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OSCULATING HYPERPLANES AND ASYMPTOTIC DIRECTIONS OF CONDIMENSION TWO SUBMANIFOLDS OF EUCLIDEAN SPACES

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Osculating hyperplanes and asymptotic directions of codimension two submanifolds of euclidean spaces

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Abstract

We define the concepts of binormal and asymptotic directions for submanifolds embedded with codimension 2 into euclidean spaces and obtain necessary conditions, in terms of the existence of such directions, for the convexity and the sphericity of these submanifolds.

Resumo

São apresentados neste trabalho os conceitos de direção assintótica e de direção binormal para mergulhos de codimensão dois de variedades em espaços euclideanos. Condições necessárias para a convexidade e a esfericidade de tais mergulhos são obtidas em termos das propriedades geométricas associadas a tais direções.

Introduction

Given an embedding, $f : M \to \mathbb{R}^{n+2}$ of a connected, closed, oriented n-manifold M, the family of height functions on M associated to f is defined as

$$\begin{array}{rcccl}\lambda(f) & : & M \times S^{n+1} & \longrightarrow & \mathbb{R} \\ & & (x,v) & \longmapsto & \langle f(x), v\rangle = f_v(x).\end{array}$$

For any fixed $v \in S^{n+1}$, we have a height function, f_v, on M, that satisfies: f_v has a singularity at $x \in M$ if and only if v is a normal vector to $f(M)$ at $f(x)$.

It has been shown by J. Montaldi ([10]) that the contact (or $\mathcal{K}$- equivalence) class of the function f_v at a point x characterizes the geometric contact of the submanifold $f(M)$ at the point $f(x)$ with the tangent hyperplane orthogonal to the vector v. For details on the notion of $\mathcal{K}$-equivalence of functions we refer to [10].

We shall say that a hyperplane H_v with orthogonal direction v has *higher order contact with* M at a point x if H_v is tangent to $f(M)$ at x and f_v has a degenerate (i.e., non Morse) singularity at x. By analogy with the case of curves in $\mathbb{R}^3$, where the binormal direction is precisely the unit vector whose corresponding height function has a degenerate singularity at the considered point, we shall call *binormal* of M at a point x to each direction v corresponding to a height function f_v having a degenerate singularity at x. The orthogonal hyperplane to such direction passing through the point x shall be called *osculating hyperplane* of M at x. Therefore, the osculating hyperplanes of M are those whose contact with M is of higher order.

In contrast with the case of curves in 3-space, for which we know that there exists a unique binormal (and hence a unique osculating plane) at each of their points of non zero curvature, submanifolds of higher codimension do not necessarily have binormals, nor they need to be unique at all their points. The case of surfaces in 4-space was studied in [9] where it was proven that the number of binormals at each point may be zero, one or two (it was called accordingly elliptic, parabolic or hyperbolic point). When M is compact, there is always an open region of M composed of hyperbolic points. Interesting examples of surfaces with boundary having no binormal directions are given by minimal surfaces in 4-space ([4]). On the other hand, convex surfaces in $\mathbb{R}^4$ illustrate the case of manifolds having binormal directions at all their points.

The first section of this paper is devoted to the study of the existence and determination of the maximum possible number of binormals at a point in a generic submanifold of codimension 2 in a euclidean space. We prove the following results:

Theorem 1

Given a generic, compact n-manifold M embedded in $\mathbb{R}^{n+2}$,

a) if n is odd, all the points of M admit at least one binormal direction, and at most n of them;

b) if n is even, there is always an open region of M all whose points admit at least one binormal direction and at most n of them.

Theorem 2

Local convexity is a sufficient condition for the existence of at least one binormal direction at every point of an even dimensional submanifold embedded in codimension 2 in $\mathbb{R}^{n+2}$.

In the second section we define the asymptotic directions. These can be seen as those directions along which the contact of M with an osculating hyperplane takes place. They provide direction fields (non necessarily globally defined) over M, whose singularities are the singularities of corank 2 of height functions on

M. We remark that in the particular case of $n = 2$, these are precisely the inflection points of M (see [9]).

Finally, in section 3 we define the concept of contact direction for a submanifold and a hypersurface in $\mathbb{R}^n$ and characterize the directions of curvature of hypersurfaces and asymptotic directions of submanifolds of codimension 2 in euclidean spaces as contact directions in appropriate settings. By means of this we show that the stereographic projection transforms lines of curvature of hypersurfaces in $(n+1)$-space into asymptotic lines of their spherical images in $(n+2)$-space, and as a consequence we stablish the following condition for the sphericity of submanifolds of codimension two,

If a generic, compact n-submanifold of $\mathbb{R}^{n+2}$ lies in a hypersphere, then it must have a field of orthonormal frames for the tangent bundle made of asymptotic directions, globally defined except over at most at subset of measure zero composed of singular points of this field.

1 Existence of osculating hyperplanes

Given an embedding, $f : M \longrightarrow \mathbb{R}^{n+2}$ of a n-manifold, M, into $\mathbb{R}^{n+2}$, we can put f in the Monge form in a neighbourhood of any point $p \in M$,

$$\begin{aligned} f : (\mathbb{R}^n, 0) &\longrightarrow (\mathbb{R}^n \times \mathbb{R}^2, (0,0)) \\ x &\longmapsto (x, f_1(x), f_2(x)). \end{aligned}$$

Then if $v = (v_1, ..., v_n)$ is a **unit vector** in $\mathbb{R}^{n+2}$, we can locally write

$$f_v(x_1, ..., x_n) = \sum_{i=1}^{n} v_i x_i + \sum_{j=1}^{2} v_{n+j} f_j.$$

Consequently, we have that p is singular point of the function f_v if and only if $v = (0, ..., 0, v_{n+1}, v_{n+2})$.

Now, if we denote the 2-jets of the components f_i (i=1,2) of the embedding f by

$$j^2 f_i(0) = \sum_{s=1}^{n} a^i_{ss} x_s^2 + 2 \sum_{s<t} a^i_{st} x_s x_t, \text{ with } \quad a_{st} = a_{ts}; \quad s, t = 1, ...n;$$

$i = 1, 2$, we obtain the Hessian of f_v at the point p as

$$\det \mathcal{H}(f_v(0)) = 2^n H^n_p(v_{n+1}, v_{n+2})$$

where H^n_p denotes a homogeneous polinomial of degree n in 2 variables that depends on the coefficients a^i_{st} of the 2-jets $j^2 f_i(0)$, $i = 1, 2, ..., n$.

It follows that p is a degenerate singularity of f_v if and only if $H_p^n(v_{n+1}, v_{n+2}) = 0$. And hence, the binormal directions of M at the point p can be characterized as the roots of polynomial H_p^n.

The *canal hypersurface* of the submanifold M in $\mathbb{R}^{n+2}$ is given by

$$CM = \{(p, v) \in M \times S^{n+1} : v \perp T_{f(p)}f(M)\}$$

Consider CM as a hypersurface of $\mathbb{R}^{n+2}$ through the natural embedding, $\tilde{f} : CM \longrightarrow \mathbb{R}^{n+2}$, and denote the family of height functions, the Gauss map and the Gaussian curvature function on CM respectively by $\lambda(\tilde{f}) : CM \times S^{n+1} \to \mathbb{R}$, $\Gamma : CM \longrightarrow S^{n+1}$ and $\mathcal{K} : CM \longrightarrow \mathbb{R}$. It is not difficult to verify that $(p, v) \in CM$ is a singular point of Γ if and only if the point $p \in M$ is a degenerate singular point of the function f_v on M.

Theorem 1
Given a generic, compact n-manifold M embedded in $\mathbb{R}^{n+2}$,

a) if n is odd, all the points of M admit at least one binormal direction, and at most n of them;

b) if n is even, there is always an open region of M all whose points admit at least one binormal direction and at most n of them.

Proof: Consider a point $p \in M$ and the height function

$$f_v(x_1, ..., x_n) = \sum_{i=1}^{n} v_i x_i + \sum_{j=1}^{2} v_{n+j} f_j$$

defined in a neighbourhood of p. Then, as we have seen, v is a binormal direction at p if and only if $v = (0, ..., 0, a, b)$ and $H_p^n(a, b) = 0$. Now, if we consider the decomposition of the polynomial H_p^n over the field $\mathbb{C}$,

$$H_p^n(a, b) = (\alpha_1 a + \beta_1 b)((\alpha_2 a + \beta_2 b)(...)(\alpha_n a + \beta_n b),$$

analising the possible real solutions of $H_p^n(a, b) = 0$ for each $p \in M$ leads to a subdivision of M into regions $M(i_1, ..., i_r) = \{p \in M : H_p^n$ admits r distint real roots with respective multiplicities $i_1, ..., i_r\}$. We denote by M_k the subset of points of M whose corresponding polynomial has only simple roots, exactly k of them being real, $k = 1, ..., n$, and by M^k those for which H_p^n has k real roots counted with their corresponding multiplicities, that is, $M^k = \cup M(i_1, \ldots, i_r), i_1 + \ldots + i_r = k$. We observe that M_k is open in M and that $M^k \subseteq cl(M_k)$ (where cl(-) denotes the closure operator). If we put $M^* = \bigcup_{k=1}^{n}(M_k)$, it follows that $M = cl(M^*) \bigcup M_0$.

We now show that if n is even, $M_0 \neq M$ and hence $M^* \neq \emptyset$. We do this by contradiction. Suppose that $M_0 = M$. This means that the Gauss map

$\Gamma : CM \longrightarrow S^{n+1}$ has maximal rank at every point and hence $\mathcal{K}(p,v) \neq 0$ for all $(p,v) \in CM$. But this implies that the connected, compact, oriented hypersurface CM is strictly convex (see [13]). Consequently CM must be homeomorphic to S^{n+1}, which is impossible for CM has the homotopy type of $M \times S^1$ and $\pi_1(CM) \neq \pi_1(S^{n+1}) = \{0\}$. So assertion b) is proven.

On the other hand, if n is odd, we have that H_p^n must admit at least one and at most n real roots, for all $p \in M$. Hence $M_0 = \emptyset$ and $M = cl(M^*)$ and there is at least a binormal direction at each point of M. Clearly, the maximum number of binormal directions at each point is n, for each one of them comes from a real root of $H_p^n(a,b) = 0$.

□

Corolary 1 (Existence of osculating hyperplanes)

Any generically embedded n-submanifold M in $\mathbb{R}^{n+2}$ admits osculating hyperplanes over some nonempty open submanifold.

A submanifold of $\mathbb{R}^{n+2}$ is said to be *locally convex* if it admits a locally support hyperplane at every point.

Theorem 2

Let M be a generic even dimensional locally convex n-submanifold of $\mathbb{R}^{n+2}$. Then M admits at least a **binormal vector** *at each one of its points.*

Proof: We shall show that M_0 is empty in this case. Suppose that $p \in M_0$ and let H be a support hyperplane of M at p. Then if v is a unit vector orthogonal to H, we have that $v \in N_{f(p)}f(M)$ and it is possible to choose local coordinates (by putting $f : M \longrightarrow \mathbb{R}^{n+2}$ in Monge form) in such way that $f_v(x) = f_2(x)$ in a neighbourhood of the origin. Moreover, we can assume that $j^2 f_2(0) = x_1^2 + ... + x_n^2$. Then if $Q(q_i)$ represents the matrix associated to the quadratic part of f_i, (i=1,2) and we take any w in $N_{f(p)}f(M)$, that must be of the form $(0,...0,w_{n+1},w_{n+2})$, we will have: $det(Q(q_1) + \frac{w_{n+2}}{w_{n+1}}Q(q_2)) = 0 \iff H_p^n(w_{n+1},w_{n+2}) = 0 \iff \frac{w_{n+2}}{w_{n+1}}$ is an eigenvalue of $Q(q_1)$. But, this implies the existence of binormal direction at p, contrary to the fact that $p \in M_0$.

□

2 Asymptotic Directions

It is well known that the parabolic set $\mathcal{K}^{-1}(0)$ of CM coincides with the singular set of the Gauss map Γ of CM, and that generically it is a regular submanifold of codimension one of CM off a subset of measure zero, $\Sigma^*(\Gamma)$, made of singularities of corank ≥ 2 of the map Γ. We denote by P the n-dimensional submanifold $\mathcal{K}^{-1}(0)\backslash\Sigma^*(\Gamma)$ of CM . Observe then that there is a unique asymptotic principal direction of curvature of CM at each point of P, whereas at the points of $\Sigma^*(\Gamma)$ we can find a whole tangent plane made of asymptotic directions.

Let $\xi : CM \longrightarrow M$ be the natural projection, taking each couple (p, v) in CM to the point p of M, and denote by $\bar{\xi}$ its restrition to the submanifold P. We then write $P_k = \bar{\xi}^{-1}(M_k)$ and $P^* = \bar{\xi}^{-1}(M^*) = \bigcup_{k=1}^{n} P_k$.

The behaviour of the map $\bar{\xi}$ over P is described by the following

Proposition

a) $\bar{\xi} : P^* \longrightarrow M^*$ *is a local diffeomorphism.*

b) $\bar{\xi}|_{P_k} : P_k \longrightarrow M_k$ it is a k-fold covering map.

c) $p \in M(i_1, ..., i_r)$ *if and only if there are unit vectors* $v_1, ..., v_r$, *such that* $(p, v_j) \in \mathcal{K}^{-1}(0)$, $j = 1, ..., r$. *Now, if* $(p, v_j) \in P$ *then* (p, v_j) *is a singularity with Boardman symbol* $\sum^{\overbrace{1, \cdots\cdots, 1}^{i_j-1}, 0}$ *of* $\bar{\xi}$.

Proof: Given $(p, v) \in P$ we have that v is a binormal direction, and either $p \in M_k$ or $p \in M^k$ for some $k = 1, ..., n$. We can choose a coordinate system $U \times W$ for CM at (p, v) in such a way that U is an orthogonal coordinate system for M at p and

i) $f(x) = (x, f_1(x), f_2(x))$

ii) $f_v(x) = f_2(x), \quad \forall x \in U$

iii) $\frac{\partial f_i}{\partial x_j}(0) = 0 \quad$ for $i = 1, 2$ and $j = 1, ..., n$

and W is a coordinate system for S^{n+1} at $v = (0, ..., 0, 1)$ obtained by identifying some open neighbourhood of v in S^{n+1} with its image through stereographic projection $s : S^{n+1} - (0, ..., 0, 1) \longrightarrow I\!R^{n+1}$. In this coordinates we have that the Jacobian matrix of the map Γ at the point (p, v) is the following

$$\begin{bmatrix} (f_1)_{x1x1}a + (f_2)_{x1x1} & \cdot \quad \cdot & (f_1)_{x1xn}a + (f_2)_{x1xn} & (f_1)_{x1} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ (f_1)_{x1xn}a + (f_2)_{x1xn} & \cdot \quad \cdot & (f_1)_{xnxn}a + (f_2)_{xnxn} & (f_1)_{xn} \\ 0 & \cdot \quad \cdot & 0 & 1 \end{bmatrix}$$

which also coincides with the Hessian matrix, $\mathcal{H}((\tilde{f}_v))(p, v)$, of the height function $\tilde{f}_v$ at the point (p, v). And hence $\mathcal{K}(p, v) = H_p^n(a, 1) \quad$ [*], where $v = (0, ..., 0, a, 1)$.

Moreover, $\mathcal{K}(0, 0) = H_0^n(a, 1) = 0$. And thus 0 is a degenerate critical point of f_v if and only if a is a root of $H_p^n(a, 1) = 0$. Furthermore, from [*] we have that a is:

i) a simple root if and only if $\frac{\partial \mathcal{K}}{\partial a}(0,0) \neq 0$

ii) a root with multiplicity r if and only if $\frac{\partial \mathcal{K}}{\partial a}(0,0) = ... = \frac{\partial \mathcal{K}^{(r-1)}}{\partial a^{(r-1)}}(0,0) = 0$ and $\frac{\partial \mathcal{K}^r}{\partial a^r}(0,0) \neq 0$.

The result now follows from observing that if $(p,v) \in P^*$, and thus $p \in M_k$, for some k, so there are k binormal vectors $v_1, ..., v_k$ with $v = v_i$, for some $i = 1, ..., k$, whose coordinates are solutions of the equation $H_p^n(a,1) = 0$. And hence we can find k points of P_k being mapped by $\bar{\xi}$ to the point p. Now, the condition i) is equivalent to $\bar{\xi}$ being a diffeomorphism in a neighbourhood of (p,v). This concludes the proof of assertions a and b.

Suppose now that $(p,v) \in P^k$, then $p \in M^k$ and thus $p \in M(i_1, ..., i_r)$ for $i_1 + ... + i_r = k$ with $i_j \neq 1$, for some i_j. And therefore we can find r unit vectors $v_1, ..., v_r$ such that $(p, v_j) \in P^k$ with $v = v_j$, for some $j = 1, ...r$. Moreover, the coordinates of v are roots of multiplicity i_j of the equation $H_p^n(a,1) = 0$ and condition ii) means then that the kernel of the projection $\bar{\xi}\,|_{P^k}$ is not contained in $T_{(p,v)}P^k$, and thus (p,v) must be a singular point of Boardman type (see [5]) $\Sigma^{\overbrace{1,........,1}^{i_j-1},0}$ of $\bar{\xi}$. The reciprocal is immediate. □

Observe in the above proof that given $(p,v) \in CM$ we can write $D\Gamma(p,v) = \mathcal{H}(\tilde{f}_v)(p,v) = \mathcal{H}(f_v)(p,v) \oplus I_1$, where I_1 represents the identity map over some 1-dimensional subspace of $T_{(p,v)}CM$.

Given any point $p \in M$ let v a binormal vector for M at p. Then, the couple (p,v) lies on the parabolic set $\mathcal{K}^{-1}(0)$ of CM. We call *asymptotic direction* of M at p the direction determined by the unit vector $\theta \in T_{f(p)}f(M)$, which is the image under $D\bar{\xi}$ of any asymptotic principal direction of curvature for CM at (p,v). Notice that the map $D\bar{\xi} : T_{(p,v)}CM \to T_pM$ takes the kernel of $D\Gamma(p,v)$ to the kernel of the Hessian of the height function f_v. And thus we can view an asymptotic direction θ at p as an element of $Ker(\mathcal{H}(f_v)(p))$, where v is some binormal direction at p.

Then, if (p,v) lies in the open and dense subset P of $\mathcal{K}^{-1}(0)$, since $Ker D\Gamma(p,v)$ has dimension one, we have that there is a unique asymptotic principal direction of curvature at (p,v). And it follows from the theorems 1 and 2,

Corollary 2 (Existence of asymptotic directions)
Any generically embedded n-submanifold M in $\mathbb{R}^{n+2}$ admits asymptotic directions over each point of some nonempty open submanifold M^*. Moreover, if n is odd or n is even and M is convex, then there is at least one asymptotic direction at each point of M.

The asymptotic directions define line fields on $cl(M^*)$ whose integral lines are called asymptotic lines. Their singular points are the images, through ξ of the points in $\Sigma^*(\Gamma)$. These singularities are isolated points in the case of surfaces in $I\!R^4$ ([4]), but they form curves in the case of 3-manifolds in $I\!R^5$ ([8]). In any case, it follows that they are singularities of corank ≥ 2 for some height function on M, and by looking to the classification of real functions ([2]) we can conclude that generically, for $n \leq 5$, they must be of umbilic type (D_k). We call them *inflection points*, by analogy with the case $n = 2$.

3 Stereographic projections

Let X_i and Y_i, $i = 1,2$ be submanifolds of $I\!R^n$, with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. The contact of X_1 and Y_1 at a point y_1 is said to be of the same type as the contact of X_2 and Y_2 at a point y_2 if there is a diffeomorphism-germ $H : (I\!R^n, y_1) \to (I\!R^n, y_2)$, such that $H(X_1) = X_2$ and $H(Y_1) = Y_2$. In this case we shall write $\mathcal{K}(X_1, Y_1; y_1) = \mathcal{K}(X_2, Y_2; y_2)$.

J. Montaldi ([10]) proved that given immersion-germs $g_i : (X_i, x_i) \to (I\!R^n, y_i)$, $i = 1,2$ and maps $f_i : (I\!R^n, y_i) \to (I\!R^p, 0)$ cutting off Y_i, we have that

$$\mathcal{K}(X_1, Y_1; y_1) = \mathcal{K}(X_2, Y_2; y_2) \Leftrightarrow f_1 \circ g_1 \sim_{\mathcal{K}} f_2 \circ g_2,$$

where $\mathcal{K}$ is the Mather's contact group. The map $\phi_i = f_i \circ g_i$ is called the *contact map* for X_i and Y_i, $i = 1,2$. Suppose now that $p = 1$, so Y_i is a hypersurface and ϕ_i is a function on $I\!R^n$, and that it has a degenerate singularity at the point x_i, $i = 1,2$. This means that the Hessian, $\mathcal{H}(\phi_i)$, defines a degenerate quadratic form, i.e., there is some unit vector $u_i \in T_{x_i}X_i$, such that $\mathcal{H}(\phi_i)(u_i, v) = 0$, $\forall v \in T_{x_i}X_i$, $i = 1,2$. We call such a vector, a *contact direction* for X_i and Y_i at $y_i = g_i(x_i)$. In fact, the contact of any curve through x_i in X_i with tangent direction u_i with the submanifold Y_i at the point x_i is of higher order (i.e., the corresponding contact map has a degenerate singularity at x_i) than that of any other curve through x_i in X_i (whose corresponding contact map has a Morse singularity at x_i).

Lemma 1 *In the above situation , the diffeomorphism H takes contact directions for the submanifolds X_1 and Y_1 at the point y_1 into contact directions for X_2 and Y_2 at y_2.*

Proof: If $H : I\!R^n \to I\!R^n$ is a diffeomorphism such that $H(X_1) = X_2$ and $H(Y_1) = Y_2$, then $h = H|_{X_1} : X_1 \to X_2$ is a diffeomorphism that satisfies $H \circ g_1 = g_2 \circ h$. We also have that $(f_2 \circ H)^{-1}(0) = f_1^{-1}(0)$, so we can write, by the Hadamard's lemma that $f_2 \circ H(y) = f_1(y)a(y)$, where $a : I\!R^n \to I\!R$ is a function such that $a(y_1) \neq 0$. In this case, we have $f_2(g_2(h(x))) = f_2(H(g_1(x))) = f_1(g_1(x))a(g_1(x))$. That is, $\phi_2(h(x)) = \phi_1(x)b(x)$ with $b : X_1 \to I\!R$, such that $b(x_1) \neq 0$.

If we differentiate now, we obtain

$$D\phi_2(h(x)) \circ Dh(x) = D\phi_1(x)b(x) \; + \; \phi_1(x)Db(x)$$

and by differentiating again,

$$D^2\phi_2(h(x))(Dh(x), Dh(x)) + D\phi_2(h(x)) \cdot D^2h(x) =$$

$$D^2\phi_1(x)b(x) + D\phi_1(x) \cdot Db(x) + \phi_1(x)D^2b(x) + D\phi_1(x) \cdot Db(x).$$

But, for $x = x_1$ we have $\phi_1(x_1) = 0$, and also $D\phi_2(h(x_1)) = D\phi_2(x_2) = 0$ (for x_2 is a singular point of ϕ_2), $D\phi_1(x_1) = 0$ (for x_1 is a singular point of ϕ_1). And thus, it follows

$$D^2\phi_2(x_2)(Dh(x_1), Dh(x_1)) = D^2\phi_1(x_1) \cdot b(x_1), \text{ with } b(x_1) \neq 0.$$

Therefore, $u_1 \in Ker(\mathcal{H}(\phi_1))(x_1) \Leftrightarrow Dh(x_1)(u_1) \in Ker(\mathcal{H}(\phi_2))(x_2)$. □

Let $g : \mathbb{R}^{n-1} \to \mathbb{R}^n$ be an embedding of a hypersurface $X = g(\mathbb{R}^{n-1})$ in $\mathbb{R}^n$, and consider the family of distance squared functions on X,

$$\begin{array}{rccl} \phi_{(c,r)} & : & \mathbb{R}^{n-1} & \longrightarrow \mathbb{R} \\ & & x & \longmapsto ||g(x) - c||^2 - r^2. \end{array}$$

with $(c, r) \in \mathbb{R}^n \times \mathbb{R}_+$.

Porteous established in [11] the correspondence between the extrinsic geometry of a generically embedded submanifold X and the singularities of the family of distance squared functions. In fact, it can be seen that the function $\phi_{(c,r)}$ has a singularity at a point x_0 if and only if the sphere, $S(c, r)$, centered at c and with radius r is tangent to X at $g(x_0)$. Moreover, the point x_0 is a degenerate singularity of $\phi_{(c,r)}$ if and only if c is a focal point of X (r being the principal curvature radius corresponding to the center c of principal curvature at $g(x_0)$).

Lemma 2 *With the above notation, a vector $u \in T_{g(x_0)}X$ is a principal curvature direction of X at $g(x_0)$ if and only if u is a contact direction for X and $S(c, r)$ at $g(x_0)$.*

Proof: By differentiating the function $\phi_{(c,r)}$ we get

$$D\phi_{(c,r)}(x) = 2(g(x) - c) \cdot Dg(x),$$

$$D\phi^2_{(c,r)}(x) = 2Dg(x) \cdot Dg(x) + 2(g(x) - c) \cdot D^2g(x).$$

Then we have that $u \in T_{g(x_0)}X$ is a contact direction for X and $S(c, r)$ at $g(x_0)$ if and only if $D\phi^2_{(c,r)}(x_0)(u, v) = 0$, $\forall v \in T_{g(x_0)}X$. Which is equivalent to saying that

$$Dg(x_0)(u) \cdot Dg(x_0)(v) = (c - g(x_0)) \cdot D^2g(x_0)(u, v), \forall v \in T_{g(x_0)}X,$$

or in other words,

$$II_{x_0}(u,v) \;=\; kI_{x_0}(u,v),\;\; \forall v \in T_{g(x_0)}X,$$

where $II_{x_0}(-,-)$ and $I_{x_0}(-,-)$ respectively represent the second and first fundamental forms of $g(X)$. But this is equivalent to saying that u is a principal direction of curvature at $g(x_0)$, being $k = \frac{1}{c-g(x_0)} = \frac{1}{r}$ its associated principal curvature. □

Consider now the family, $\lambda(g)$ of height functions associated to the embedding $g : X^{n-1} \to I\!R^n$. As we said before, the function g_v has a degenerate singularity at x if and only if x is a singular point of the Gauss map $\gamma : X \to S^{n-1}$. It is always possible to find a local coordinate system at x, for which $D\gamma(x) = \mathcal{H}(g_v)(x)$. Consequently, the kernel of the Hessian of g_v at the point x is given by the eigenvectors of $D\gamma(x)$ corresponding to null eigenvalues (i.e. vanishing principal curvatures) and we get the following,

Lemma 3
With the above notation, u is an asymptotic direction at a parabolic point $g(x)$ of a hypersurface X if and only if u is a contact direction for X and the affine hyperplane orthogonal to v through the point x.

Suppose now that $f : I\!R^{n-2} \to I\!R^n$ is an embedding of a submanifold M of codimension two in $I\!R^n$. Remember that the asymptotic directions at a point x were obtained as images under the differential, $D\xi(x,b)$, of asymptotic principal directions of curvature at the points (x,b) of the canal hypersurface CM of M, where b represents a binormal vector of M at x. And since the Gauss map $\Gamma : CM \to S^{n-1}$ of CM satisfies $D\Gamma(x,b) = \mathcal{H}(f_b)(x) \oplus I_1$, we obtain the following result:

Lemma 4
The unit vector $u \in T_xM$ is an asymptotic direction at $x \in M$ if and only if u is a contact direction for M and some osculating hyperplane of M at x.

Consider now the inverse of the stereographic projection,

$$\chi : I\!R^{n+1} \to S^{n+1} - \{(0,0,...,0,1)\}.$$

Observe that, given a hypersphere $S(a,r)$, centered at a point a of $I\!R^{n+1}$ with radius r, its image by χ is a hypercircle $C(a,r)$ in S^{n+1} whose contact with $\chi(M)$ is the same than that of $S(a,r)$ with M. Now, if we take $S(a,r)$ to be the osculating hypersphere of M, we have that the hyperplane $H(a,r)$ of $I\!R^{n+2}$ containing $C(a,r)$ must be the osculating hyperplane of $\chi(M)$ at $\chi(a)$ (see [9]). Then lemma 2 tells us that χ carries the contact direction of $S(a,r)$ and M at a to the contact direction of $C(a,r)$ and $\chi(M)$ at $\chi(a)$. But the first one is a curvature direction of M, whereas the second one is an asymptotic direction of $\chi(M)$. So we have shown

Theorem 3

The inverse of the stereographic projection carries the lines of curvature of any hypersurface M of $\mathbb{R}^{n+1}$ into the asymptotic lines of the n-submanifold $\chi(M)$ of $\mathbb{R}^{n+2}$.

An immediate consequence of this is

Corollary 3

A necessary condition for a n-submanifold M of $\mathbb{R}^{n+2}$ to be spherical is that it has an orthonormal basis of its tangent space composed of asymptotic directions at every point except at most at a subset of measure zero made of singular points of corank ≥ 2 of the generalized Gauss map over M.

We wonder whether this condition is also sufficient. In any case it follows that any spherical n-submanifold of $\mathbb{R}^{n+2}$ must have the maximum possible number (n) of osculating planes at each point, whereas locally convex n-manifolds need to have at least one osculating plane at each point (as we have shown in theorem 2).

And from the above corollary we get

For $n \leq 5$, generic nonparallelizable n-manifolds spherically embedded in $\mathbb{R}^{n+2}$, must have umbilic points for some of their height functions.

As a final example, we look at the 3-dimensional ellipsoid with distinct axes in $\mathbb{R}^4$. It was shown independently in [7] and [6], that it has 4 closed curves of hyperbolic umbilics. Applying Theorem 3, we then obtain an example of a spherical ellipsoid in $\mathbb{R}^5$, which has 4 closed curves of inflection points. It would be interesting to know whether this property also holds for any 3-sphere generically embedded as a convex submanifold of $\mathbb{R}^5$, in the sense that it is contained in the boundary of its convex hull. The case $n = 2$ was studied in [4] and it was shown that generically embedded convex spheres in $\mathbb{R}^4$ have at least 4 inflection points.

This example can be extended and formulated as a conjecture for any dimension n. In fact, the following conjecture, due to Arnol'd [1],
The set of umbilic points of the n-ellipsoid with distinct axes in $\mathbb{R}^{n+1}$ is made of $2(n-1)$ spheres of dimension $n-2$,
after Theorem 3, is equivalent to the following:
The set of inflection points of the corresponding spherical n-ellipsoid in $\mathbb{R}^{n+2}$ is made of $2(n-1)$ spheres of dimension $n-2$.

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