 0$ large enough, and for every $x \in \Omega$. Thus, from now on we will work only with the inequalities to $w(t;w_0)$ as given on Lemmas 4.1 and 4.2.

Since $|w_0| \geq 0$, the Corollary 2.2 item (i) implies that $\overline{w}(t;|w_0|)(x) \geq 0$ for every $t \geq 0$ and $x \in \Omega$. Therefore, $\overline{w}(t;|w_0|)(x)$ satisfies the problem

$$\begin{array}{ll} \overline{w}_t = \Delta \overline{w} + L_f \overline{w} + C_f & \text{in } \ \Omega, \\ \frac{\partial \overline{w}}{\partial \vec{n}} = +L_g \overline{w} + C_g - 2k\overline{w} & \text{in } \ \Gamma, \end{array} \tag{4.8}$$

with initial data $|w_0|$.

Now consider the change of variables $\tilde{w} = \overline{w} - \phi_1$, that is, $\overline{w} = \tilde{w} + \phi_1$, where ϕ_1 is solution of the elliptic problem

$$\begin{array}{ll} -\Delta \phi_1 = +L_f \phi_1 + C_f & \text{in } \ \Omega, \\ \frac{\partial \phi_1}{\partial \vec{n}} = +L_g \phi_1 + C_g - 2k\phi_1 & \text{in } \ \Gamma. \end{array} \tag{4.9}$$

Thus, (4.8) and (4.9) imply that $\tilde{w}(t;|w_0| - \phi_1(0))$ is solution of the problem

$$\begin{array}{ll} \tilde{w}_t = \Delta \tilde{w} + L_f \tilde{w} & \text{in } \ \Omega, \\ \frac{\partial \tilde{w}}{\partial \vec{n}} = +L_g \tilde{w} - 2k\tilde{w} & \text{in } \ \Gamma. \end{array} \tag{4.10}$$

To show some properties concerning synchronization of solutions of the problem (4.10), we need to study your associated eigenvalue problem. Consider now the following eigenvalues problems

$$\begin{cases} -\Delta w = \mu w & \text{in } \ \Omega, \\ \frac{\partial w}{\partial \vec{n}} + (2k - L_g)w = 0 & \text{in } \ \Gamma, \end{cases} \tag{4.11}$$

and

$$\begin{cases} -\Delta w = \mu w & \text{in } \ \Omega, \\ w = 0 & \text{in } \ \Gamma. \end{cases} \tag{4.12}$$

To apply the results contained in [9], we need to put some restriction on the boundary condition in (4.11), expressed as follow

$$\theta := 2k - L_g > 0. \tag{4.13}$$

Such restriction is the first condition on the constant k, of the problem (1.1), to obtain the desired synchronization.

Lemma 4.3 *Consider θ as in (4.13), and let $\mu_1(\theta)$ and μ_∞ respectively, be the first eigenvalue of the problems (4.11) and (4.12). Then,*

$$\mu_1(\theta) = \inf\{\int_\Omega |\nabla\phi|^2 + \theta\int_\Gamma |\phi|^2 \mid \phi \in H^1(\Omega) \ \text{ and } \ \|\phi\|_{L^2(\Omega)} = 1\}.$$

Furthermore,

$$\mu_1(\theta) \leq \mu_\infty \ \text{ and } \ \lim_{\theta\to+\infty} \mu_1(\theta) = \mu_\infty.$$

Proof: The first part of this lemma is a result well known, see [9]. An immediate consequence of this result is that $\mu_1(\theta_1) \leq \mu_1(\theta_2)$ for $\theta_1 \leq \theta_2$. Also, since $H_0^1(\Omega) \subset H^1(\Omega)$, we obtain that for every $\theta > 0$,

$$\begin{aligned} \mu_1(\theta) &= \inf\{\int_\Omega |\nabla\phi|^2 + \theta\int_\Gamma |\phi|^2 \mid \phi \in H^1(\Omega) \ \text{ and } \ \|\phi\|_{L^2(\Omega)} = 1\} \\ &\leq \inf\{\int_\Omega |\nabla\phi|^2 \mid \phi \in H_0^1(\Omega) \ \text{ and } \ \|\phi\|_{L^2(\Omega)} = 1\} \\ &= \mu_\infty. \end{aligned}$$

But $\mu_1(\theta)$ is increasing and bounded, then the next limit exists and $\lim_{\theta\to+\infty}\mu_1(\theta) \leq \mu_\infty$. Thus, there exist sequences $\{\theta_n, n \geq 1\} \subset \mathbb{R}$, $\theta_n \to +\infty$ and $\{\phi_n, n \geq 1\} \subset H^1(\Omega)$ with $\|\phi_n\|_{L^2(\Omega)} = 1$, for every $n \geq 1$ such that

$$\int_\Omega |\nabla\phi_n|^2 + \theta_n\int_\Gamma |\phi_n|^2 \leq \mu_\infty.$$

Therefore, $\{\phi_n, n \geq 1\}$ is bounded in $H^1(\Omega)$. Then, there exist a subsequence, which we also denote by $\{\phi_n, n \geq 1\}$, and $\phi \in H^1(\Omega)$ such that

$$\phi_n \rightharpoonup \phi \ \text{ in } \ H^1(\Omega) \ \text{ and } \ \phi_n \to \phi \ \text{ in } \ L^2(\Omega),$$

and similarly,

$$\gamma(\phi_n) \rightharpoonup \gamma(\phi) \ \text{ in } \ L^2(\Gamma),$$

where γ denotes the trace operator. Thus, $\|\phi\|_{L^2(\Omega)} = 1$, and since $\theta_n \to \infty$ and $\theta_n\|\phi_n\|_{L^2(\Gamma)}$ is bounded for every $n \geq 1$, we obtain that

$$\int_\Gamma |\phi_n|^2 \to 0, \quad \text{as} \quad n \to +\infty.$$

Hence, $\phi \in H_0^1(\Omega)$. Now, we have

$$\int_\Omega |\nabla\phi|^2 \leq \liminf_{n\to+\infty}\int_\Omega |\nabla\phi_n|^2,$$

then

$$\lim_{n\to+\infty}\int_\Omega |\nabla\phi_n|^2 + \theta_n\int_\Gamma |\phi_n|^2 \geq \int_\Omega |\nabla\phi|^2 \geq \mu_\infty.$$

Since the argument utilized is independent of the sequence we take, we obtain the result. ■

An immediate consequence of the Lemma 4.3 is that the first eigenvalue, $\mu_1(k)$, of the eigenvalue problem

$$\begin{cases} -\Delta\psi - L_f\psi = \mu\psi & \text{in } \Omega, \\ \frac{\partial\psi}{\partial\vec{n}} + (2k - L_g)\psi = 0 & \text{in } \Gamma, \end{cases} \tag{4.14}$$

can be made positive if and only if

$$L_f < \mu_\infty. \tag{4.15}$$

With all these, we can prove the next result.

Theorem 4.1 *For $i = 1, 2$ let $f_i, g_i : \mathbb{R}\to\mathbb{R}$ be locally Lipschitz functions satisfying* **(C1)** [**(C2)**], (1.12) *and* (1.2). *Suppose that the first eigenvalue of* (1.13) *is positive. Also, assume that the constants L_f, L_g in* (4.2), *verify* (4.13) *and* (4.15). *Then, there exists a positive constant $k_0 = k_0(\Omega, f_1, f_2, g_1, g_2)$ such that for every $k > k_0$, a.a. $x \in \Omega$ and for every $w_0 = u_0 - v_0$ with $(u_0, v_0) \in L^q(\Omega, \mathbb{R}^2)$ $[(u_0, v_0) \in W^{1,q}(\Omega, \mathbb{R}^2)]$ we have*

$$\limsup_{t\to\infty} \overline{w}(t; |w_0|)(x) \leq \|\phi_1\|_{L^\infty(\Omega)},$$

where ϕ_1 is the solution of the elliptic problem (4.9).

Proof: Taking k satisfying (4.13) and (4.15), it is easy to see that

$$\|\tilde{w}(t; |w_0| - \phi_1(0))\|_{L^\infty(\Omega)}\to 0, \quad \text{as} \quad t\to+\infty,$$

where $\tilde{w}(t; |w_0| - \phi_1(0))$ is solution of problem (4.10).
Since $\overline{w}(t; |w_0|) = \tilde{w}(t; |w_0| - \phi_1(0)) + \phi_1$ and $\overline{w}(t; |w_0|) \geq 0$, we have completed the proof. ■

In a similar way we have

Theorem 4.2 *Assume that all assumptions of* Theorem 4.1 *hold. Let k_0 be a constant found on* Theorem 4.1. *Then, for $k > k_0$, a.a. $x \in \Omega$ and for all $w_0 = u_0 - v_0$ with $(u_0, v_0) \in L^q(\Omega, \mathbb{R}^2)$ $[(u_0, v_0) \in W^{1,q}(\Omega, \mathbb{R}^2)]$, we have*

$$\liminf_{t\to\infty} \underline{w}(t; -|w_0|)(x) \geq -\|\phi_2\|_{L^\infty(\Omega)},$$

where ϕ_2 is solution of the elliptic problem

$$\begin{array}{ll} -\Delta\phi_2 = +L_f\phi_2 - C_f & \textit{in } \Omega, \\ \frac{\partial\phi_2}{\partial\vec{n}} = +L_g\phi_2 - C_g - 2k\phi_2 & \textit{in } \Gamma. \end{array} \tag{4.16}$$

Therefore, the Lemmas 4.1 and 4.2 and the Theorems 4.1 and 4.2 imply that,

$$\begin{gathered} \underline{w}(t;-|w_0|)(x) \leq w(t;w_0)(x) \leq \overline{w}(t;|w_0|)(x), \\ -\|\phi_2\|_{L^\infty(\Omega)} \leq \liminf_{t\to\infty} w(t;w_0)(x) \leq \limsup_{t\to\infty} w(t;w_0)(x) \leq \|\phi_1\|_{L^\infty(\Omega)} \end{gathered} \tag{4.17}$$

for every $x \in \Omega$ and for any $w_0 = u_0 - v_0$ with $u_0, v_0 \in X$, where $X = L^q(\Omega, \mathbb{R}^2)$ or $X = W^{1,q}(\Omega, \mathbb{R}^2)$.

The next result is an adaptation of the Lemma 8.1 item (iii) from [5].

Proposition 4.1 *Assume that all assumptions of Theorem 4.1 hold. Then, there exists a positive constant $k_0 = k_0(\Omega, f_1, f_2, g_1, g_2)$ such that for every $k \geq k_0$, we have, for $i = 1, 2$,*

$$\|\phi_i\|_{L^\infty(\Omega)} \leq C(|C_f| + |C_g|),$$

where $C = C(\Omega, f_1, f_2, g_1, g_2) > 0$ and ϕ_1, ϕ_2 are solutions of (4.9) and (4.16) respectively.

Proof: First we will show a bound, independent of k for k large enough, for the $L^2(\Omega)$-norm of solutions of the problems (4.9) and (4.16).

The restrictions on k imply that $L_k := 2k - L_g > 0$. Thus, multiplying the first equation of (4.9) by ϕ_1 and integrating by parts we obtain that

$$\int_\Omega |\nabla\phi_1|^2 - L_f \int_\Omega \phi_1^2 + L_k \int_\Gamma \phi_1^2 - C_f \int_\Omega \phi_1 - C_g \int_\Gamma \phi_1 = 0. \tag{4.18}$$

Taking $\phi = \phi_1 - \frac{C_g}{2L_k}$, and using that the first eigenvalue, $\mu_1(k)$, of (4.14) is positive, (4.18) implies that

$$\mu_1(k) \int_\Omega \phi^2 + C_1 \int_\Omega \phi + C_2 \leq 0,$$

where $C_1 := -(C_f + \frac{L_f C_g}{L_k})$ and $C_2 := -(|\Gamma|\frac{C_g^2}{4L_k} + |\Omega|(\frac{3C_g^2 L_f}{4L_k^2} + \frac{C_f C_g}{2L_k}))$.

From Young inequality, with $\varepsilon = \frac{|C_1|}{\mu_1(k)}$, we have that

$$\|\phi\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\mu_1(k)}} \sqrt{2|C_2| + \frac{|C_1|^2|\Omega|}{\sqrt{\mu_1(k)}}}.$$

Thus,

$$\|\phi_1\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{\mu_1(k)}} \sqrt{2|C_2| + \frac{|C_1^2|\Omega|}{\sqrt{\mu_1(k)}}} + \frac{C_g|\Omega|^{1/2}}{2L_k}.$$

Now, the Lemma 4.3, and the expression of C_1 and C_2 imply that for $k > k_0$, there exists a positive constant $\overline{C} = C(|\Omega|, |\Gamma|, f_1, f_2, g_1, g_2)$ such that

$$\|\phi_1\|_{L^2(\Omega)} \leq \overline{C}(|C_f| + |C_g|).$$

With the same arguments we obtain that the expression above holds for ϕ_2, the solution of (4.16).

Next we obtain estimates in $L^\infty(\Omega)$ for ϕ_i, $i = 1, 2$. In (4.9) choosing $k_1 > 0$ such that $2k_1 - L_g = L_g$, then for every $k \geq k_1$ we have $2k - L_g \geq L_g$. Let us define

$$r_0 := \max\{\frac{C_g}{L_g}, 1\}.$$

For every $r \geq r_0$, consider $\phi := (\phi_1 - r)^+$ and

$$A_r := \{x \in \Omega \;\; : \;\; \phi_1(x) > r\},$$

where ϕ_1 is the solution of elliptic problem (4.9).

Multiplying the first equation of (4.9) by ϕ, integrating by parts and using the definition of ϕ, it follows that

$$\int_\Omega |\nabla\phi|^2 + (2k - L_g)\int_\Gamma \phi^2 - L_f\int_\Omega \phi^2 = \int_\Omega (C_f + L_f r)\phi + \int_\Gamma (C_g - (2k - L_g r))\phi.$$

The definitions of r_0 and ϕ imply that

$$\int_\Omega |\nabla\phi|^2 + (2k - L_g)\int_\Gamma \phi^2 - L_f\int_\Omega \phi^2 \leq \int_\Omega (C_f + L_f r)\phi. \tag{4.19}$$

Since the first eigenvalue, $\mu_1(k)$, of (4.14) is positive for every $k > k_0$ satisfying the conditions on Theorem 4.1, we have that

$$\begin{aligned} \min\{1, L_g\}(\textstyle\int_\Omega |\nabla\phi|^2 + \int_\Gamma \phi^2) &\leq \textstyle\int_\Omega |\nabla\phi|^2 + (2k - L_g)\int_\Gamma \phi^2 \\ &\leq \frac{(1+L_f)(C_f+L_f r)}{\mu_1}\textstyle\int_\Omega \phi. \end{aligned}$$

Denoting by $M := \frac{(1+L_f)(C_f+L_f r)}{\mu_1 \min\{1,L_g\}}$, we obtain that

$$\int_\Omega |\nabla\phi|^2 + \int_\Gamma \phi^2 \leq M\int_\Omega |\phi|.$$

Now, an inequality due to Maz'ja [13] implies that there exists a positive constant $C_3 = C_3(|\Omega|, N)$ such that

$$\begin{aligned} ||\phi||^2_{L^{\frac{2N}{N-1}}(\Omega)} \leq C_3(\textstyle\int_\Omega |\nabla\phi|^2 + \int_\Gamma \phi^2) &\leq C_3 M \textstyle\int_\Omega |\phi| \\ &\leq C_3 M |\Omega|^{1/p} ||\phi||_{L^{p'}(\Omega)} \\ &\leq C_3 M |\Omega|^{1/p} ||\phi||_{L^{\frac{2N}{N-1}}(\Omega)} |A_r|^{1/2-1/p+1/N}. \end{aligned}$$

Thus,

$$||\phi||_{L^{\frac{2N}{N-1}}} \leq C_3 M |\Omega|^{1/p} |A_r|^{1/2-1/p+1/N}.$$

Taking $p = 2N$, we obtain, from Lemma 5.1 in [12], that

$$\max\{\phi_1(x),\ \ x \in \Omega\} \leq C_3 M |\Omega|^{1/2N} r_0 \|\phi_1\|_{L^1(A_{r_0})} \frac{N+1}{N}.$$

Changing ϕ_1 by $-\phi_1$ we have that

$$\|\phi_1\|_{L^\infty(\Omega)} \leq C_3 M |\Omega|^{1/2N} r_0 \frac{N+1}{N} \|\phi_1\|_{L^1(A_{r_0})}.$$

Using the first part of the proof of this result, we obtain that

$$\|\phi_1\|_{L^\infty(\Omega)} \leq C_3 M |\Omega|^{1/2N} r_0 \frac{N+1}{N} \overline{C}(|C_f| + |C_g|).$$

The expressions of M and C_3, as given above, imply that there exists a positive constant $C = C(\Omega, |\Gamma|, N, f_1, f_2, g_1, g_2)$ such that

$$\|\phi_1\|_{L^\infty(\Omega)} \leq C(|C_f| + |C_g|),$$

for k large enough. Similarly, we have the same estimate for ϕ_2, solution of (4.16). ■

Finally we have

Corollary 4.1 *Suppose that all assumptions of* Theorem 4.1 *hold. Then, for every* $(u_0, v_0) \in L^q(\Omega, \mathbb{R}^2)$ $[(u_0, v_0) \in W^{1,q}(\Omega, \mathbb{R}^2)]$ *there exists a positive constant* $k_0 = k_0(\Omega, f_1, f_2, g_1, g_2)$ *such that for every* $k \geq k_0$,

$$\limsup_{t \to \infty} \|u(t; u_0) - v(t; v_0)\|_{L^\infty(\Omega)} \leq C(|C_f| + |C_g|),$$

where C_f *and* C_g *are given in* (1.2) *and* $C \geq 0$ *is the constant from* Proposition 4.1. *Moreover, if* $f_1 = f_2$ *and* $g_1 = g_2$, *then*

$$\lim_{t \to \infty} \|u(t; u_0) - v(t; v_0)\|_{L^\infty(\Omega)} = 0,$$

for every $k > k_0$.

5 An Example

In this section, for $\Omega = (0,1)$ and functions f and g satisfying the conditions of previous sections, we will calculate in a precise way, the value of $k_0 = k_0(\Omega, f, g)$ as given in Section 4. For simplicity we will work in $L^2(\Omega, \mathbb{R}^2)$.

Let be $\Omega = (0,1)$ and consider the problem

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega, \\ v_t = \Delta v + f(v) & \text{in } \Omega, \\ \frac{\partial u}{\partial \vec{n}} = g(u) - k(u - v) & \text{in } \Gamma, \\ \frac{\partial v}{\partial \vec{n}} = g(v) + k(u - v) & \text{in } \Gamma. \end{cases} \tag{5.1}$$

From previous sections, we know that to obtain sychronization of solutions of the problem (5.1) it is enough to show that the first eigenvalue, $\mu_1(k)$, associated with the problem

$$\begin{cases} w_t = \Delta w + L_f w \quad \text{in} \quad \Omega, \\ w_x - (2k - L_g)w = 0 \quad \text{in} \quad x = 0, \\ w_x + (2k - L_g)w = 0 \quad \text{in} \quad x = 1, \end{cases} \tag{5.2}$$

is positive.

Let us consider $\theta = 2k - L_g$. The first restriction is that θ must be positive, therefore we must have $k > L_g > \frac{L_g}{2}$.

Let $A_\theta : D(A_\theta) \subset L^2(\Omega) \to L^2(\Omega)$ be the operator defined by

$$\begin{array}{ll} D(A_\theta) & = \{\phi \in H^2(\Omega) : \phi_x(0) = \theta\phi(0) \quad \text{and} \quad \phi_x(1) = -\theta\phi(1)\} \\ A_\theta(\phi) & = -\phi_{xx}, \end{array} \tag{5.3}$$

and consider the following eigenvalue problem $A_\theta \phi = \mu\phi$. If $\mu = \gamma^2$, then γ is given by $\cot\ \gamma = G(\frac{\gamma}{\theta})$, where $G(s) = \frac{1}{2}(\frac{\gamma}{\theta} - \frac{\theta}{\gamma})$. Notice that fixed $0 < \gamma < \pi$, we have that $\lim_{\theta \to +\infty} G(\frac{\gamma}{\theta}) = -\infty$.

Now, if $L_f \in (0, \pi^2)$, there exists $\theta_0 > 0$ such that $\cot\ \sqrt{L_f} = \frac{1}{2}(\frac{\sqrt{L_f}}{\theta_0} - \frac{\theta_0}{\sqrt{L_f}})$, observe that θ_0 is given by

$$\theta_0 = -\cot\ (\sqrt{L_f})\sqrt{L_f} + \sqrt{L_f}\sqrt{(\cot\ (\sqrt{L_f}))^2 + 1}.$$

Therefore, for every $\theta > \theta_0$, that is, for

$$k > L_g + \frac{\theta_0}{2} := k_0(\Omega, f, g),$$

the first eigenvalue, $\mu_1(k)$, associated with problem (5.2), is positive. Thus, the Corollary 4.1 implies that for every $k > k_0(\Omega, f, g)$,

$$\lim_{t \to +\infty} \|u(t; u_0) - v(t; v_0)\|_{L^\infty(\Omega)} = 0,$$

for any $(u_0, v_0) \in L^2(\Omega, \mathbb{R}^2)$.

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