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SEMINAR ON FLAG MANIFOLDS
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Abstract

In these notes we present the state of the span problem for flag manifolds up to now, the technics involve Lie groups theory, characteristic classes, spectral sequences and representation rings of classical groups. We touch the main points of these too.

Resumo

Nestas notas apresentamos o estado atual do problema do posto linear para as variedades flâmula até o momento. As técnicas envolvem a teoria dos grupos de Lie, classes características, seqüências espectrais e os anéis de representações dos grupos clássicos. Nós tocamos em cada um dos pontos principais destes.

Introduction

These Lectures (1-8) deal mainly with flag manifolds and the question of find which are parallelizable. Lectures 1-5, are independent of each other. However Lectures 6-8 use representationn theory of compact. Lie groups, so a reader not fammiliar with this material would probably want to read say [4]. A working knowladge of spectral sequences is also useful in reading these notas, as well as some basics of differential toppology and vector bundles.

Although the primary goal, the parallelizability (and stable parallelizability) of flag manifolds is fairly specific, the lectures include a variety of techniques that are applicable to many other questions. These are vector bundle tecniques, multilinear algebra, characteristic classes, homotopy theory, k-theory (the Atiyah-Hirzebruch and the Hodgkin spectral sequences), and others. One method not treated in these notas (but mentioned briefly in Lecture 5) is the application of Lie algebras.

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1 Lecture

Flag Manifolds and Their Tangent Bundles

The main reference for this leiture is Lam [14]. To set up notation, let k be the field $I\!R$ or $C\!\!\!\!I$ ($I\!H$ can also be treated with a little extra care). The vector space k^n has the Hermitian inner product $< x, y >= \sum_{i=l}^{n} x_i \overline{y}_i$, and $G_k(n)$ is the group of k-linear transformations preserving the inner product.

k	d	$G_l(n)$	$\dim_{I\!R} G_l(n)$
$I\!R$	1	$O(n)$	$\binom{n}{2}$
$C\!\!\!\!I$	2	$U(n)$	n^2

$(d = \dim_{I\!R} k)$

1.1 Definition: $F = F_k(n_1, \ldots, n_s), n_1 + \cdots + n_s = n,$ is the flag manifold (variedade flâmula) whose elements are "flags" $\sigma = (\sigma_1, \ldots, \sigma_s)$ where $\sigma_1, \ldots, \sigma_s$ are mutually orthogonal subspaces of k^n and $\dim \sigma_i = n_i$.

Of course this implies $\sigma_l \oplus \cdots \oplus \sigma_s = k^n$. As examples of flag manifolds one has the Grassmann manifolds $F_k(n_l, n_2) = G_k(n_l, n_2) = G^k_{n,n_1}$ of n_l-planes in k^n (in particular $F_k(1, n-l) = kP^{n-l}$, and the "classical" flag manifolds $F_k(1, 1, \ldots, 1)$.

1.2 Let $A \in G_k(n)$ and write it in terms of its (mutually orthonormal) row vectors $A = \begin{pmatrix} a_l \\ \vdots \\ a_n \end{pmatrix}$. Then $\psi(A) = (\sigma_l, \ldots, \sigma_1)$, where $\sigma_1 = \text{span}\{a_l, \ldots, a_n\}$, $\sigma_2 = \text{span}\{a_{n_l+1}, \ldots, a_{n_l+n_2}\}$, ... gives a map $\varphi : G_k(n) \to F$ with isotropy $H = G_k(n_l) \times \cdots \times G_k(n_s)$, and this induces a bijection φ as shown by the commmutative diagram

$$\begin{tikzcd} G_k(n) \arrow[r, "\varphi"] \arrow[dr] & F \arrow[d, "\varphi"] \\ & \dfrac{G_k(n)}{H} \end{tikzcd}$$

We can we φ to topologize F, indeed this makes F into a smooth manifold called a homogeneous space.

1.3 It is casy to calculate $\dim_{I\!R} F = \dim G_k(n) - \dim H = (d/2)(n^2 - \sum n_i^2,$ whence $\dim_k(F) = (1/2)(n^2 - \sum n_i^2)$. Note also that $n^2 = (\sum n_i)(\sum nj) = \sum n_i^2 + 2\sum_{i<j} n_i nj$, hence $\sum_{i<j} n_i nj = \dim_k F$.

1.4 There are natural k-vector bundles $\xi_1, \ldots, \xi_s$ over F of ranks respectively $n_1, \ldots, n_s$. The fibre of ξ_i, at a point $\sigma = (\sigma_1, \ldots, \sigma_s) \in F$, is simply σ_i.

1.5 Proposition: $\xi_1 \oplus \cdots \oplus \xi_s \approx n\varepsilon$, where $n\varepsilon$ denotes the trivial bundle of rank n.

Proof: $(\sigma, v_1, \ldots, v_s) \longrightarrow (\sigma, v_1 + \ldots v_s$, where $\sigma \in F$ and $v_i \in \sigma i$, gives a linear isomorphism of the two vector bundles.

1.6 Recall, from elementary linear algebra, the row echelon form (REF) and reduced row echelon form (RREF) obtained by Gaussian elimination on any $m \times n$ matrix A. And one has the following basic theorem from linear algebra.

Theorem: Let A, B be $m \times n$ matrices over any field k. Then row span $(A) =$ row span (B) iff RREF$(A) =$ RREF(B), where row span (A) is the subspace of k^n spanned by the m row vectors of A, and similarly for B.

1.7 For a $\mathbb{C}$-vector space V, the conjugate $\overline{V}$ is given $\overline{V} = V$ with the scalar product $z * v = \overline{z} \cdot v$, $z \in \mathbb{C}, v \in V$. For an $\mathbb{R}$-vector space V, $\overline{V} = V$ with the some scalar product.

Similarly one defines $\overline{\alpha}$ for a k-vector bundle α over a space X. We assume throughout that all the vector bundles we deal with have a hermitian metric, for example this is alwags true if the base space X is paracompact ([ ], p.23).

Proposition: For vector bundles α, β and $k = \mathbb{R}, \mathbb{C}$,

$$\mathrm{Hom}_k(\alpha, \beta) \approx \overline{\alpha} \oplus \beta.$$

Proof: For an inner product space V (hermitian inner product) let $\varphi : \overline{V} \oplus_k W \to \mathrm{Hom}_k(V, W)$ be given by $\varphi(v \oplus w)(v') =< v', v > w$. Note that φ is linear in v and in w. To see that φ is an isomorphism, let $v_1, \ldots, v_m$ be an orthonormal basis of V and $w_1, \ldots, w_n$ any basis of W. A basis for $\mathrm{Hom}_k(V, W)$ is $\{f_{ij}\}$, where $f_{ij}(v_\ell) = \delta_{i\ell} w_j$. Since $fij = \varphi(v_i \oplus w_j)$, φ is surjective and hence an isomorphism (the spaces are finite dimensional). One should also note that $\varphi(v \oplus w)$ is linear in v', thus $\varphi(v \oplus w) \in \mathrm{Hom}_k(V, M)$.

Now φ is clearly natural with respect to inner product preserving linear transformations $f : V \to V'$, and any linear transformations $g : W \to W'$. So using the methods of [20] §3 or [9] Ch.5, φ extends to a vector bundle isomorphism $\overline{\alpha} \oplus_k \beta \xrightarrow{\approx} \mathrm{Hom}_k(\alpha, \beta)$ as long as α is endowed with a hermitian metric. □

1.8 We now turn to the tangent bundle of $F = F_k(n_1, \ldots, n_s)$. Some intuition for the main result Theorem 1.10, can be obtained by imagining a "small" change in $\sigma = (\sigma_1, \ldots, \sigma_s) \in F$. For any vector $v \in \sigma_i$, the "'small"' change in v will be a vector orthogonal to σ_i, which "gives" a linear transformation in $\mathrm{Hom}_k(\sigma_i, \sigma_i^\perp) = \oplus_{j \neq i} \mathrm{Hom}_k(\sigma_i, \sigma_j)$. To make this precise, let $i < j$ and $f : \sigma_i \to \sigma_j$ linear. For convenience take $i = 1$, $j = 2$. For any $t \in \mathbb{R}$,

$$tf \in \mathrm{Hom}_k(\sigma_1, \sigma_2) = \mathrm{Hom}_k(\xi_1 \xi_2)_\sigma \,.$$

Let its graph be $\gamma(tf) = \{(v, t \cdot f(v)) : v \in \sigma_1, \} \subset \sigma_1 \oplus \sigma_2$. Note that $\dim_k \gamma(tf) = n$ and $\gamma(of) = \sigma_l \oplus o$. Also let $\gamma^\perp(tf)$ be the orthogonal complement of $\gamma(tf)$ in $\sigma_1 \oplus \sigma_2$, clearly $\dim \gamma^\perp(tf) = n_2$ and $\gamma^\perp(of) = o \oplus \sigma_2$.

Definition: Set $\sigma(t) = (\gamma(tf), \gamma^\perp(tf), \sigma_3, \ldots, \sigma_s$. Clearly this defines a smooth curve in F with $\sigma(o) = \sigma$.

Definition: $L(f) = D_t\sigma(o) \in (TF)_\sigma$, where TF is the total space of the tangent bundle T_F of F, and $(TF)_\sigma$ is the fibre over σ.

1.9 Our main task is to prove the follewing lemma.

Lemma: $L : \mathrm{Hom}_k(\xi_i, \xi_j) \to TF$, $i < j$, is injective.

Proof: As above set $i = 1$, $j = 2$ for convenience. Choose an orthonormal basis $v_1, \ldots, v_n$ for k^n such that

$$\begin{aligned} v_1, \ldots, v_n &= \text{basis for } \ \sigma_1, \\ v_{n_l+l}, \ldots, v_{n_l+n_2} &= \text{basis for } \ \sigma_2, \quad \text{etc.} \end{aligned}$$

For any $n \times n$ matrix A, we write A in block form using $n = n_1 + n_2 + \ldots + n_s$, both in rows and calumns. We consider now those A having block form

$$A = \begin{bmatrix} I_{n_1} & * & * & * & \ldots & * \\ 0 & I_{n_2} & * & * & \ldots & * \\ 0 & 0 & I_{n_1} & * & \ldots & * \\ & & \vdots & & & \\ 0 & 0 & 0 & 0 & \ldots & I_{n_s} \end{bmatrix}$$

Note first that any such A is in REF (not in general RREF). Let V_i = span of first $n_1 + \ldots + n_i$ rows of A with respect to the chosen basis $v_1, \ldots, v_n$. Since A is in REF, $\dim V_i = n_1 + \cdots + n_i$. Next sat $\sigma_1^A = V_1$ and, for $i > 1$, $\sigma_i^A = V_i \cap V_{i-1}^{\perp}$. Clearly $\dim \sigma i^A = n_i$ and $\sigma_1^A, \ldots, \sigma_s^A$ are mutually orthogonal. So define $\sum(A) := (\sigma_1^A, \ldots, \sigma_s^A) \in F$. For exemplo, $\sum(I_n) = \sigma$.

We now clain $\sum$ is injective. For write A in the form

$$A = \begin{bmatrix} A_l & & & & \\ 0 & A_2 & & & \\ 0 & 0 & A_3 & & \\ & \vdots & & & \\ 0 & 0 & 0 & \cdots & A_s \end{bmatrix}$$

where

$$\begin{aligned} A_1 &= \text{first } \ n_1 \ \text{ rows of } \ A = [I_{n_1} ** \ldots *] \ \text{ is } \ n_1 \times n, \\ A_2 &= \text{next } \ n_2 \ \text{ rows of } \ A = [I_{n_2} * \ldots *] \ \text{ is } \ n_2 \times (n - n_1), \\ &\vdots \\ A_s &= \text{last } \ n_s \ \text{ rows of } \ A = [I_s] \ \text{ is } \ n_s \times (n - n_1 - \ldots - n_{s-1}). \end{aligned}$$

Notice that each A_i is in RREF, $1 \leq i \leq s$. So by 1.6, A_1 is uniquely determined by $V_1 = \sigma_1^A$, hence also by $\sum(A)$. Now consider V_2 = span of first $n_1 + n_2$ rows of A. Since $A_1 = [I_{n_1} * \ldots *]$, it is clear that $\sigma_2^A = V_2 \cap V_1^{\perp}$ satisfies $\sigma_2^A \subset \mathrm{span}\,\{V_{n_1+1}, \ldots, V_{n_1+n_2}\}$. In this space A_2 is in RREP, hence σ_2^A (and also $\sum(A)$) determines A_2 uniquely, again by 1.6. Thus $\sum(A)$ determines A_1, A_2 uniquely and continuing this way clearly determines $A_1, A_2, \ldots, A_s$ uniquely, i.e. determines A uniquely.

Thus $\sum$ is injective and clearly smooth. Further, we have

$$\begin{aligned}
\dim\ \mathrm{Im}\ \sum &= n_1(n-n_1)+n_2(n-n_1-n_2)+\cdots+n_{s-l}(n-n_1-\cdots-n_{s-l}) \\
&\quad + n_s(n-n_1-\cdots-n_s) = \\
&= n^2-\sum n_i^2-n_2n_1-n_3(n_1+n_2)-\cdots-n_s(n_1+\cdots+n_{s-l}) = \\
&= n^2-\sigma n_i^2-\sum_{i>j} n_in_j = n^2-\sum n_i^2-\sum_{i<j} n_in_j = \\
&= \dim_k F \quad \text{by} \quad 1.3.
\end{aligned}$$

By the inverse function theorem $\sum$ is locally a diffeomorphism, hence is a homeomorphism onto Im $\sum$. By the invariance of domain, Im $\sum$ is open in F. Since $\sigma \in$ Im $\sum$, this forms a coordinate chart for the open neighborhood $U =$ Im $\sum$ of σ, with $0 \in k^n$ corresponding to σ. Now let

$$A_t = \begin{bmatrix} I_{n_1} & t\cdot M(f) & 0 & \cdots & 0 \\ 0 & I_{n_2} & 0 & \cdots & 0 \\ 0 & 0 & I_{n_3} & \cdots & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & \cdots & I_{n_s} \end{bmatrix},$$

where $M(f)$ is the matrix of f with respect to $V_1,\ldots,V_{n_1+n_2}$. From the above definitions $\sum(A_t)=\sigma_t$, and by differentiating with respect to t it is clear that $L(f)$ corresponds to

$$\begin{bmatrix} 0 & M(f) & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and this L is injective. □

1.10 Theorem: $\tau_F \approx \oplus_{1\le i<j\le s}\mathrm{Hom}_k(\xi_i,\xi_j)$.

Proof: Using the neighborhood U of σ as in the proof of 1.9, we see that L embedds $\oplus_{i<j}\mathrm{Hom}(\xi_i,\xi_j)$ into τ_F, since $L(\mathrm{Hom}_k(\xi_i,\xi_j))$ is in the ij block. Again, since $\sum n_in_j =$ $\dim_k F$ by 1.3, this map is an isomorphism. □

Corollary: $\tau_F \approx \oplus_{1\le i<j\le s}\xi_i\otimes\xi_j$.

We now make a few applications with $k = \mathbb{R}$, in which case $\tau_F \approx \oplus_{i<j}\xi_i\otimes\xi_j$. First recall a few properties of exterior powers of a vector bundle. One has rank $\lambda^i_k = \binom{\text{rank }\alpha}{i}$, $\lambda^0\alpha\approx\varepsilon$, $\lambda^1\alpha\approx\alpha$, $\lambda^2(\alpha\oplus\beta)\approx\oplus_{p+g=i}(\lambda^p\alpha\otimes\lambda^g\beta)$. Also, for a rank n orientable bundle α, there is the Hodge duality $\lambda^i\alpha\approx\lambda^{n-i}\alpha$.

1.11 Application to immersions: From 1.5, and with $k=\mathbb{R}$, $\xi_1\oplus\cdots\oplus\xi_s\approx n\varepsilon$,

$$\binom{n}{2}\varepsilon \approx \lambda^2(\xi_1\oplus\cdots\oplus\xi_s)\approx\lambda^2\xi_1\oplus_1\cdots\oplus\lambda^2\xi_s\oplus\sum_{i<j}\xi_i\oplus\xi_j\ ,$$

$$\binom{n}{2}\varepsilon \approx \eta \oplus \tau, \quad \text{where} \quad \eta = \sum_{i=1}^{s} \lambda^2 \xi_i \,.$$

By the Hirsch immersion theorem F immerses in $I\!R^{\binom{n}{2}}$, with codimension $=$ rank $\eta = \sum \binom{n_i}{2}$ (as long as $\sum \binom{n_i}{2} > 0$, i.e. $n_i \geq 2$ for some i). For further information about this, especially with regard to when c $\binom{n}{2}$ is the best possible immersion dimension, see also [28], [18].

1.12 We now consider the span of manifolds and state some of the basic definitions and theorems. The attached historical summary 1.16 may also be of interest.

(a) Write $\alpha \approx \beta$ for isomorphism of vector bundles, and $\alpha \sim \beta$ for stable isomorphism (or stable equivalence), meaning $\alpha \oplus m\varepsilon \approx \beta \oplus \eta\varepsilon$ for some m, n.

Definition: span$\alpha = \max\{r : \alpha \approx r\varepsilon \oplus r$ for some $\gamma\}$. The rank of γ, i.e. rank $\alpha -$ span α, is often called the geometric dimension (gd) of α.

(b) Definition: For a smooth M^n, span $M =$ span τ_M. Clearly $0 \leq$ span $M \leq n$.

(c) The lower extreme span $M = 0$ is settled by Hopf's theorem: For M connected, smooth, compact, without boundary, span $M = 0$ iff $X(M) \neq 0$. Equivalenty, span $M > 0$ iff $X(M) = 0$. Since any add dimensional manifold has Euler characteristic $\chi(M) = 0$, positive span is the case for "ver half" the manifolds, in as approximate sense.

(d) The upper extreme is span $M = n$, M is then called parallelizable (written par in these notes). Equivalently, $\tau_M \approx n\varepsilon$ is trivial. A move general condition is $\tau_M \sim 0$, i.e. τ_M is stably trivial; M is then called stably parallelizable (sp). An example of a sp manifold that is not par is S^2. Our main interest in these lectures will be determining which flag manifolds are par (or sp).

(e) For $n = 2^{4a+b}x$ (odd), where $0 \leq b \leq 3$ (a, b are unique), let $\rho(n) = 8a + 2^b$, the Hurwitz-Radon function of n. Then span $S^{n-1} = \rho(n) - 1$.

Exercise: Show span $S^{n-1} < \frac{n-1}{2}$ unless $n = 2, 4, 8, 16$, and hence only S^1, S^3, S^7 are par.

(f) Theorem (Bredon Kosinski): If M^n is stably parallelizable, and $n \neq 1, 3, 7$, then either span $M^n =$ span S^n $(= \rho(n+l) - l)$, or M is parallelizable.

The theorem contains further information to distinguish the two cases, which we omit here.

(g) Although the Bredon-Kosinski theorem effectively gives the span of sp manifolds, we shall see in the work that follows that relatively few flag manifolds (and other homogeneous spaces) are sp. For such manifolds (i.e. "most" manifolds), little is known about the span.

1.13 The classical flag manifold $F = F_{I\!R}(1, 1, \ldots, 1)$.

Proceding as in 1.11 and noting rank $\xi_i = l$, hence $\lambda^2\xi_i = 0$, we get $\tau_F \approx \binom{n}{2}\varepsilon$ and thus F is par.

Remark: $F_{\mathbb{C}}(1,\ldots,1)$ is sp but not par, this is proved in [14] using a variant of the λ^2 construction for the complex case.

We will also be interested, in case $k = I\!R$, in the case where each (or later some) subspace σ_i carries an orientation. This is written $\widetilde{F}(n_1,\ldots,n_s)$, clearly $\widetilde{F}(n_1,\ldots,n_s) \cong SO(n)/H$, where $H = SO(n_1)\times\cdots\times SO(n_s)$ (We also requive. That the orientations here on $\sigma_1,\ldots,\sigma_s$ induce the standard orientation on $I\!R^n$, otherwise the space would have two components). Note that $\widetilde{F}(n_1,\ldots,n_s)$ is thus a 2^{s-l}-fold covering space of $F(n_1,\ldots,n_s$.

2 Lecture

Stiefel Manifolds

We make a small digression from the main path of the lectures here to consider one specific family of manifolds, the Stiefel manifolds, in more detail. The Stiefel manifolds, $V_{n,r}$ are flag manifolds in the sense of Lect. 4, that is, "partially oriented" flag monifolds. Our goal is to show that for $r \geq 2$, $V_{n,r}$ is par. For references see [29], [6], [14], [32], [10].

2.1 Definition: $V_{n,r}\cdots$, for $1 \leq r \leq n$, is the space of orthonormal r-frames of vectors in $I\!R^n$.

We denote a point of $V_{n,r}$ as $\begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}$, where $a_i \in I\!R^n$ and $\langle a_i, a_j\rangle = \delta_{ij}$. One gives this space the structure of a smooth manifold by writing it as a homogeneous space

$$V_{n,r} = O(n)/(I_r \times 0(n-r))$$

Familiar examples are $V_{n,1} = S^{n-l}$, $V_{n,n} = 0(n)$, $V_{n,n-1} \cong S0(n)$. There are smooth fibrations $S^{n-r-1} \hookrightarrow V_{n,r+1} \xrightarrow{\pi} V_{n,r}$, where

$$\pi\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_{r+1} \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}.$$

2.2 There is also a smooth fibration, in fact a 2^r-fold covering projection, $\varphi : V_{n,r} \twoheadrightarrow F_{I\!R}(1,\ldots,1,n-r)$, where

$$\varphi\begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} = (\text{span }\{a_1\},\ldots\text{span }\{a_r\},\ \text{span}^{perp}\{a_1,\ldots,a_r\})$$

.

Now $F_{I\!R}(1,\ldots,1,n-r)$ has the canonical bundles $\xi_1,\ldots,\xi_r,\xi_{r+1}$. Clearly $\varphi^*(\xi_i) \approx \varepsilon$, $1 \leq i \leq r$, and we call $\beta_{n,r} := \varphi^*(\xi_{r+1})$ the orthogonal complement bundle (of rank $n-r$). The next lemma is an easy application of 1.3, 1.5, and Cor. 1.10, its proof is left as an exercise. Heve $\tau_{n,r} = \tau(V_{n,r})$ result, so all such $\binom{V}{V_r} \in (\tau_{n,r})_A$. On the other hand these equations reduce to the $r+\binom{r}{2}=\binom{r+1}{2}$ equations homotopic. Using same basic topology of the Stiefel manifolds, one knows $V_{2n,3}$ is $(2n-4)$-connected.

Lemma:

(a) $\beta_{n,r}\oplus r\varepsilon\approx n\varepsilon\,,$

(b) $\tau_{n,r}\approx r\beta_{n,r}\oplus\binom{r}{2}\varepsilon\,,$

(c) $\dim V_{n,r}=nr-\binom{r+1}{2}.$

Remark: The complex and quaternionic Stiefel manifolds of orthonormal r-frames in $\mathbb{C}^n,\mathbb{H}^n$ can be defined similarly. There is no change in the above lemma expect for (c), which becomes real dim $=dnr-r-d\binom{r}{2}$, $d=1,2,4$ respectively.

2.3 Theorem: $V_{n,r}$ is parallelizable, $r\geq 2$.

This theorem was first proved by Sutherland [29]. A little later Handel [6] and Lam [14] noticed that for $r\geq 3$ the theorem could be proved by elementary vector bundle techniques. Alternatively, notice that. Lemma 2.2 trivially shows $V_{n,r}$ is sp, $r\geq 1$, and the Bredon-Kosinski theorem will then give a quick proof that $V_{n,r}$ is par, $r\geq 2$. Here, however, we shall follow [32] to give a complete proof of Theorem 2.3 using nothing move than a basic theorem of homotopy theory, the Hopf classification theoremm: for $n\geq 2$, $\dim X\leq n$, $(n-1)$-connected,

$$[X,Y]\approx H^n(X;\pi_n(Y))\,.$$

First, an alternative description of the tangent bundle $\tau_{n,r}$.

2.4 Lemma: Let $A=\begin{pmatrix}a_1\\ \vdots\\ a_r\end{pmatrix}\in V_{n,r}$. Then

$$(\tau_{n,r})_A=\left\{\begin{pmatrix}v_1\\ \vdots\\ v_r\end{pmatrix}: v_i\in\mathbb{R}^n,\ \langle a_i,v_j\rangle+\langle v_i,a_j\rangle=0\,,\ 1\leq i,j\leq r\right\}.$$

Proof: Consider a smooth curve $A(t)=\begin{pmatrix}a_1(t)\\ \vdots\\ a_r(t)\end{pmatrix}$, with $a_i(0)=v_i$. Differentiating the equations $\langle a_i(t),a_j(t)\rangle=0$ with respect to t and setting $a_i(0)=v_i$ gives the desired result, so all such $\begin{pmatrix}v_1\\ \vdots\\ v_r\end{pmatrix}\in(\tau_{n,r})_A$. On the other hand these equations reduce to the $r+\binom{r}{2}=\binom{r+1}{2}$ equations $\langle a_i,v_i\rangle=0$, $1\leq i\leq r$, and $\langle a_i,v_j\rangle+\langle a_j,v_i\rangle=0$, $1\leq i<j\leq r$, so the solution space has dimension $\geq nr-\binom{r+1}{2}=\dim V_{n,r}$ by 2.2 (c). This shows that the solution

space is $(\tau_{n,r})_A$, and also that the set of equations are linearly independent, i.e. have rank $\binom{r+1}{2}$. $\square$

Corollary: For $r \geq 2$, $\tau_{n,r}$ admits $\binom{r}{2}$ linear sections

$$w_{ij}(A) = \begin{pmatrix} v_1(A) \\ \vdots \\ v_r(A) \end{pmatrix} \quad , \quad \text{where} \quad v_k(A) = \begin{cases} 0 \quad , & k \neq i,j \\ -a_j \quad , & k = i \quad , \quad i < j \\ a_i \quad , & k = j \end{cases}$$

2.5 We now turn to the case $r = 2$, and follow some ideas of Eckmann. The 3 equations $\langle a_1, v_1 \rangle + \langle a_2, v_2 \rangle = \langle a_1, v_2 \rangle = \langle a_2, v_1 \rangle = 0$ are considered as a map $f : V_{n,2} \to V_{2n,3}$,

$$f\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ a_2/\sqrt{2} & a_1/\sqrt{2} \end{pmatrix} \quad ,$$

and a unit vector field on $V_{n,2}$ is simply a lifting g of f to $V_{2n,4}$,

$$\text{i.e.} \quad g\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ a_2/\sqrt{2} & a_1/\sqrt{2} \\ v_1 & v_2 \end{pmatrix} \quad ,$$

indead an r-field (orthonormal) will be a lifting g as in

$$\begin{tikzcd} & V_{2n,3+r} \arrow[d, "\pi"] \\ V_{n,2} \arrow[ur, "g"] \arrow[r, "f"'] & V_{2n,3} \end{tikzcd}$$

Since π is a Serre fibration, such a lifting always exists (for any $r \leq \dim V_{n,2} = 2n-3$) if $f \simeq 0$, and this is what we shall show, this proving $V_{n,2}$ parallelizable.

2.6 Theorem: $V_{n,2}$ is parallelizable.

Proof: As shown in 2.5, it suffices to prove f null homotopic. Using some basic topology and the $Z/2$-cohomology of the Stiefel manifolds, one knows $V_{2n,3}$ is $(2n-4)$-connected, $\dim V_{n,2} = 2n-3$, so the Hopf classification theorem applies. Also $\pi_{2n-3}(V_{2n,3}) \approx H_{2n-3}(V_{2n,3}) \approx H_{2n-3}(\mathbb{R}P^{2n-1}/\mathbb{R}P^{2n-4}) \approx Z/2$. So by the Hopf classification it suffices to show

$$0 = f^* : H^{2n-3}(V_{2n,3}; Z/2) \to H^{2n-3}(V_{n,2}; Z/2) \; .$$

Using the Cor. to 2.4, we have a lift g

$$\begin{tikzcd} & V_{2n,4} \arrow[d, "\pi"] \\ V_{n,2} \arrow[ur, "g"] \arrow[r, "f"] & V_{2n,3} \end{tikzcd} , \quad \text{where} \quad g\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ a_2/\sqrt{2} & a_1/\sqrt{2} \\ -a_1/\sqrt{2} & a_2/\sqrt{2} \end{pmatrix}.$$

Now $H^{2n-3}(V_{2n,3}; Z/2) \approx Z/2$, generated by a class we write y_{2n-3}. Since $\pi^*(y_{2n-3}) = y_{2n-3} \in H^{2n-3}(V_{2n,4}) \approx Z/2$, it suffices to show $g^*(y_{2n-3} = 0$. Now fairly obvious homotopies of $V_{2n,4}$ show $g \simeq g_1 \simeq g_2 \ldots$, where

$$g_1(A) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ a_2 & 0 \\ 0 & a_1 \end{pmatrix} , \quad g_2(A) = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \\ 0 & a_1 \\ 0 & a_2 \end{pmatrix}$$

Let $\sigma : V_{n,2} \times V_{n,2} \to V_{2n,4}$, $\sigma(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, so that $g_2 = \sigma\Delta$, where $\Delta : V_{n,2} \to V_{n,2} \times V_{n,2}$ is the diagonal. Consider the involution $\omega : V_{2n,4} \to V_{2n,4}$ which interchanges the first two rows with the last two and similarly for the columns. Clearly $\omega\sigma = \sigma\tau$, where $\tau : V_{n,2} \times V_{n,2} \to V_{n,2} \times V_{n,2}$ switches the two factors. Now ω being an involution and $H^{2n-3}(V_{2n,4}; Z/2) \approx Z/2$, generated by $y_{2n-3}, \ldots$ implies $\omega^*(y_{2n-3}) = y_{2n-3}$. It follows that $\tau^*\sigma^*(y_{2n-3}) = \sigma^*\omega^*(y_{2n-3}) = \sigma^*(y_{2n-3})$, i.e. $\sigma^*(y_{2n-3})$ is invariant under τ^*. But $H^*(V_{n,2}; Z/2)$ has only the classes $x_{n-2}, x_{n-1}, x_{n-1}x_{n-2}$ in respective dimensions $n-2, n-1, 2n-3$. Thus $\sigma^*(y_{2n-3}) = \alpha(x_{n-2} \times x_{n-1} + x_{n-1} \times x_{n-1}x_{n-2}) + \beta(x_{n-1}x_{n-2} \times 1 + 1 \times x_{n-1}x_{n-2})$, for some $\alpha, \beta \in Z/2$. Taking Δ^* of this gives $0 = \Delta^*\sigma^*(y_{2n-3}) = g_2^*(y_{2n-3}) = g^*(y_{2n-3})$, as required. □

2.7 Lemma: Under $p : V_{n,r+1} \to V_{n,r}$, $p^*(\tau_{n,r}) \oplus \beta_{n,r+1} \approx \tau_{n,r+1}$.

The proof is left as an exercise, using either 2.2 or 2.4

2.8 Theorem: Vn, r is parallalizable, $r \geq 2$.

Proof: Note $n \geq r + 1$ and $r \geq 2$ imply $2r \geq r + 2$ and hence $2nr \geq (r + 2)(r + l)$, or equivalently $d_{n,r} = nr - \binom{r+1}{2} \geq r + 1$.

Now use induction, $2 \leq r \leq n - 1$, and suppose $\tau_{n,r}$ trivial.
Theorem 2.6 starts the induction, and

$$\begin{aligned} \tau_{n,r+1} \approx p^*(\tau_{n,r}) \oplus \beta_{n,r+1} &\approx d_{n,r}\varepsilon \oplus \beta_{n,r+1} \text{ (inductive hypothesis)} \\ &\approx (d_{n,r} - r - 1)\varepsilon \oplus (r+1)\varepsilon \oplus \beta_{n,r+1} \\ &\approx (d_{n,r} - r - 1)\varepsilon \oplus n\varepsilon \approx d_{n,r+1}\varepsilon \end{aligned}$$

using that $d_{n,r} - r - 1 \geq 0$ as shown above. □

Remark: The proofs for $k = \mathbb{C}, \mathbb{H}$ are similar, except that the case $r = 2$ (Theorem 2.6) turns out to be much easier as far as showing $f^* = 0$. See [32] for details.

2.9 Questions: (a) Can one give a fully elementary proof that $V_{n,2}$ is par? For $n \equiv 0 \pmod{4}$, yes (cf. [14], or try as an Exercise).

(b) Can one exhibit the vector fields on $V_{n,2}$, $V_{n,r}$, say with polynomial formulae? For $n = 4, 8$, done in [32].

(c) (Daciberg Lima Gonçalves): When does $\beta_{n,r}$ admit a section. Answer: $r = 1$ and n even, $r = n-1$, or $(n,r) = (7,2), (8,3)$.

(d) (Daciberg Lima Gonçalves): When does $\ell\beta_{n,r}$ admit a section, $\ell > 1$? What is span $(\ell\beta_{n,r})$?

2.10 Definition: $X_{n,r} = V_{n,r} / (A \sim -A)$, $1 \le r \le n-1$ is the projective Stiefel manifold. One has obvious covering projections

$$V_{n,r} \xrightarrow{(2)} X_{n,r} \xrightarrow[(2^{r+1})]{q} F(1, \dots, 1, n-r) ,$$

so $\dim X_{n,r} = \dim V_{n,r} = nr - \binom{r+1}{2}$, also there is a Hopf line bundle $\xi_{n,r}$ over $X_{n,r}$. One has $q^*(\xi_i) \approx \xi_{n,r}$, $1 \le i \le r$, and defines $q^*(\xi_{r+1}) := \beta_{n,r}$.

3 Lecture

Parallelizability of Standard Flag Manidolds

The main references here are [12], [21].

3.1 Preliminary facts (to be dealt with in Lecture 5).

(a) For $k = I\!R$, all Grassmann manifolds $F_{I\!R}(p,q) = G_{p+q,p} \cong G_{p+q,q}$ are not sp, except for $p = 1, q = 1, 3, 7$ (or vice-versa) which are par, namely $G_{2,1} = I\!RP^1$, $G_{4,1} = I\!RP^3$, $G_{8,1} = I\!RP^7$. For $k = \mathbb{C}, I\!H$, only $F_k(1,1)$ are sp (and not par).

(b) The projective Stiefel manifolds $X_{n,2}$ are not sp unless $n = 3, 4, 8$, and $X_{3,2}$, $X_{4,2}$, $X_{8,2}$ are par.

3.2 Inclusion method Many inclusion and fibration maps exist between flag manifolds. The inclusions are obtained by increasing the size of the ambient space k^n, and the fibrations by collapsing some of the subspaces of the subspaces of a flag into their (direct) sum, i.e. into the single subspace they span.

Example: Let $F' = F_k(n_1, \dots, n_{s-1}) \dots$, $F = F_k(n_1, \dots, n_s)$, and θ be the span $\{e_{n-n_s+1}, \dots, e_n\} \subseteq k^n$. Then $i : F' \hookrightarrow F$, where $i(\sigma_1, \dots, \sigma s - 1 = (\sigma_1, \dots, \sigma_{s-l}, \theta)$ is an inclusion of flag manifolds.

Let $p : F \to F(m, n_s)$, where $m = n_1 + \dots + n_{s-1} = n - n_s$, be given by $p(\sigma_1, \dots, \sigma_{s-1}, \sigma_s) = (\sigma_1 \oplus \dots \oplus \sigma_{s-1}, \sigma_s)$. Then p is a fibration and its fibre is precisely F' with nclusion i.

Exercise: Show this same fact using the homogeneous space definition for $F', F, F(m, n_s)$. The key to using the inclusion method is the nest lemma.

3.3 Lemma: $i^*(\tau_F) \sim \tau_{F'}$.

Proof: Clearly $i^*(\xi_i) \approx \xi_i'$, $1 \le i \le s-1$, while $i^*(\xi_s) \approx n_s\varepsilon$. By Theorem 1.10,

$$\begin{aligned} i^*(\tau_F) &= i^*\left(\sum_{1\le i<j\le s-1} \mathrm{Hom}_k(\xi_i,\xi_j) \oplus \sum_{i=1}^{s-1} \mathrm{Hom}_k(\xi_i,\xi_s)\right) \\ &\approx \sum_{1\le i<j\le s-1} \mathrm{Hom}_k(\xi_i',\xi_j' \oplus \sum_{i=1}^{s-1} \mathrm{Hom}_k(\xi_i', n_s\varepsilon) \\ &\approx \tau_{F'} \oplus \mathrm{Hom}_k\left(\sum_{i=1}^{s-1}\xi_i', n_s\varepsilon\right) \approx \tau_{F'} \oplus \mathrm{Hom}_k((n-n_s)\varepsilon, n_s\varepsilon) \sim \tau_{F'}.\square \end{aligned}$$

Remark: This is a special case of the fact that for any smooth fibration, the inclusion of the fibre has trivial normal bundle (for consider a small enough coordinate neighborhood U of a point $x \in X$, the base space, then $p^{-1}(U)$ is diffeomorphic to $U\times$ fibre).

3.4 Theorem: Let $s \ge 3$, $k = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.
(i) $F_k(1,\dots,1)$ is sp, and is par iff $k = \mathbb{R}$.
(ii) If $n_i > 1$ for some i, then $F_k(n_1,\dots,n_s)$ is not sp.

Proof: Without loss of generality suppose $n_1 \ge n_2 \ge \dots \ge n_s$. For (i) the case $k = \mathbb{R}$ is already done in 1.13, and it was remarked there that for $k = \mathbb{C}$ (also $\mathbb{H}$) $F_k(1,\dots,1)$ is sp. In the cases $k = \mathbb{C}, \mathbb{H}$, $\chi(F_k(1,\dots,1)) \ne 0$ so it cannot be par, indeed has span $= 0$ by 1.12 (c).

For (ii), suppose $F_k(n_1,\dots,n_s)$ is sp, then by the inclusion method $F_k(n_1,n_2)$ is also sp. By 3.1(a), the only possibilities are $k = \mathbb{R}$, $(n_1,n_2) = (1,1),(3,1),(7,1)$, or $k = \mathbb{C},\mathbb{H}$ and $(n_1,n_2) = (1,1)$. So, if also $n_1 > 1$, it is only possible that $k = \mathbb{R}$ and $(n_1,n_2) = (3,1),(7,1)$. The proof will then be completed, again by the inclusion method, if we can show $F_{\mathbb{R}}(3,1,1)$ and $F_{\mathbb{R}}(7,1,1)$ not sp. The double cover $X_{5,2} \to F_{\mathbb{R}}(3,1,1)$ and $X_{5,2}$ not sp (see 3.1 (b)) shows $F_{\mathbb{R}}(3,1,1)$ not sp, and similarly for $F_{\mathbb{R}}(7,1,1)$ using $X_{9,2}$. $\square$

4 Lecture

Parallelizability of Partially Oriented Flag Manifolds

The main reference for this lecture is [22], and $k = \mathbb{R}$ throughout.

4.1 Definition: $F(n_1,\dots,n_r \mid n_{r+1},\dots,n_s)$ is the usual flag manifold $F(n_1,\dots,n_s)$ except that the subspaces $\sigma_1,\dots,\sigma_r$ also carry an orientation, with the convention that if $r = s$ then the orientations on $\sigma_1,\dots,\sigma_s$ induce the standard orientation of $\mathbb{R}^n$.

Thus $F(n_1,\dots,n_r \mid n_{r+1},\dots,n_s)$ is a 2^r convering space of the standard flag manifold $F(n_1,\dots,n_s)$, $r < s$ (or a 2^{r-1} fold covering if $r = s$). Note that with the above convention $F(n_1,\dots,n_{s-1} \mid n_s) \cong F(n_1,\dots,n_s \mid)$, and we write this space $\widetilde{F}(n_1,\dots,n_s)$ in analogy to the oriented Grassmann manifolds $\widetilde{G}_{n,r}$. As a homogeneous space $F(n_1,\dots,n_r \mid n_{r+1},\dots,n_s)$, for $r \le s-2$, is $O(n)\,/SO(n_1)\times\ldots\times SO(n_r)\times O(n_{r+1})\times\cdots\times O(n_s))$, while $\widetilde{F}(n_1,\dots,n_s) \cong SO\,/S0(n_1)\times\cdots\times SO(n_s)$.

4.2 Examples: (a) $\tilde{F}(n_1, n_2) = \tilde{G}_{n,n_1} = \tilde{G}_{n,n_2}$, in particular $\tilde{F}(1, n-1) = S^{n-1}$

(b) $F(1, \dots, 1 \mid n-r) = \tilde{F}(1, \dots, 1, n-r) = V_{n,r}$

(c) $F(\mid 1, 1) \cong X_{m+2,2}$

4.3 The tangent bundle τ of $F(n_1, \dots, n_r \backslash n_{r+1}, \dots, n_s)$ is given by the same formula as 1.10, Cor., since it pulls back under the covering projection. The $\tau \approx \sum_{i<j} \xi_i \oplus \xi_j$, with the extra information that ξ_i is an oriented bundle of rank n_i, $1 \leq i \leq r$.

The tecniques used now are for the most part (but not entirely) the same as in the previous work: the λ^2-construction for positive results, and the inclusion method for negative results.

4.4 Preliminary facts (to be dealt with in Lec. 5)

(a) $\tilde{F}(p, q) = \tilde{G}_{p+q,p} = \tilde{G}_{p+q,q}$ is sp iff $p = 1$, or $q = 1$, or $(p, q) = (2, 2), (3, 3)$, and of these only $\tilde{G}_{2,1} = S^1$, $\tilde{G}_{4,1} = S^3$, $\tilde{G}_{8,1} = S^1$, $\tilde{G}_{6,3}$ are par.

(b) $X_{n,2}$ is not sp unless $n = 3, 4, 8$ (as in 3.1 (b), also $X_{10,4}$ is not sp. $X_{3,2}, X_{4,2}, X_{8,2}$ are all par.

We now study the par (sp) of these manifolds. We consider only $s \geq 3$, since the case $s = 2$ is already done in 4.4 (a). And, without loss of generality, we always suppose $n_1 \geq \dots \geq n_r, n_{r+1} \geq \dots \geq n_s$.

4.5 Theorem: Suppose $n_2 > 1$. Then $\tilde{F}(n_1, n_2, \dots, n_s)$ is not sp unless $\{n_1, \dots, n_s\} \subseteq \{3, 1\}$ or $\{n_1, \dots, n_s\} \subseteq \{2, 1\}$.

Proof: Write $n_1 \geq n_2 \geq \cdots n_p > n_{p+1} = \cdots = n_s = 1$. By the inclusion method, $\tilde{F}(n_1, \dots, n_s)$ sp implies $\tilde{F}(n_i, n_j)$ sp for any $1 \leq i < j \leq s$. In particular, for $1 \leq i < j \leq p$, using 4.4(a), we see $n_i = n_j = 2$ or $n_i = n_j = 3$. This completes the proof. $\square$

Corollary: Suppose at least two of $n_1, \dots, n_s$ are > 1. Then $F(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$ is not sp unless $\{n_1, \dots, n_s\} \subseteq \{3, 1\}$ or $\{2, 1\}$.

This follows because the sp of $F(n_1, \dots, n_r \mid n_{r+1}, \dots, n_s)$ would imply the sp of its covering space $\tilde{F}(n_1, \dots, n_s)$.

Remark: We are always assuming $r \geq 1$, ortherwise we are back in the case of standard flag manifolds dealt with in Lec. 3.

4.6 Lemma: The following are sp: (a) $F(\underbrace{1, \dots, 1}_{r}) \mid 1, \dots, 1)$,

(b) $F(2, \dots, 2, 1. \dots, 1 \mid 1, \dots, 1)$,

(c) $\tilde{F}(3, \dots, 3, 1, \dots, 1)$.

Proof: (a) Here we have a covering space of $F(1, \dots, 1)$ so it is in fact par by 1.13.

(b) Left as an exercise - use λ^2.

(c) Here $\xi_1 \oplus \cdots \ni \xi_k \oplus \xi_{k+1} \oplus \cdots \oplus \xi_s \approx n\varepsilon$, with $\xi_1, \dots, \xi_k$ oriented rank 3, $\xi_{k+1}, \dots, \xi_s$ oriented rank 1, i.e. trivial line bundles. So

$\xi_1 \oplus \cdots \oplus \xi_k \oplus (s-k)\varepsilon \approx n\varepsilon$. The λ^2 construction gives

$\tau \oplus \lambda^2\xi_1 \oplus \cdots \oplus \lambda^2\xi_k \oplus \binom{s-k}{2}\varepsilon \approx \binom{n}{2}\varepsilon$, now Hodge duality gives

$$\tau \oplus \lambda^1\xi_1 \oplus \cdots \oplus \lambda^1\xi_k \oplus \binom{s-k}{2}\varepsilon \approx \binom{n}{2}\varepsilon\,, \text{ and } \lambda^1\xi_i \approx \xi_i \text{ gives}$$

$$\tau \oplus (\xi_1 \oplus \cdots \oplus \xi_k) \oplus \binom{s-k}{2}\varepsilon \approx \binom{n}{2}\varepsilon\,,$$

$$\tau \oplus (\xi_1 \oplus \cdots \oplus \xi_k \oplus (s-k)\varepsilon) \oplus \binom{s-k}{2}\varepsilon \approx \binom{n}{2}\varepsilon\,,$$

$$\tau \oplus \quad n\,\varepsilon \quad \oplus \binom{s-k}{2}\varepsilon \approx \binom{n}{2}\varepsilon\,, \text{ i.e. } \tau \sim O. \qquad \square$$

Notice that 4.4 and 4.5 completely determine which $\tilde{F}(n_1, \ldots, n_s)$ are sp, so we may suppose $1 \leq r \leq s-2$ henceforth. The next theorem gives most of the remaining negative results.

4.7 Theorem: Consider $F = F(n_1, \ldots, n_r \mid n_{r+1}, \ldots, n_s)$, $1 \leq r \leq s-2$, and suppose either precisely one $n_q > 1$, or $\{n_1, \ldots, n_s\} = \{3, 1\}$, or $\{n_1, \ldots, n_s\} = \{2, 1\}$ with $n_{r+1} = 2$. Then F is not sp in the following cases.

(i) $F(n_1, \ldots, n_r \mid 1, 1, \ldots, 1)$, $n_i \neq 1, 2, 6$ for some $i \leq r$,

(ii) $F(n_1, \ldots, n_r \mid 1, 1, \ldots, 1)$, $r \leq s-4$, $n_1 \neq 1, 2$ for some $i \leq r$,

(iii) $n_{r+1} > 1$, $r < s-2$,

(iv) $n_{r+1} > 1$, $r = s-2$, and $(n_{s-1}, n_s) \neq (3, 1, (7, 1)$.

Proof: For (i), the inclusion $F(n_1 \mid 1, 1) = X_{n_i+2,2} \hookrightarrow F$ shows F is not sp unless $n_i = 1, 2, 6$ (using 4.4 (b)).

For (ii), we may suppose using (i) that $n_i = 6$, so by inclusion it suffices to show $F(6 \mid 1, 1, 1, 1)$ not sp. But $X_{10,4} \longrightarrow F(6 \mid 1, 1, 1, 1)$ is a (8-fold) covering and $X_{10,4}$ is not sp (4.4(b)), hence $F(6 \mid 1, 1, 1, 1)$ is not sp.

For (iii) the inclusion $F(n_{r+1}, n_{r+2}, \ldots, n_s) \hookrightarrow F$ and Theorem 3.4 suffices.

For (iv), if $(n_{s-1}, n_s) \neq (3, 1), (7, 1)$, the inclusion $F(n_{s-1}, n_s) \hookrightarrow F$ shows F not sp. $\square$

4.8 If $r = s-2$, and $(n_{s-1}, n_s) = (7, 1)$, and $n_1 > 1$ then F is not sp by Cor. 4.5. However we need one extra negative result if $(n_{s-1}, n_s) = (3, 1)$ (Cor. 4.5 does not help here).

Theorem: For $k \geq 1$, $F = F(\underset{k}{3, \ldots, 3}, 1, \ldots, 1 \mid 3, 1)$ is not sp.

Proof: One shows $F(3 \mid 3, 1)$ not sp by computing its Stiefel-Whitney classes, see [22] for details. The inclusion method then completes the proof.

4.9 Proposition: $F(6 \mid 1, 1)$ and $F(6, 1 \mid 1, 1)$ are par.

Proof: The first equals $X_{8,2}$ and is covered by 4.4 (b). The second is more delicate, see [22] for details.

At this stage only a few questions remain. First, in the positive cases 4.6 (b), (c), which are par (in 4.6(a) all were observed to be par). Using results of Miatello-Miatello [19] and Stong [27], one can see that all manifolds in 4.6(b), (c) are par except for $\tilde{F}(2, 2, \ldots, 2)$ and $\tilde{F}(2, \ldots, 2, 1)$. As for the negative results (for sp), there are four unsolved families, as follows.

4.10 Unsolved: $F(1,\ldots,1 \mid 3,1)$, $s \geq 3$,
$F(1,\ldots,1 \mid 7,1)$, $s \geq 3$,
$F(6,1,\ldots,1 \mid 1,1)$, $s \geq 5$,
$F(6,1,\ldots,1 \mid 1,1,1)$, $s \geq 4$.

To see this, first consider any case with at least two $n_i, n_j > 1$. Using 4.5, $\{n_1,\ldots,n_s\} =$ either $\{2,1\}$ or $\{3,1\}$. If $\{2,1\}$, then $n_{r+1} = 2$ is eliminated by 4.7 (iii), (iv), leaving $F(2,\ldots,2,1,\ldots,1 \mid 1,\ldots,1)$ which is sp by 4.6. If $\{n_1,\ldots,n_s\} = \{3,1\}$, then 4.7 (i), (iii) and 4.8 eliminate all possibilities except $\widetilde{F}(3,\ldots,3,1,\ldots,1)$, and this is known to be sp by 4.6.

This leaves only the cases where precisely one $n_q > 1$, and these are easily seen to be the families listed using 4.7.

4.11 For completeness, we list what is now known.

Theorem: Let $F = F(n_1,\ldots,n_r \mid n_{r+1},\ldots n_s)$, with $r \geq 1$ and $s \geq 3$. The following are all stably parallelizable and apart from $\widetilde{F}(2,2,\ldots,2)$, $\widetilde{F}(2,\ldots,2,1)$ are parallelizable:

(i) $F(1,\ldots,1 \mid 1,\ldots,1)$,
(ii) $\widetilde{F}(n_1,\underbrace{1,\ldots,1}_{s}) = V_{n_1+1,s}$,
(iii) $wtF(3,\ldots,3,1,\ldots,1)$,
(iv) $F(2,\ldots,2,1,\ldots,1 \mid 1,\ldots,1$,
(v) $F(6 \mid 1,1) = X_{8,2}$, $F(6,1 \mid 1,1)$.

Any other flag manifold of this type is not stably parallelizable, apart from the four undecided families listed in 4.10. Our main goal in the following lectures is to introduce the tool of K-theory and show how it led to solutions for the undecided families.

5 Lecture

Introducing K-theory

5.1 In this lecture we shall consider the Stiefel and projective Stiefel manifolds, the Grassmann and oriented Grassman manifolds. These were already used in the "preliminaries" of Lec. 3,4. We shall now see how the use of K-theory can give quick proofs of parallelizability results for these manifolds. The Stiefel manifolds $V_{n,r}$ will be omitted, they have already been dealt with in Lec. 2 and in any case are all sp (par for $r \geq 2$).

5.2 To define K-theory (see also [9], Ch. 9), one starts with the Grothendieck completion of an abelian semi-group A, the same process by which one obtains the integers from the non-negative integers in elementary algebra. So let A be an abelian semi-group.

Definition: $\overline{A} = \{(a,b) : a,b \in A\} / \sim$, where $(a,b) \sim (a',b')$ iff $a + b' = a' + b$. One checks that $\sim$ is an equivalence relation. Then define $(a,b)+(c,d) = (a+c, b+d)$, check this is well defined, and $\overline{A}$ is now an abelian group with $0 - (a,a)$ for any $a \in A$ and $-(a,b) = (b,a)$. Further $A \subseteq \overline{A}$ with $a \longrightarrow (a,0)$.

5.3 Now consider A = abelian semigroup of isomorphism classes of (constant rank) F-vector bundles over a space X, with $[\alpha] + [\beta] = [\alpha \oplus \beta]$.

Definition: $\overline{A} = K_F(X)$.

Remarks: (a) The requirement of constant rank is redundant if X is path connected.
(b) $\text{rank} : A \to Z$ induces a well defined homomorphism $\text{rank} : K(X) \to Z$.
(c) Recall that for vector bundles of finite type (for X paracompact all vector bundles are of finite type), say α, there exists a vector bundle β such that $\alpha \oplus \beta$ is trivial (cf. [9], p. 32).

5.4 Definition: $\widetilde{F}(X) = \ker(\text{rank} : K_F(X) \to Z)$.

Notice that $K_F(X)$ also has a ring structure given by $[\alpha] \cdot [\beta] = [\alpha \otimes \beta]$, this induces a ring structure on $\widetilde{K}_F(X)$ as well. Since $\alpha \oplus \beta \approx \beta \oplus \alpha$, it is a commutative ring.

Remark: $\widetilde{K}_F(X)$ can equivalently be defined as the equivalence classes of stable vector bundles over X, but this definition needs care for multiplication since $\alpha_1 \sim \alpha_2$, $\beta_1 \sim \beta_2$ does not in general imply $\alpha_1 \otimes \beta_1 \sim \alpha_2 \otimes \beta_2$.

Remark: Notice that for a vector bundle α of rank n, $\alpha \sim 0$ iff $[\alpha] = 0 \in \widetilde{K}_F(X)$ iff $[\alpha] - n = 0 \in K_F(X)$. This K-theory is an ideal tool for detecting stable parallelizability.

5.5 One makes K-theory into a cohomology theory by defining $\widetilde{K}_F^{-p}(X) = \widetilde{K}(\Sigma^p X)$, where ΣX is the reduced suspension of X. According to the remarkable Bott periodicity theorems, $\widetilde{K}^{-p}(X) \approx \widetilde{K}^{-p+2}(X)$ and $\widetilde{KO}^{-p}(X) \approx \widetilde{KO}^{-p+8}(X)$; these allow us to define $\widetilde{K}_F^i(X)$ and hence also $K_F^i(X)$ for all $i \in Z$. Also, for $f : X \to Y$, the induced bundle construction $f^*(\beta)$, for any vector bundle β over Y, gives rise to a well defined homomorphism

$$\begin{array}{rcl} f^* : K_F(Y) & \longrightarrow & K_F(X) \\ [\beta] & \longmapsto & [f^*\beta] \end{array} ,$$

similarly $\Sigma^p f$ gives rise to $f^* : K_F^{-p}(Y) \longrightarrow K_F^{-p}(X)$. Altogether, these definitions turn the graded object $K_F^* = \{, \ldots, K_F^{-1}, K_F, K_F^1, K_F^2, \ldots, \}$ into a cohomology theory.

5.6 Recall (Lec. 3) the projective Stiefel manifold $X_{n,r}$, the covering projections $V_{n,r} \xrightarrow{(2)} X_{n,r} \underset{q}{\xrightarrow{(2^r-1)}} F(1, \ldots, 1, n-r)$ the Hopf bundle $\xi_{n,r} \approx q^*(\xi_i)$, $1 \le i \le r$, and $\beta'_{n,r} = q^*(\xi_{r+1})$ of rank $n-r$. Also define $\beta_{n,r} = \beta'_{n,r} \otimes \xi_{n,r}$ (also rank $n-r$), and recall $\xi \otimes \xi \approx \varepsilon$ for any real line bundle ξ. We have the following basic isomorphisms.

Lemma: (a) $\beta'_{n,r} \oplus r\,\varepsilon \approx n\,\xi_{n,r}$,
(b) $\beta_{n,r} \oplus r\,\xi n, r \approx n\,\varepsilon$,
(c) $\tau_{n,r} \approx r\beta'_{n,r} \oplus \binom{r}{2}\varepsilon$,
(d) $\tau_{n,r} \oplus \binom{r+1}{2}\varepsilon \approx nr\,\xi_{n,r}$ $\quad (\Longrightarrow \tau_{n,r} \sim nr\,\xi_{n,r})$
(e) $\tau_{n,r} \oplus \lambda^2\beta_{n,r} \approx \binom{n}{2}\varepsilon$.

Exercise: Derive (b) from (a), and derive (e) by taking λ^2 of (b), then using (c).

Deriving (a), (c), (d) is also not difficult using the known facts about $F(1,\ldots,1,n-r)$, cf. [14] or [32] for details.

5.7 One can already show that many $X_{n,r}$ are not sp, using Stiefel-Whitney classes and the $Z/2$-cohomology, $H^*(X_{n,r};Z/2)$, first found by Gitler-Handel ( ). There is always a generator $x \in H^1(X_{n,r};Z/2)$, and the Stiefel-Whitney class of $\xi_{n,r}$ is $w(\xi_{n,r}) = 1+x$. Then, since stably equivalent vector bundles have the same Stiefell-Whitney classes,

$$w(\tau_{n,r}) = w(nr\xi_{n,r}) = (w(\xi_{n,r})^{nr} = (1+x)^{nr} .$$

To compute the binomial coefficients mod 2. arising this way, the following result of Lucas [16], for any prime p, is useful.

Theorem: Let $n = \sum_{i\geq 0} d_i\, p^i$, $r = \sum_{i\geq 0} c_i p^i$, where $0 \leq c_i, d_i < p$, be the p-ary expansions of n, r. Then

$$\binom{n}{p} \equiv \prod_{i\geq 0} \binom{d_i}{c_i} \pmod p .$$

Proof: $(1+t)^n = (1+t)^{\Sigma d_i p^i} = \prod_{i\geq 0}(1+t)^{d_i p^i} \equiv \prod_{i\geq 0}(1+t^{p^i})^{d_i}$, since $(a+b)^{p^i} \equiv (a^{p^i}+b^{p^i})$ (mod p). Now simply compare the coefficient of t^r on both sides. $\square$

5.8 Examples: (a) For $X_{10,3}$, the work of Gitler-Handel shows $H^*(X_{10,3};Z/2)$ contains $x, x^2, \ldots, x^7 \neq 0$, $x^8 = 0$, and other generators x_8, x_9 that do not concern us here. So

$$w(\tau_{10,3} = (1+x)^{30} = (1+x)^{2\cdot 15} = (1+x^2)^{15} = 1 + x^2 + x^4 + x^6$$

since $15 = 1111$ in binary expansion. So $w(\tau) \neq 1$, hence $X_{10,3}$ not sp.

(b) For $X_{10,4}$, we similarly have $x^8 = 0$, so here $w(\tau_{10,4}) = (1+x)^{40} = (1+x^8)^5 = 1^5 = 1$, giving no information. Cases like (10,4), being more delicate, were not settled until the K-theory of $X_{n,r}$ was calculated, cf. [1], [2]. We will not go into $K^*(X_{n,r})$ here, but will quote the final answer.

5.9 Theorem: $X_{n,r}$ is par for $(n, n-1)$, $(2m, 2m-2)$, $(4,r)$, $(8,r)$, and (16,8). No other is sp except possibly $X_{12,8}$ which is undecided.

5.10 We now turn to the Grassmann manifolds $G_{n,r} = F_{\mathbb{R}}(r, n-r)$. Since $G_{n,r} \cong G_{n,n-r}$, we assume without loss of generality that $r \leq n/2$. The next result is just a repeat of 3.1 (a).

Theorem: $G_{n,r}$ is par for $r = 1$ and $n = 2,4,8$. Any other $G_{n,r}$ (with $r \leq n/2$) is not sp.

This theorem was first proved in [31] and a little later in [7], [11], and [30]. We will study the proof given in [30] since it is a very easy application of K-theory and also covers the complex and quaternionic Grassmann manifolds.

5.11 First recall the result of Adams (cf. [9], p. 237) :

$$\widetilde{KO}(\mathbb{R}P^n) \approx Z/2^{\varphi(m)}, \text{ where } \varphi(m) = |\{i : 1 \leq i \leq, \ i \equiv 0,1,2,4 \ (mod.8)\}|.$$

Furthermore the generator of this cyclic group is $[\xi_m] - 1$, or equivalently $2^{\varphi(m)}\xi_m \sim 0$. Here are some sample values.

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$m-1$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\varphi(m-1)$	1	2	2	3	3	3	3	4	5	6	6	7	7	7	7	8	9

Notice that $(\varphi(n)$ is approximately $n/2$, so that $2^{\varphi}(m-1)$ is much larger than $m-1$ except for the small values $m-1 \leq 8$. This makes it easy to prove the next lemma, we omit the proof.

Lemma: $m\,\xi_{m-1} \sim 0$ iff $m = 2,4,8$, and $(m-i)\xi_{m-1} \not\sim 0, \ i \geq 1$.

5.12 Proof of Theorem 5.10: For convenience we write the canonical bundles ξ_1, ξ_2 on $G_{n,r} = F_{\mathbb{R}}(r,n,n-r)$ as $\xi_1 = \beta, \ \xi_2 = \gamma$. On $G_{n-1,r-1}$ write these as β', γ', and under the inclusion $i : G_{n-1,r-1} \hookrightarrow G_{n,r}$, where $i(A) = \operatorname{span}(A, e_n)$, observe that $i^*(\gamma) \approx \gamma'$, $i^*(\beta) \approx \beta' \oplus \varepsilon$. It follows that

$$i^*(\tau_{n,r}) \approx i^*(\beta \otimes \gamma) \approx i^*\beta \otimes i^*\gamma \approx \tau_{n-1,r-1} \oplus \gamma' \ .$$

Iterating the inclusion i, $(r-1)$ times, gives an inclusion

$$\mathbb{R}P^{n-r} = F_{\mathbb{R}}(1, n-r) \overset{j}{\hookrightarrow} G_{n,r} \quad \text{with}$$

$$j^*(\tau_{n,r}) \approx \tau_{\mathbb{R}P^{n-r}} \oplus (r-1)\gamma_{\mathbb{R}P^{n-r}} \approx \xi_{n-r} \otimes \gamma_{n-r} \oplus (r-1)\gamma_{n-r} \ ,$$

where we write $\gamma_{n-r} = \gamma_{\mathbb{R}P^{n-r}}$.

Now recall (Lec. 1) $\xi_{n-r} \oplus \gamma_{n-r} \approx (n-r+1)\varepsilon$ implies $\varepsilon \oplus (\xi_{n-r} \otimes \gamma_{n-r}) \approx (n-r+1)\xi_{n-r}$, i.e. $\xi_{n-r} \otimes \gamma_{n-r} \sim (n-r+1)\xi_{n-r}$ and $\gamma_{n-r} \sim -\xi_{n-r}$. Substituting,

$$j^*(\tau_{n,r}) \approx (n-r+1)\xi \oplus (r-1)(-\xi) = (n-r+1-(r-1))\xi_{n-r} \ .$$

By Lemma 5.11 this can be stably equivalent to O iff $r = 1$ and $n - r + 1 = 2,4,8,$ i.e. $n = 2,4,8$.

Remark: The cases $k = \mathbb{C}, \mathbb{H}$ can be handled in the same way, and the final step is in fact easier because the Hopf bundle ξ_{n-r} turns out to have infinite order, in $\widetilde{K}(k\,P^{n-r})$.

5.13 We now turn to $\widetilde{G}_{n,k}$, the oriented Grassmann manifolds. Since $\widetilde{G}_{n,1} \cong S^{n-1} \cong \widetilde{G}_{n,n-1}$, we write $\widetilde{G}_{n,k} = \widetilde{F}(k,\ell)$ with $k + \ell = n$ and henceforth suppose $k, \ell > 1$ to dispense of the spheres. In Lec. 1 we saw that $\widetilde{F}(2,2)$ was sp and $\widetilde{F}(3,3)$ par, our aim now is to show that no other $\widetilde{F}(k,\ell)$, $k, \ell > 1$, is sp. This theorem was first stated and partially proved in [19], fully proved in [22], and again proved (using characteristic classes, the Pontrjagin classes) in [13]. In [23] $K^*(\widetilde{G}_{n,k})$ was computed and this leads to a very short proof of this theorem, which we now present at least in part. The work in [23] also settled which $\widetilde{G}_{n,k}$ admit an almost complex structure, a problem going back to the 50s, but we only consider the parallelizability here.

5.14 Let $k = 2s$ or $2s+1$, $\ell = 2t$ or $2t+1$. The description of $K(\tilde{G}_{n,k}) = K^o(\tilde{G}_{n,k})$ shows that this ring always contains a subalgebra $H_{s,t} \subseteq K(\tilde{G}_{n,k})$ defined as

$$H_{s,t} = Z[p_1, \ldots, p_s, q_1, \ldots, q_t] / ((1 + p_1 + \ldots + p_s)(1 + q_1 + \cdots + q_t) = 1) \ .$$

Further $H_{s,t}$ is graded via $\deg p_i = \deg q_i = i$. Thus one obtains $p_1 + q_1 = 0$ in degree 1, $p_2 + p_1 q_1 + q_2 = 0$ in degree 2, etc.

Remark: In the case k, ℓ both odd, $H_{s,t} = K(\tilde{G}_{n,k})$, for other parities $H_{s,t} \subset K(\tilde{G}_{n,k})$.

5.15 Lemma: $[c\tau] = p_1 q_1 + k q_1 + \ell p_1 + k\ell \in H_{s,t}, \subseteq K(\tilde{G}_{n,k})$.

Proof: Two tochnical things must be assumed here, first that the "K-theory Pontrjagin classes" p_1, q_1 are related to the exterior power representations $\lambda_k^1 \in R_{I\!R}\mathrm{Spin}(k)$, $\lambda_\ell^1 \in R_{I\!R}\mathrm{Spin}(\ell)$, by $c\lambda_k^1 = p_1 + k$, $c\lambda_\ell^1 = q_1 + \ell$, and second that the α-construction here satisfies $\alpha(\lambda_k^1 = [\beta]$, $\alpha(\lambda_\ell^1) = [\gamma]$ in $KO(\tilde{G}_{n,k})$.

$$\begin{array}{lrcl} \text{Then} & \alpha(\lambda_k^1 \otimes \lambda_\ell^1 & = & [\beta \otimes \gamma] = [\tau] \in KO(\tilde{G}_{n,k}), \ \text{ so} \\ & \alpha(c\lambda_k^1 \otimes c\lambda_\ell^1) & = & [c\tau] \in K(\tilde{G}_{n,k}) \, . \\ \text{Thus} & [c\tau] & = & \alpha(c\lambda_k^1) \cdot \alpha(c\lambda_k^1) \\ & & = & (p_1 + k)(q_1 + \ell) = p_1 q_1 + k q_1 + \ell p_1 + k\ell \ . \end{array}$$

□

Remark: If one is not familiar with the technicalities in the above proof, Lemma 5.15 can just be assumed and Lemma 5.16 can be directly studied.

5.16 Theorem: For $k, \ell \geq 2$, $\tilde{G}_{n,k}$ $sp \Longrightarrow (n,k) = (2,2)$ or $(3,3)$.

Proof: From 5.15, if also $\tilde{G}_{n,k}$ is sp, we have (since $\dim \tilde{G}_{n,k} = k\ell$) $k\ell = [\tau]$, so $k\ell = [c\tau] = p_1 q_1 + k q_1 + \ell p_1 + k\ell$, giving $k q_1 + \ell p_1 = 0$ in degree 1, and $p_1 q_1 = 0$ in degree 2. The second relation gives $0 = p_2 + p_1 q_1 + q_2 = p_2 + q_2$ which can only hold in $H_{s,t}$ if $p_2 = q_2 = 0$, i.e. $s = t = 1$. Since $p_1 + q_1 = 0$ in $H_{s,t}$, the first relation gives $(k - \ell) q_1 = 0$ which implies $k = \ell$. This leaves only $k = \ell = 2$ or $k = \ell = 3$. □

6 Lecture

Outline of the Hodgkin Spectral Sequence

6.1 The Hodgkin spectral sequence is a powerful tool to compute the (complex) K-theory of a homogeneous space X. Unfortunately steps 2,3 deal with the representation rings of compact Lie groups cf.[Brocker-Tom Dieck], [Husemoller], it is not available for KO-theory. The basic reference for it is the book [8]. In this lecture we give an 8-step "user's guide" to using this tool. Steps 4-6 involve some homological algebra, and details for these steps are given in Lec. 7.

Step 1 Write $X = G/H$ where G = compact, connected Lie group with $\pi_1(G) = 0$ and H is a closed subgroup. Frequently $G = \mathrm{Spin}(n)$ is used. We remark that $\pi_1(G)$ has no 2-torsion suffices, and write $i : H \hookrightarrow G$.

Step 2 Determine RH, RG, the (complex) representation rings of H, G. It is useful if RG is a polynomial algebra, this is the case for $G = \mathrm{Spin}(n)$.

Step 3 Calculate the "restriction" homomorphism $i^* : RG \to RH$. This also makes RH into an RG-module.

Step 4 $E_2 = \mathrm{Tor}_{RG}(RH, Z)$ gives the E_2 term of the Hodgkin spectral sequence.

Step 5 Simplify the Tor calculation, using the "change of rings" theorem, to get rid of redundant terms. One gets $\mathrm{Tor}_{RG}(RH, Z) \approx Tor_A(B, Z)$, with A, B having fewer generators, and tries to retain A = polynomial algebra.

Step 6 Use the Koszul resolution for $Tor_A(B, Z)$ calculation.

Step 7 Now having E_2, calcule the differentials in the spectral sequence to get $E_3, \ldots, E_\infty$. We remark that in all examples known to the author the spectral sequence has "callapsed", i.e. $0 = d_2 = d_3 = \cdots$, whence $E_2 = E_3 = \cdots = E_\infty$ and thus this step is trivial in these circumstances. See also 6.2 below.

Step 8 Reassemble $K^*(X)$, as a $Z/2$-graded algebra (due to Bott periodicity), from E_∞. An excellent source for this step is [1], 6.1-6.5.

6.2 Remark: Here are a few further details about the Hodgkin spectral sequence.

(a) $E_2^p = \mathrm{Tor}_{RG}^p(RH, Z)$ gives the p-grading, the q-grading is the $Z/2$-grading (periodic) induced by regarding $(RG)^0 = RG$, $(RG)^1 = 0$, similarly for RH and Z.

(b) $d_r : E_r^{p-r} \to E_r^p$, and $d_r = 0$ for r even.

(c) $E_2^{*,*}$ gives rise to a multiplicative spectral sequence, in particular if $\mathrm{Tor}_{RG}^*(RH, Z)$ is generated as an algebra by its elements in degree ≤ 1 then the spectral sequence collapses.

7 Lecture

Some Homological Algebra

In this lecture we review some homological algebra that is very useful towards performing Steps 4,5,6 of the Hodgkin spectral sequence calculation, as outlined in Lecture 6. Let R he a commutative ring with unit and $A, B, \ldots$ (left) R-modules.

7.1 Definition: A free resolution of A is an exact sequence

$$\cdots \longrightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0 \ ,$$

where each P_i is a free R-module.

Note that ε, called the augmentation,is necessarily surjective. One then obtains a chain complex

$$P \otimes_R B = \cdots \xrightarrow{d_3 \otimes 1} P_2 \otimes_R B \xrightarrow{d_2 \otimes 1} P_1 \otimes_R B \xrightarrow{d_1 \otimes 1} P_0 \otimes_R B \ ,$$

but in general this sequence is not exact.

7.2 Definition: $\mathrm{Tor}_A^R(A, B) = H_n(P \otimes_R B)$, $n \geq 0$.

7.3 Remarks: (a) It must be verified that this is well defined, i.e. independent of the resolution P.

(b) The modules in P can be taken to tbe projective, a generalization of free.

(c) $\mathrm{Tor}^R(A, B)$ can also be computed by a proj. resolution Q o f B, using $A \otimes_R Q$, giving the same result to $\approx$.

All of 7.1 – 7.3 are basic facts and may be found in any text of homological algebra. In general the main difficulty in a Tor calculation is to find a convenient projective resolution P. When R = a polynomial algebra, say $R = K[x_1, \ldots, x_m]$ where K = comm ring with 1 (usually $K = Z$ in applications), there is a convenient free resolution called the Koszul resolution, of K. Suppose each x_i has degree 0, so $\varepsilon : R \twoheadrightarrow K$ with $\varepsilon(1) = 1$, $\varepsilon(x_i) = 0$, gives an augmentation to start the resolution.

Let $E = E_R[\xi_1, \ldots, \xi_m]$ be the exterior algebra over R on generators $\xi_1, \ldots, \xi_m$ of degree 1. Thus each E_r is the free R-module on $\{\xi_{i_1} \cdots \xi_{i_r} = i_1 < \cdots < i_r\}$, and $E_0 = R$, $E_m \approx R$, $E_r = 0$ for $r > m$. Setting $\partial(\xi_j) = x_j$ makes E into a differential graded algebra over R, i.e. the property

$$\partial(uv) = \partial u \cdot v + (-1)^p \quad u \cdot \partial v \quad ,$$

for u homogeneous of degree p, and linearity, defines ∂ on all of E.

Example: Take $u = \xi_1$, $v = \xi_2$, so $\deg u = 1$.

$$\partial(uv) = \partial(\xi_1\xi_2) = \partial\xi_1 \cdot \xi_2 - \xi_1 \cdot \partial\xi_2 = x_1\xi_2 - \xi_1 x_2 = x_1\xi_2 - x_2\xi_1 \; .$$

7.4 Proposition: $0 \longrightarrow E_m \xrightarrow{\partial} E_{m-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} E_1 \xrightarrow{\partial} E_0 \stackrel{\varepsilon}{\twoheadrightarrow} K$ is a free R-resolution of K (here K is regarded as a trivial R-module, i.e. $x_i \cdot k = k \quad \forall k \in K$, $1 \leq i \leq m$).

For a proof see [17], p. 204.

The next theorem serves to simplify the calculation $\mathrm{Tor}^{RG}(RH, Z)$ occurring in the Hodgkin spectral sequence by "cancelling" common parts of RH, RG as for as possible (we make this precise in Remark 7.6(b)). It is a straightfoward application of techniques in [5] involving supplemented algebras and the change of rings theorem.

Given $i : H \hookrightarrow G$, suppose $RG \approx Z[\gamma_1, \ldots, \gamma_n]$ with $\varepsilon(\gamma_i = 0$. Let $\Lambda = Z[\gamma_1, \ldots, \gamma_r]$ for some $r \leq n$, and $i^*(\gamma_i) = h_i \in RH$, $1 \leq i \leq r$. Also let $A = RG/(\gamma_1, .., \gamma_r)$, $B = RG/(h_1, .., h_r)$ and note that just as i^* makes RH an RG-module, it induces an A-module structure on B.

7.5 Theorem: If RH is free as a Λ-module, then $\mathrm{Tor}^{RG}(RH, Z) \approx \mathrm{Tor}^A((B, Z)$.

7.6 Remarks: (a) RH flat as a Λ-module suffices (free $\Rightarrow$ flat),

(b) the idea of "cancelling as for as possible" means taking r as large a possible with RH free as a Λ-module.

8 Lecture

Completion of Parallelizability Theorem

8.1 Recall from 4.9 that there were four families of partially oriented flag manifolds whose parallelizability was still undecided:

$$\begin{array}{lll} F(1,\ldots,1 \mid 3,1) & , & s \geq 3 \, , \\ F(1,\ldots,1 \mid 7,1) & , & s \geq 3 \, , \\ F(6,1,\ldots,1 \mid 1,1) & , & s \geq 5 \, , \\ F(6,1,\ldots,1 \mid 1,1,1) & , & s \geq 4 \, . \end{array}$$

The object in this lecture is to apply K-theory to show that none of these manifolds is even sp, although in the end there is a single undecided case $F(6,1,1,|\,1,1)$ (of dim 30).

8.2 By the inclusion method, to prove all the assertions in 8.1, it will suffice to show that none of the following are sp:

$$\begin{aligned} X &= F(1 \mid 3,1)\ , \\ Y &= F(1 \mid 7,1)\ , \\ Z &= F(6,1,1,1 \mid 1,1)\ , \\ W &= F(6 \mid 1,1,1)\ , \end{aligned}$$

with respective dimensions $7, 15, 40, 21$. In this lecture we will outline the non-sp proofs for X, using the Atiyah-Hirzebruch spectral sequence, and for W using the Hodgkin spectral sequence. The proof for Y is similar to that for X, and the proof for Z similar (actually much easier) to that for W. The main reference is [24].

8.3 Remark: The above conclusions together with those in Lec. 3,4 decide the sp and par of all standard and all partially oriented flag manifolds, apart from the one exception $F(6,1,1,|\,1,1)$.

8.4 We determine first the tangent bundles of X, W. Using the usual notation, we have
Proposition: $[\tau_x] = 4[\xi_3] + 3 \in KO(X))$,
$[\tau_w] = 8[\xi_2 \oplus \xi_3 \oplus \xi_4] - 3 \in KO(W)$.

Proof: For X we have $\xi_1 = \varepsilon$, ξ_2 rank 3, ξ_3 rank 1, and $\xi_1 \oplus \xi_2 \oplus \xi_3 \approx 5\varepsilon$. Since $\xi_1 = \varepsilon$ this gives $[\xi_2] + [\xi_3] = 4$. Now $\tau_x \approx \xi_1 \otimes \xi_2 \oplus \xi_1 \otimes \xi_3 \oplus \xi_2 \otimes \xi_3 \approx \xi_2 \oplus \xi_3 \oplus \xi_2 \otimes \xi_3$. So in $KO(X)$,

$$[\tau_x] = [\xi_2] + [\xi_3] + (4 - [\xi_3])[\xi_3] = 4 + 4[\xi_3] - 1 = 4[\xi_3] + 3\ .$$

For W, let $\xi := \xi_2 \oplus \xi_3 \oplus \xi_4$, it is rank 3 orientable. Also $\xi_1 \oplus \xi \approx 9\varepsilon$. Then $\xi \approx \lambda^2\xi \approx \xi_2 \otimes \xi_3 \oplus \xi_3 \otimes \xi_4 \oplus \xi_4 \otimes \xi_2$, from which $\xi^2 = \xi \otimes \xi \approx \xi_2^2 \oplus \xi_3^2 \oplus \xi_4^2 \oplus 2(\xi_2 \otimes \xi_3 \oplus \xi_3 \otimes \xi_4 \oplus \xi_4 \otimes \xi_2) \approx 3\varepsilon \oplus 2\xi$.

This
$$\begin{aligned} [\tau_w] &= [\xi_1 \otimes \xi] + [\lambda^2\xi] \\ &= [9 - \xi][\xi] + [\xi] = 9[\xi] - [\xi^2] + [\xi] \\ &= 10[\xi] - 3 - 2[\xi] = 8[\xi] - 3\ . \end{aligned}$$
$\square$

Exercise: Show similarly $[\tau_Y] = 8[\xi_3] + 7$, $[\tau_Z] = 16[\xi_6] + 24$ (for Z first show $\xi_5 \approx \xi_6$).

8.5 We now turn to the non-sp of X. By 8.4 it will suffice to show $4\xi_3 \not\sim 0$. Note the fibre map $q : X = F(1 \mid 3,1) \to F(4,1) = P^4$ (cf. 3.2), clearly $q^*(\xi_{P^4}) \approx \xi_3$. Since $8\xi_{p^4} \sim 0$. (cf. 5.11) it follows that $8\xi_3 \sim 0$, our goal will be to use the Atiyah-Hirzebruch spectral sequence to show $4\xi_{p^4} \not\sim 0$ implies the same for $4\xi_3$, this completing the proof.

To use the $A - H$ spectral sequence for KO-theory one needs to know both $H^*(X;Z)$ and $H^*(X;Z/2)$. For this there is another useful fibration

$$F(3,1) = P^3 \hookrightarrow X = F(l \mid 3,1) \xrightarrow{\widetilde{q}} \widetilde{F}(1,4) = S^4\ ,$$

to which the Serre spectral sequence may be applied. We omit the details and simply give the results, see [24] for all the details.

dim	$H^*(X;Z) \xleftarrow{q^*} H^*(P^4;Z)$	$H^*(X;Z/2) \xleftarrow{q^*} H^*(P^4;Z/2)$
0	$Z \xleftarrow{\approx} Z$	$Z/2 \xleftarrow{\approx} Z/2$
1		$Z/2 \xleftarrow{\approx} Z/2$
2	$Z/2 \xleftarrow{\approx} Z/2$	$Z/2 \xleftarrow{\approx} Z/2$
3		$Z/2 \xleftarrow{\approx} Z/2$
4	$Z/4 \xleftarrow{2} Z/2$	$Z/2 \xleftarrow{0} Z/2$
5		$Z/2$
6	$Z/2$	$Z/2$
7	Z	$Z/2$

8.6 The Atiyah-Hirzebruch spectral sequence is summarized by

$$E_2^{p,-q} = H^p(Y; KO^q(pt)) \Longrightarrow KO^{-p-q}(Y),$$

which means that the E_2 term is the ordinary cohomology of Y with coefficients in $KO^*(pt)$, converging to E_∞ which represents the successive quotients in a filtration of $KO^{-p-q}(Y)$. Further facts are as follows.

(a) d_r has bidegree $(r, r-1)$,

(b) the spectral sequence lies in quadrants $1, 4$,

(c) the coefficients are $KO^0(pt) \approx KO^4(pt) \approx Z$, $KO^1(pt) \approx KO^2(pt) \approx Z/2$, $KO^3(pt) = KO^5(pt) = KO^6(pt) = KO^7(pt) = 0$, $KO^{i+8} \approx kO^i$ (so the spectral sequence is periodic in q).

In general it is not too dificult to find the E_2 term, but determining the d_r and the possible group extensions of the E_∞ terms may be quite tricky. We are primarily interested in $K^0(Y)$, so will consider the diagonal $p+q=0$. This always (for Y connected) has the term $Z = H^0(Y; KO^0(pt)) = E_2^{0,0}$, which always lasts to E_∞ and gives rise to the direct summand Z in $KO^0(Y) \approx Z \oplus \widetilde{KO}^0(Y)$, so it is the $E_\infty^{p,-p}$ terms with $p > 0$ that are of interest.

Remark: There is a similar (and simpler) spectral sequence for $K^*(Y)$, with $K^0(pt) = Z$, $K^1(pt) = 0$, $K^{i+2} \approx K^i$ giving periodicity of order 2 in q.

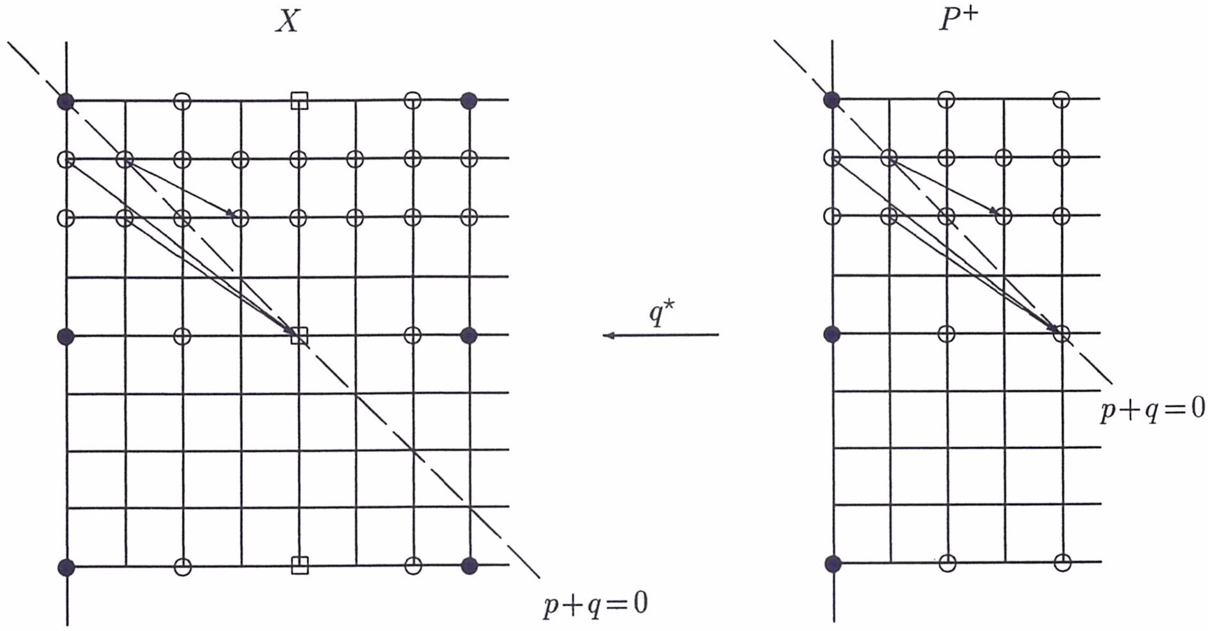
8.7 The real and complex K-theory of $\mathbb{R}P^n$ and $\mathbb{C}P^n$ was found in 1961 by J.F. Adams (cf. [9], p.237 or 5.11 in these notes). In particular all differentials in the Atiyah-Hirzebruch spectral sequence are known to vanish, and the groups $K^0(\mathbb{F}P^n)$ are cyclic so all extension of the $E_\infty^{p,-p}$ terms are non-trivial. For example, for $\mathbb{R}P^4 = P^4$, we have already seen in 5.11 that $\widetilde{KO}^0(P^4) \approx Z/2^{\varphi(4)} = Z/8$. Since (ignoring the $E_2^{0,0} = Z$ term as discussed in 8.6) the diagonal $E_2^{p,-p}$ has just three non-zero entries, $E_2^{1,-1} \approx E_2^{2,-2} \approx E_2^{4,-4} \approx Z/2$, these three $Z/2$'s must last to E_∞ and then recombine to give $Z/8$ by the non-trivial extensions

$$\begin{array}{ccccccccc} 0 & \to & 0 & \to & Z/2 & \to & Z/2 & \to & 0 \\ 0 & \to & Z/2 & \to & Z/4 & \to & Z/2 & \to & 0 \\ 0 & \to & Z/4 & \to & Z/8 & \to & Z/2 & \to & 0. \end{array}$$

This is shown in the right hand spectral sequence in Diagram 8.8 below, the E_2 term for $Y = P^4$. The notation $\bullet = Z$, $0 = Z/2$, $\square = Z/4$ is used, the arrows indicate all possible d_r into or from $E_r^{p,-p}$ terms, $r \geq 2$, and these are all 0 as mentioned above.

The left hand spectral sequence is the E_2 term for X. Further, the map $q : X \to P^4$ induces a map of the spectral sequences, and this is completely described at the E_2 level by 8.5.

8.8 Diagram $KO^*(X) \xleftarrow{q^*} KO^*(P^4)$ at E_2 level.



The only possible non-zero differentials coming from or going to $E_r^{p,-p}$ are shown, there are 3 such in each spectral sequence. As described in 8.7 these 3 differentials equal 0 in the P^4 spectral sequence. Then using naturality and the fact that $q\star : H^i(P^4; Z/2) \xrightarrow{\approx} H^i(X; Z/2)$, $i = 1, 2$, it follows that the 3 differentials also vanish in the X spectral sequence. Thus $E_\infty^{p,-p} = E_2^{p,-p}$ for both spectral sequences. The maps $q^* : E_\infty^{p,-p}(P^4) \longrightarrow E_\infty^{p,-p}(X)$ are therefore the same as q^* at the E_2 level, so are isomorphisms $Z/2 \xrightarrow{\approx} Z/2$ for $p = 1, 2$ and the injection $Z/2 \xrightarrow{2} Z/4$ for $p = 4$. If follws that $q^* : KO^0(P^4) \longrightarrow KO^0(X)$ is injective, so ξ_3 has order 8, $4\xi_3 \not\sim 0$, and X is not sp.

8.9 We now turn to $W = F(6 \mid 1,1,1)$, and will outline (for full details cf. [24]) the 8 steps for the Hodgkin spectral sequence determining the complex K-theory $K^*(W)$.

Step 1: Recall $W = F(6 \mid 1,1,1) = SO(9)/SO(6) \times D$, where $D = \{\mathrm{diag}(1,1,1), \mathrm{diag}(1,-1,-1), \mathrm{diag}(-1,-1,1)\} \subset SO(3)$ is the diagonal subgroup of $SO(3)$. Let ϕ : $\mathrm{Spin}(9) \longrightarrow\!\!\!\!\rightarrow SO(9)$ be the usual double cover, $Q = \phi^{-1}(D)$, then $W = \mathrm{Spin}(9)/H$ where $H = \phi^{-1}(SO(6) \times D) \xhookrightarrow{j} \mathrm{Spin}(9)$.

8.10 Step 2: $R\,\mathrm{Spin}(9) = Z[P_1, P_2, P_3, \Delta_4]$, and $R\,\mathrm{Spin}(6) = Z[p_1, \Delta_3^+, \Delta_3^-]$ is standard (cf. [9]). Here P_1, P_2, P_3, p_1 are the "K-theory pontrjagin classes" which are certain linear combinations of the corresponding exterior powers $\lambda^1, \lambda^2, \lambda^3$ adjusted to have augmentation (or total degree) 0, e.g. $P_1 = \lambda^1 - \begin{cases} 2r \\ 2r+1 \end{cases}$ in $R(SO(2r))$,$R(SO(2r+1))$ resp, similarly in R Spin$(2r)$, R Spin$(2r+1)$, and in general $P_k = \lambda^k + \cdots$ lin. comb. of $\lambda^{k-1}, \ldots, \lambda^1, \lambda^0 = 1$. Using the standard description of Spin(9) in terms of the Clifford algebra C_9[9]], p.169-170), we see

$$Q = \{\pm 1, \pm e_7e_8, \pm e_8e_9, \pm e_9e_9\} \subset \text{Spin }(9)\ .$$

Since $(e_7e_8)^2 = e_7e_8e_7e_8 = -e_7e_7e_8e_8 = (-1)^3 = -1$, and $(e_7e_8)(e_8e_9) = e_7e_8^2e_9 = -e_7e_9 = e_9e_7$, etc., Q is clearly isomorphic to the group $\{\pm 1, \pm i, \pm j, \pm k\}$ of unit quarternions. The representation ring RQ is well known, one has $RQ = Z[x,y,q]/\!\sim$, where $\sim$ is the relations $x^2 = y^2 = 1,\ \ q^2 = (1+x)(1+y),\ \ qx = q = qy$.
Here x, y are the 1-dimensional representations of the Klein 4-group D, composed with $Q \twoheadrightarrow D$, while q is the obvious representation of Q on the quarternions $I\!H$, regarded as a 2-dimennsional complex representation.

Now $Z/2$ acts on Spin$(6) \times Q$ via $(a,b) \longmapsto (-a,-b)$, and clearly $H \approx \text{Spin}(6) \times_{\mathbf{Z}/2} Q$. Thus

$$\begin{aligned} RH &= R(\text{Spin}(6) \times Q)^{Z/2} \\ &\approx (R\text{Spin}(6) \otimes RQ)^{Z/2}\ . \end{aligned}$$

A calculation of this gives

$$RH \approx Z[p_1, p_2, p_3]\left[\left(\Delta_3^+,\right)^2,\ \Delta_3^+q,\ \Delta_3^-q,\ x, y\right] \Big/ \sim\ ,$$

where $\sim$ consists of 9 relations, for details se [24].

Caution: $q \notin RH$, so Δ_3^+q and Δ_3^-q are not products in RH, similarly for $\left(\Delta_3^+\right)^2$.

8.11 Step 3: Using the maximal torus of Spin (9) one can compute the restriction homomorphism $j^\#$. The result is

$$\begin{aligned} &j^\#(P_1) = p_1 + \omega := p_1' \ (\text{where } \omega = x + y + xy - 3 = q^2 - 4 \in RH\,), \\ &j^\#(P_2) = p_2 + \omega p_1 := p_2'\,, \\ &j^\#(P_3) = p_3 + \omega p_2 := p_3'\,, \\ &j^\#(P_4) = p_3\omega\,, \\ &j^\#(\Delta_4) = \Delta_3^+q + \Delta_3^-q. \end{aligned}$$

8.12 Step 4,5: To apply the change of rings as in 7.5, let
$\Lambda = Z[P_1, P_2, P_3]$. Also let
$A = R\text{Spin}(9)\,/(\Lambda^+) = Z[\Delta_4]$
$B = RH\,/(p_i', p_2', p_3')$.
With some difficulty one can show that RH is Λ-free, so $\text{Tor}^{RG}(RH, Z) \approx \text{Tor}^A(B, Z)$.

8.13 Step 6: Since A is a polynomial algebra on just a single generator Δ_4, the Koszul resolution (and this step) are in this case quite easy, the resolution consisting of chains in degrees 1,0 only. After tensoring with B one obtains the chain complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & BD & \xrightarrow{d} & B & \longrightarrow & 0 \qquad (BD \approx B) \\ & & D & \rightarrowtail & \Delta_3^+q + \Delta_3^-q - 16 & & \end{array}$$

In particular only $\text{Tor}_0^A(B,Z)$, $\text{Tor}_1^A(B,Z) \neq 0$.

8.14 Step 7-8: These are also quite easy here. Indeed for any Hodgkin spectral sequence with only $\mathrm{Tor}_0, \mathrm{Tor}_1$ possibly non-zero, the spectral sequence automatically collapses, $E_2 = E_\infty$, and furthermore $K^0 = \mathrm{Tor}_0, K' = \mathrm{Tor}_1$. So for the case at hand we find

$$\begin{aligned} K^0(W) = \mathrm{Tor}_0 = B\,/\mathrm{Im}(d) &= B\Big/\left(\Delta_3^+ q + \Delta_3^- q - 16\right) \\ &\approx Z\left[\left(\Delta_3^+\right)^2, \Delta_3^+ q, x, y\right]\Big/ \sim \end{aligned}$$

for a suitable set of relations $\sim$.

8.15 To complete the proof it remains to show, from 8.4, that $8\omega \neq 0 \in K^0(W)$ (see also the definition of ω in 8.11). To do this one constructs an epimorphism

$$\begin{aligned} r : K^0(W) &\longrightarrow R := (Z/16)[u, v]\,/(u^2 + 2u, v^2 + v) \\ \left(\Delta_3^+\right)^2 &\longmapsto 0 \\ \Delta_3^+ q &\longmapsto 0 \\ x &\longmapsto u + 1 \\ y &\longmapsto v + 1\,. \end{aligned}$$

Of course one must check that all the relations $\sim$ map to 0 under r, this is routine. Next, noting that $r(\omega) = r(x + y + xy - 3) = u+1+v+1+(u+1)(v+1)-3 = uv+2u+2v$, it suffices to show. $0 \neq r(8\omega) = 8\,r(\omega) = 8\,uv$. This is done by a representation of R in $M_{4,4}(Z/16)$:

$$u \longmapsto U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \qquad v \longmapsto V = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}.$$

One checks this respects $U^2 + 2U = 0, V^2 + 2V = 0$, also $UV = VU = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -2 & -2 & 4 \end{bmatrix}.$

Clearly $8\,U\,V \neq 0$, completing the proof that W is not sp. □

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