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NEWTON FILTRATIONS, GRADED ALGEBRAS AND CODIMENSION OF NON-DEGENERATE IDEALS

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Resumo

Neste trabalho é feita uma generalização do método introduzido por Kouchnirenko para o cálculo da codimensão de um ideal I sob uma condição de não degeneração para um dado sistema de geradores de I.

É discutido também, usando a noção de ideais reduzidos, a relação entre ideais Newton não degenerados e ideais não degenerados em relação a algum poliedro de Newton.

Abstract

We investigate a generalization of the method introduced by Kouchnirenko to compute the codimension (colength) of an ideal under a certain non-degeneracy condition on a given system of generators of I. We also discuss Newton non-degenerate ideals and give characterizations using the notion of reductions and Newton polyhedra of ideals.

1. Introduction

The computation of $\dim_{\mathbb{C}} \mathcal{O}_n/I$, the codimension of an ideal I in the ring $\mathcal{O}_n$ of holomorphic map germs at the origin in $\mathbb{C}^n$, is one of the main tools used to calculate geometric invariants of singularities. In general, this calculus is done using the theory of standard basis of the ring $\mathcal{O}_n/I$. For weighted homogeneous map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ it is known that some geometric invariants are given in terms of its weights and degrees. For instance, Milnor and Orlik gave a formula in [17] for the Milnor number of a weighted homogeneous map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin. There are other invariants that are also computed for weighted homogeneous map germs in terms of its weights and degrees, Gaffney and Mond in [10] show how to compute the number of cusps in a generic deformation of a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and Mond [18] shows how to compute the number of cross caps in a generic deformation of a map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$.

On the other side, Kouchnirenko considered in [15] the ring $\mathcal{F}_n$ of formal power series in the variables $x_1, \ldots, x_n$ around the origin and gave a formula, in terms of the Newton polyhedron $\Gamma_+(f)$, for the Milnor number of a Newton non-degenerate germ of function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin. The key object needed to

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obtain his result is the Newton filtration of $\mathcal{F}_n$ constructed from $\Gamma_+(f)$. Therefore, we observe that also in this case, the Milnor number of a Newton non-degenerate germ of function only depends on such a basic object like its Newton polyhedron.

Here we show a generalization of the method introduced by Kouchnirenko in order to formalize the above idea. In Section 2 we use the notion of Newton filtration to introduce a new non-degeneracy condition. This gives rise to a class of ideals that contains the Newton non-degenerate ideals in the sense of [21] and the ideals of finite codimension generated by weighted homogeneous germs. When an ideal of this class has finite codimension and is a complete intersection, we use the Koszul complex to obtain a formula for its codimension that only depends on a suitable fixed Newton polyhedron Γ_+ and the levels (or degrees) of a system of generators with respect to the filtration induced by Γ_+. This formula also characterizes complete intersection ideals which satisfy this non-degeneracy condition. In Section 3 we show the relationship between the multiplicity of a Newton non-degenerate ideal and the volume of its Newton polyhedron. Moreover, we give a relation between non-degenerate systems and Newton non-degenerate ideals. We also show in this section an explicit proof of the local version of Bernstein's theorem, or in other words, we use the notion of non-degeneracy for map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ given by Khovanskii [13] to show a formula for the codimension of a complete intersection ideal through the mixed covolume of the Newton polyhedra constructed from a given generating system.

In Section 4, we reproduce Kouchnirenko's argument for algebras for which we know a free resolution coming from a free resolution of a perfect module, which is explicitly described. From this resolution we consider some induced complexes using the graded ring and the Rees ring associated to the filtration. We compare the acyclicity of these complexes there and obtain that acyclicity is equivalent to non-degeneracy of the matrices (g_{ij}). In this setup, we show that if (g_{ij}) is a non-degenerate matrix defining a morphism $\varphi : \mathcal{F}_n^{e_1} \to \mathcal{F}_n^{e_0}$ and if $\operatorname{Im}\varphi$ has finite codimension in $\mathcal{F}_n^{e_0}$, we can compute this codimension in terms of the volume of a suitable Newton polyhedron.

We shall consider in this article the ring $\mathcal{F}_n$, but we assume that all map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and all germs of functions $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are holomorphic. We remember that if I denotes an ideal generated by holomorphic germs of functions $g_1, \dots, g_k$ in $\mathcal{O}_n$ and $\widehat{I}$ denotes the ideal generated by the same $g_1, \dots, g_k$ in $\mathcal{F}_n$ it follows from the Artin Theorem (see [24], Theorem 4.2), that $\dim_{\mathbb{C}} \mathcal{O}_n/I = \dim_{\mathbb{C}} \mathcal{F}_n/\widehat{I}$, if they are finite.

In the language of commutative algebra, it is proper to say that the number $\dim_{\mathbb{C}} \mathcal{O}_n/I$ is the colength of the ideal I, to avoid confusion with Krull codimension (i.e., height). But we apologize that we sometimes prefer to say this is just codimension, since this is actually the codimension of an appropriate orbit in some geometric setup. However, we sometimes use the word 'codimension' as Krull codimension (height $\operatorname{ht}(I)$). In such cases, we mention this like 'codimension (or height)' to avoid the reader's confusion.

2. The codimension of non-degenerate complete intersection ideals

The notion of Newton non-degeneracy of germs of functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is used by several authors. For instance Kouchnirenko computes in [15] the Milnor number of Newton non-degenerate functions with an isolated singularity at the origin and

Yoshinaga [28] gave conditions for the topological triviality of families defined by Newton non-degenerate functions. Varchenko [25] also dealt with the Newton polyhedron of a Newton non-degenerate germ of function to compute the zeta function of its monodromy. More recently Wall gave in [27] a weaker condition of non-degeneracy for germs of functions in order to characterize the Milnor number of any isolated singularity germ which satisfies this condition.

Motivated by the algebraic characterization of Newton non-degeneracy of functions given by Yoshinaga [28], Saia gave in [21] a definition of Newton non-degeneracy for ideals and showed how to compute the integral closure of Newton non-degenerate ideals in terms of its Newton polyhedron.

Another important class of ideals that often appears in Singularity Theory is the class of semi weighted homogeneous ideals. These ideals are not Newton non-degenerate in general. We shall give in this section a more general definition of non-degeneracy. We remark that the family of ideals that satisfy this condition includes the class of Newton non-degenerate ideals and the class of semi weighted homogeneous ideals.

We shall denote by $\mathcal{A}$ the ring $\mathcal{F}_n$ of formal power series in the variables $x_1, \dots, x_n$ around the origin.

Let $g = \sum_k a_k x^k$ be a series in $\mathcal{A}$, the *support of* g, denoted by $\operatorname{supp} g$, is the set of points $k \in \mathbb{Z}^n$ such that $a_k \neq 0$. If I is an ideal of $\mathcal{A}$, we define the *support of* I as $\operatorname{supp} I = \cup_{g \in I} \operatorname{supp} g$.

Definition 2.1. The *Newton polyhedron* of an ideal I, denoted by $\Gamma_+(I)$, is the convex hull in $\mathbb{R}^n_+$ of $\displaystyle\bigcup_{k \in \operatorname{supp} I} \{k + v : v \in \mathbb{R}^n_+\}$.

We denote by $\Gamma(I)$ the union of all compact faces of $\Gamma_+(I)$ and call $\Gamma_-(I) = \mathbb{R}^n_+ \smallsetminus \Gamma_+(I)$. Given a face $\Delta \subseteq \Gamma(I)$, we denote the union of half-rays emanating from the origin and passing through Δ by $C(\Delta)$. The set $C(\Delta) \cap \mathbb{Z}^k$ is a subsemigroup of $\mathbb{Z}^k$, hence the subset of $\mathcal{A}$ given by $\mathcal{A}_\Delta = \{g \in \mathcal{A} : \operatorname{supp} g \subseteq C(\Delta) \cap \mathbb{Z}^k\}$ is a subring with unity of $\mathcal{A}$. Moreover, if D is a fixed subset of $\Gamma_+(I)$ and $g = \sum a_k x^k \in \mathcal{A}$, we set $g_D = \sum_{k \in D} a_k x^k$.

Definition 2.2. Let I be an ideal of finite codimension in $\mathcal{A}$. We say that I is *Newton non-degenerate* if there exists a system of generators $g_1, \dots, g_s$ of I such that for each compact face $\Delta \subseteq \Gamma(I)$, the ideal I_Δ generated by $g_{1_\Delta}, \dots, g_{s_\Delta}$ has finite codimension in $\mathcal{A}_\Delta$.

By Theorem 6.2 of [15], we have that the ideal I_Δ generated by $g_{1_\Delta}, \dots, g_{s_\Delta}$ has finite codimension in $\mathcal{A}_\Delta$ if and only if for each compact face Δ_1 of Δ the equations $g_{1\Delta_1} = \cdots = g_{s\Delta_1} = 0$ have no common solution in $(\mathbb{C} \smallsetminus \{0\})^n$.

We remark that this definition does not depend on the system of generators of the ideal I. We say that I is *Newton degenerate* when it is not Newton non-degenerate.

A series $g = \sum_k a_k x^k$ in $\mathcal{A}$ is said *Newton non-degenerate* if the ideal generated by the system $x_1 \partial g/\partial x_1, \dots, x_n \partial g/\partial x_n$ is Newton non-degenerate.

Next, we show some preliminary concepts which allow us to give the new definition of non-degeneracy. The key point is that we can consider any Newton polyhedra and not only the Newton polyhedron of the ideal that we want to compute the codimension.

We say that $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a *Newton polyhedron* if there exists $k_1, \dots, k_r \in \mathbb{Q}^n_+$ such that Γ_+ is the convex hull in $\mathbb{R}^n_+$ of the set $\{k_i + v : v \in \mathbb{R}^n_+,\ i = 1, \dots, r\}$ and Γ_+ intersects the coordinate axis. The sets Γ and Γ_- are defined as when $\Gamma_+ = \Gamma_+(I)$. Moreover, we denote the n-dimensional volume of a compact subset $K \subseteq \mathbb{R}^n_+$ by $\mathrm{V}_n(K)$. When an ideal $J \subset \mathcal{A}$ has finite codimension, the n-dimensional volume of $\Gamma_-(J)$ is finite, in this case we denote the number $\mathrm{V}_n(\Gamma_-(J))$ by $v(J)$.

From the boundary of a Newton polyhedron Γ_+ we construct a filtration of $\mathcal{A}$ (see [15, 2.1]), defined via a piecewise-linear function $\phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ with the following properties:

(i) ϕ is linear on each cone $C(\Delta)$, where Δ is a compact face of Γ;
(ii) ϕ takes positive integer values on the lattice points of $\mathbb{R}^n_+ \setminus \{0\}$;
(iii) there exists a positive integer M such that $\phi(k) = M$, for all $k \in \Gamma$.

We define
$$\mathcal{A}_q = \{g \in \mathcal{A} : \operatorname{supp} g \subseteq \phi^{-1}(q + \mathbb{N})\}, \qquad \text{for all } q \geq 0,$$
then
$$\mathcal{A} = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \cdots$$
is a filtration of ideals of $\mathcal{A}$, that is $\mathcal{A}_p\mathcal{A}_q \subseteq \mathcal{A}_{p+q}$. This is called the *Newton filtration of $\mathcal{A}$ associated to* Γ_+. For any compact face Δ of Γ, this filtration induces a filtration on $\mathcal{A}_\Delta$ in a natural way.

Let $g = \sum_k a_k x^k$ be a series in $\mathcal{A}$, the *level of g* with respect to the filtration defined above is defined as
$$\mathrm{d}(g) = \min\{\phi(k) : k \in \operatorname{supp} g\} = \max\{q : g \in \mathcal{A}_q\}.$$
The *principal part of* g, denoted by $\operatorname{in} g$, is the sum of terms $a_k x^k$ such that $\phi(k) = \mathrm{d}(g)$. Given a face Δ of Γ, *the principal part of g over* Δ, denoted by $\operatorname{in}_\Delta g$ is the polynomial
$$\operatorname{in}_\Delta g = \sum \left\{a_k x^k : k \in \operatorname{supp} g \cap C(\Delta) \text{ and } \phi(k) = \mathrm{d}(g)\right\}.$$

Definition 2.3. Let I be an ideal of finite codimension of $\mathcal{A}$. We say that a system of generators $g_1, \dots, g_s$ of I is *non-degenerate on* Γ_+ if for each compact face $\Delta \subseteq \Gamma$, the ideal $\operatorname{in}_\Delta I$ generated by $\operatorname{in}_\Delta g_1, \dots, \operatorname{in}_\Delta g_s$ has finite codimension in $\mathcal{A}_\Delta$.

Analogously to Definition 2.2, the ideal $\operatorname{in}_\Delta I$ generated by $\operatorname{in}_\Delta g_1, \dots, \operatorname{in}_\Delta g_s$ has finite codimension in $\mathcal{A}_\Delta$ if and only if for each compact face $\Delta_1 \subseteq \Delta$ the equations $\operatorname{in} g_{1\Delta_1}(x) = \cdots = \operatorname{in} g_{s\Delta_1}(x) = 0$ have no common solution in $(\mathbb{C} \setminus \{0\})^n$.

Remark 2.4. (i) This definition is quite different from Definition 2.2, since it depends on the given system of generators. It is clear that if I is Newton non-degenerate, then any system of generators for I is non-degenerate on $\Gamma_+(I)$. Nevertheless, the ideal I generated by a non-degenerate system with respect to $\Gamma_+(I)$ could be Newton degenerate. For example, this occurs with the system $\{x^3 + xy + y^3, x^2y^2\} \subset \mathbb{C}[[x, y]]$.
(ii) Observe that if I is an ideal of finite codimension and Γ_+ is any Newton polyhedron, then I always admits a generating system which is non-degenerate on Γ_+. To see this, we know from the finite codimension of the ideal I that there exists an integer k such that $m_n^k \subseteq I$. But this implies that we can add all the monomials of degree k to a given generating system $\mathcal{S}$ of I to obtain a non-degenerate system on Γ_+.

The problem is that if we do this, the number of generators of the system $\mathcal{S}$ increases. For example, the ideal $I = \langle x^5+x^2y, xy^2+y^5\rangle$ is Newton degenerate and the given generating system of I is degenerate on any Newton polyhedron. This fact is a consequence of Theorem 3.12. On the other hand, we have that the system $\{x^5+x^2y, xy^2+y^5, x^4y^2, x^2y^4\}$ also generates I and is non-degenerate on $\Gamma_+(I)$.

We shall show in Section 4 that if we have a system of generators $\{g_1, \dots, g_r\}$ of an ideal I which is non-degenerate on some Newton polyhedron and if we have a free resolution of $\mathcal{F}_n/\langle g_1, \dots, g_r\rangle$, which comes from a free resolution of a perfect module, then we can compute the codimension of I in terms of the volume of this Newton polyhedron.

Here we give the main result of this section.

Theorem 2.5. *Let $g_1, \dots, g_n$ be a system of generators of an ideal I with finite codimension in $\mathcal{A}$ and Γ_+ a Newton polyhedron. If M is the value on Γ of the filtration induced by Γ_+ and $d_1 = \mathrm{d}(g_1), \dots, d_n = \mathrm{d}(g_n)$ are the levels of the given set of generators of I with respect to this filtration, then*

(i) *we have the inequality* $\dim_{\mathbb{C}} \mathcal{A}/I \geq \dfrac{d_1 \dots d_n}{M^n} n!\, \mathrm{V}_n(\Gamma_-)$,

(ii) *the equality holds if and only if the system $g_1, \dots, g_n$ is non-degenerate on Γ_+.*

Proof of part (i). We first observe that each set $\mathcal{A}_\ell$ with $\ell \geq 0$, of the Newton filtration induced by Γ_+ is a monomial ideal, then for $d = d_1 \cdots d_n$, we have

$$\frac{d^n}{M^n} n! \mathrm{V}_n(\Gamma_-) = n! \mathrm{V}_n(\phi^{-1}([0, d])) \leq n! v(\mathcal{A}_d).$$

Now consider the ideal J generated by the system $\{g_1^{d/d_1}, \dots, g_n^{d/d_n}\}$, since $J \subseteq \mathcal{A}_d$, we conclude that

$$n! v(\mathcal{A}_d) \leq n!\, v(J) \leq \dim_{\mathbb{C}} \mathcal{A}/J = \frac{d^n}{d_1 \cdots d_n} \dim_{\mathbb{C}} \mathcal{A}/\langle g_1, \dots, g_n\rangle, \tag{1}$$

where the last equality is given in [1, 12.3], and the equality comes from (2) (see the proof of Proposition 3.4). □

The proof of part (ii) of the above theorem will be given in Section 4.

In order to compute the Milnor number of a germ of a function $f : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ with isolated singularity at 0, Kouchnirenko considered in [15] the ideal I generated by $\{x_1 \partial f/\partial x_1, \dots, x_n \partial f/\partial x_n\}$. He showed in the Theorem AI that, if I has finite codimension and is Newton non-degenerate, then $\dim_{\mathbb{C}} \mathcal{A}/I = n! v(I)$. In the next corollary we show that the same result holds for Newton non-degenerate ideals generated by n elements.

Corollary 2.6. *Suppose that $I \subseteq \mathcal{A}$ is a Newton non-degenerate complete intersection ideal of finite codimension. Then, the codimension of I is expressed as*

$$\dim_{\mathbb{C}} \mathcal{A}/I = n!\, v(I).$$

Example 2.7. We consider the ideal $I = \langle xy + z^3, xz + y^3, yz + x^3\rangle$ in $\mathbb{C}[[x, y, z]]$. Since I is Newton non-degenerate and $v(I) = 11/6$, we obtain $\dim_{\mathbb{C}} \mathbb{C}[[x, y, z]]/I = 11$.

A map germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, is *weighted homogeneous of weights* $w_1, \ldots, w_n$ *and degrees* $d_1, \ldots, d_p$ if

$$g(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \left(\lambda^{d_1} g_1(x_1, \ldots, x_n), \ldots, \lambda^{d_p} g_p(x_1, \ldots, x_n)\right), \qquad \text{for all } \lambda \in \mathbb{C}.$$

We say that a map germ $f = (f_1, \ldots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is *semi weighted homogeneous* of weights $w_1, \ldots, w_n$ and degrees $d_1, \ldots, d_p$ if f can be expressed as a sum $f = g + h$, where g is a finitely determined weighted homogeneous map germ of weights $w_1, \ldots, w_n$ and degrees $d_1, \ldots, d_p$, $h = (h_1, \ldots, h_p)$ and each h_i is a germ of function such that $w_1 k_1 + \ldots + w_n k_n > d_i$ for all $k = (k_1, \ldots, k_n) \in \operatorname{supp} h_i$.

Observe that the component functions $f_1, \ldots, f_p$ of any semi weighted homogeneous map germ f constitute a non-degenerate system on the polyhedron determined by the plane in $\mathbb{R}^n$ with equation $w_1 k_1 + \ldots + w_n k_n = 1$, where $w_1, \ldots, w_n$ denote the weights.

The *multiplicity of a finite map germ* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is defined as the codimension (i.e., colength) in $\mathcal{O}_n$ of the ideal generated by the component functions of f. For a semi weighted homogeneous map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ we see in the following proposition that the multiplicity of f only depends on its weighted initial part.

Proposition 2.8 ([1, 12.3]). *Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $g = (g_1, \ldots, g_n)$ be a semi weighted homogeneous map germ of weights $w_1, \ldots, w_n$ and degrees $d_1, \ldots, d_n$. Suppose that $g_1, \ldots, g_n$ is a system of generators of an ideal I of finite codimension, then*

$$\dim_{\mathbb{C}} \mathcal{O}_n / I = \frac{d_1 \cdots d_n}{w_1 \cdots w_n}.$$

Moreover, as a particular case of Theorem 2.5, we have that if the component functions of a finite map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ give a non-degenerate system with respect to some Newton polyhedron Γ_+, then the multiplicity of f only depends on the ideal generated by the principal parts of its coordinate functions with respect to the filtration induced by Γ_+.

Example 2.9. Consider the ideal in $\mathcal{A} = \mathbb{C}[[x, y]]$ given by $I = \langle x^\alpha + x^r y^s + y^\beta,\ x^{\lambda r} y^{\lambda s} \rangle$, where $\lambda > 0$. The Newton polyhedron of I has two faces if and only if $\alpha s < \beta(\alpha - r)$. Suppose that this is the case. The associated Newton filtration of $\mathcal{A}$ is given via the map $\phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ defined by

$$\phi(a, b) = \begin{cases} r\beta(sa + (\alpha - r)b), & \text{if } b \leq \frac{s}{r} a \\ \alpha s((\beta - s)a + rb), & \text{if } b \geq \frac{s}{r} a. \end{cases}$$

The value of ϕ along $\Gamma(I)$ is $M = \alpha\beta r s$. We observe that this ideal is Newton degenerate but the system of generators of I given above is non-degenerate on $\Gamma_+(I)$. Then, by Theorem 2.5, we have that $\dim_{\mathbb{C}} \mathcal{A}/I = \lambda(\alpha s + \beta r)$.

Next we show some applications of Theorem 2.5 to Singularity Theory.

2.1. **Thom-Boardman singularities.** The Thom-Boardman singularities which appear in a generic deformation of a finitely determined map germ give one of the main invariants used in Singularity Theory. In some cases, see [9, p.141], these invariants can be defined algebraically by means of the iterated jacobian of the map germ.

In [10] the authors considered finitely determined weighted homogeneous map germs $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and gave an algebraic formula, in terms of the weights and degrees of f, for the number of *cusps* (or $\Sigma^{1,1,0}$ singularities) that appear in a generic deformation of f.

Next we shall consider the case of corank one map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ to illustrate how to compute these invariants in terms of a suitable Newton polyhedron, for map germs that are not weighted homogeneous.

We denote by $c_{\mathfrak{i}}(f)$ the number of singularities of type $\Sigma^{\mathfrak{i}}$, $\mathfrak{i} = (1, ..., 1)$, ($n$ times), that appear in a one parameter generic deformation of f.

Example 2.10. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a corank one map germ of the form

$$f(x_1, \dots, x_{n-1}, y) = \left(x_1, \dots, x_{n-1}, y^{n+s} + g_1(x)y^{n-1} + g_2(x)y^{n-2} + \dots + g_{n-1}(x)y\right).$$

If the ideal $I = \langle y^s, g_1, g_2, \dots, g_{n-1}\rangle$ is Newton non-degenerate, then $c_{\mathfrak{i}}(f) = n!v(I)$.

As f is corank one, it follows from Proposition 2.5 of [9] that $c_{\mathfrak{i}}(f) = \dim_{\mathbb{C}} \mathcal{A}/I$. Since I is Newton non-degenerate, the result is a consequence of Corollary 2.6.

Example 2.11. Let $f(x, y, z) = (x, y, z^{p+3} + z^2(x^{r+2} + x^2y^2) + z(y^{s+2} + x^2y^2))$.

The number of swallowtails ($\Sigma^{1,1,1,0}$ singularities) that appear in a 1-parameter generic deformation of f is the codimension of the ideal $I = \langle z^p, x^{r+2} + x^2y^2, y^{s+2} + x^2y^2\rangle$. Using Corollary 2.6 we obtain $\#\left(\Sigma^{1,1,1,0}(f)\right) = 2p(4+r+s)$, since I is Newton non-degenerate.

2.2. **Multiplicities of Implicit Differential Equations.** Bruce and Tari considered in [3] the problem of the computation of the local multiplicity of a germ of a singularity of an implicit differential equation $F(x, y, dy/dx) = 0$. They showed that when the discriminants are simple curve singularities, these multiplicities are equal to the codimension of ideals in $\mathcal{O}_2$ generated by two elements. We shall show here how to calculate the codimension of a family of ideals that appears in this work. This is given by $I_{k,\ell,m} = \langle y^2 + x^k, y^2 + x^\ell y + x^m\rangle$, we compute the codimension of $I_{k,\ell,m}$ for some integer values of k, ℓ, m, with k and $m > 2$.

(i) Suppose first that $\ell \geq r/2$, where $r = \min\{k, m\}$. The Newton polyhedron $\Gamma_+(I_{k,\ell,m})$ has only one face Δ of maximal dimension with vertices $(0, 2)$ and $(r, 0)$. From Proposition 2.8 we have $\dim_{\mathbb{C}} \mathcal{O}_2/I_{k,\ell,m} = 2r$, if $k \neq m$ and $\dim_{\mathbb{C}} \mathcal{O}_2/I = 2\ell + k$, if $k = m$.

(ii) Suppose now that $\ell < r/2$. Then, $\Gamma_+(I)$ has two 1-dimensional compact faces Δ_1 and Δ_2 with vertices $(0, 2), (\ell, 1)$ and $(r, 0)$.

If $k \leq m$, we have $I_{\Delta_1} = \langle y^2, y^2 + x^\ell y\rangle$ and $I_{\Delta_2} = \langle x^k, x^\ell y\rangle$, hence I is Newton non-degenerate and by Corollary 2.6, $\dim_{\mathbb{C}} \mathcal{O}_2/I = n!v(I) = 2\ell + k$.

If $m < k$, the ideal I is Newton degenerate and the system of generators $\{y^2 + x^k, y^2 + x^\ell y + x^m\}$ is also degenerate with respect to $\Gamma_+(I)$. If we prolong the face Δ_2, we obtain another polyhedron Δ that intersects the Y axis at $(0, \frac{m}{m-\ell})$. The filtration induced by Δ is given by the map $\phi : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that $\phi(a, b) = a + (m - \ell)b$. Observe that the above system is non-degenerate on this polyhedron if and only if $\phi(0, 2) \leq \phi(k, 0)$, which is equivalent to say that $2m - k \leq 2\ell$. In this case, $\dim_{\mathbb{C}} \mathcal{O}_2/I = 2m$, by Theorem 2.5. We observe that we cannot apply Theorem 2.5 in the remaining cases.

3. The multiplicity of Newton non-degenerate ideals

The integral closure of an ideal I in the ring $\mathcal{O}_n$ is a key ingredient to study the problem of Whitney equisingularity for families of complex analytic hypersurfaces with isolated singularities. For ideals with finite colength, the invariant which controls the integral closure is the multiplicity of I. In this section we show how to calculate the multiplicity of any Newton non-degenerate ideal I with finite colength in terms of $\Gamma_+(I)$. We also give a result relating Definitions 2.2 and 2.3. To end this section, we show how to compute the multiplicity of products of ideals and the local version of Bernstein's theorem, which is very useful in our context.

Given an ideal I in a ring R, an element $h \in R$ is *integral* over I if it satisfies a relation

$$h^k + a_1 h^{k-1} + \cdots + a_{k-1}h + a_k = 0, \quad \text{with } a_i \in I^i.$$

The set of such elements is the ideal $\overline{I}$ in R, called the integral closure of I in R.

When $R = \mathcal{O}_n$, Teissier gave in [22, p.288] various equivalent notions to the above concept.

Proposition 3.1. *Let I be an ideal in $\mathcal{O}_n$. The following statements are equivalent:*

(i) $h \in \overline{I}$,

(ii) *For each system of generators $g_1, \ldots, g_s$ of I there exists a neighbourhood U of 0 and a constant $C > 0$ such that $|h(x)| \leq C \sup\{|g_1(x)|, \ldots, |g_s(x)|\}$, for all $x \in U$.*

(iii) *For each analytic curve $\varphi : (\mathbb{C}, 0) \to (\mathcal{O}_n, 0)$, $h \circ \varphi$ lies in $(\varphi^*(I))\,\mathcal{O}_1$.*

Since these conditions are essential to prove some results of this section, we shall consider in this section the ring $\mathcal{O}_n$. We make an abuse of notation and also denote this ring by $\mathcal{A}$.

Definition 3.2 ([16, p.109]). The multiplicity of an ideal I of finite codimension in $\mathcal{A}$ is defined as

$$e(I) = \lim_{k \to \infty} \frac{n!}{k^n} \dim_{\mathbb{C}} \mathcal{A}/I^k.$$

Now, we quote the following result of Teissier about the multiplicity of monomial ideals.

Lemma 3.3 ([23, p.131]). *If I is an ideal of $\mathcal{A}$ of finite codimension that is generated by monomials, then $e(I) = n!v(I)$.*

Next, we show that the Newton non-degeneracy condition characterizes all ideals I satisfying the condition $e(I) = n!v(I)$.

We define the set $\mathrm{C}(\overline{I})$ as the convex hull in $\mathbb{R}^n_+$ of $\{k \in \mathbb{Z}^k : x^k \in \overline{I}\}$.

Proposition 3.4. *Let $I \subseteq \mathcal{A}$ be an ideal of finite codimension. Then, the relation $e(I) = n!v(I)$ holds if and only if $\Gamma_+(I) = \mathrm{C}(\overline{I})$.*

Corollary 3.5. *Let I be an ideal of finite codimension of $\mathcal{A}$, then I is Newton non-degenerate if and only if $e(I) = n!v(I)$.*

This corollary is a consequence of Proposition 3.4 and the following theorem.

Theorem 3.6. [21, 3.4] *Let I be an ideal of $\mathcal{A}$ of finite codimension, then* $\mathrm{C}(\overline{I}) \subseteq \Gamma_+(I)$. *The equality $\Gamma_+(I) = \mathrm{C}(\overline{I})$ holds if and only if I is Newton non-degenerate.*

We give now some results relating the integral closure and reductions of an ideal which are needed to prove the Proposition 3.4.

Lemma 3.7. *Let I be an ideal of $\mathcal{A}$, then $\Gamma_+(I) = \Gamma_+(\overline{I})$.*

Proof. Let I_0 be the ideal in $\mathcal{A}$ generated by all monomials x^k, where k belongs to supp I. Since I_0 is a monomial ideal, we have that $\Gamma_+(I_0) = \Gamma_+(\overline{I_0})$ (see [23, p. 129]). Then,

$$\Gamma_+(I) \subseteq \Gamma_+(\overline{I}) \subseteq \Gamma_+(\overline{I_0}) = \Gamma_+(I_0) = \Gamma_+(I).$$

□

Given two ideals $J \subseteq I$ in an arbitrary ring R, J is said to be a *reduction* of I when there is an integer $r > 0$ such that $I^{r+1} = JI^r$. From [4] Lemma 4.6.5 and the work of Rees [20] we have the following characterization of reductions in $\mathcal{A}$.

Theorem 3.8 (Rees, see [23, p.132]). *Let $J \subseteq I$ be a pair of ideals of $\mathcal{A}$ with I of finite codimension. Then, the following conditions are equivalent:*

(i) *J is a reduction of I;*
(ii) *$e(I)$=$e(J)$;*
(iii) *J and I have the same integral closure.*

The following proposition is immediate from Theorems 3.6, 3.8 and Lemma 3.7.

Proposition 3.9. *If I is a Newton non-degenerate ideal of $\mathcal{A}$, then any reduction of I is also Newton non-degenerate.*

Proof of Proposition 3.4. We denote by $K_I \subseteq \overline{I}$ the ideal of $\mathcal{A}$ generated by all monomials $x^k \in \overline{I}$, observe that the Newton polyhedron $\Gamma_+(K_I)$ is equal to the polyhedron $\mathrm{C}(\overline{I})$. We denote the number $\mathrm{V}_n(\Gamma_-(K_I))$ by $w(I)$. From Lemma 3.3 we obtain the following inequalities for any ideal I of finite codimension of $\mathcal{A}$

$$n!w(I) = e(K) \geq e(\overline{I}) = e(I) \geq e(I_0) = n!\mathrm{V}_n(\Gamma_-(I)) = n!v(I), \tag{2}$$

here we use the fact that $e(J) \geq e(I)$ when $J \subseteq I$ and the equality $\Gamma_+(I) = \Gamma_+(I_0)$.

We shall show that I is a Newton non-degenerate ideal if and only if one of the above inequalities becomes an equality.

Suppose that $e(I) = n!v(I)$. Since the multiplicity of I_0 is also equal to $n!v(I_0) = n!v(I)$ and $I \subseteq I_0$, we find that I is a reduction of I_0, by Theorem 3.8. Then, both ideals have the same integral closure and this implies that $\overline{I}$ is a monomial ideal since I_0 is a monomial ideal. In particular we have that $\Gamma_+(I) = \mathrm{C}(\overline{I})$, since $\Gamma_+(I) = \Gamma_+(\overline{I})$.
Suppose that $\Gamma_+(I) = \mathrm{C}(\overline{I})$, then $v(I) = w(I)$ and by (2) we deduce that $e(I) = n!v(I)$. □

As a consequence of the previous results, any Newton degenerate ideal $I \subseteq \mathcal{A}$ satisfies the strict inequality $e(I) > n!v(I)$. Moreover, the inclusion $\mathrm{C}(\overline{I}) \subseteq \Gamma_+(I)$ must be strict. Hence, there arises the question of determining the cases in which the multiplicity of a Newton degenerate ideal can be calculated as $n!w(I)$. As we shall see in the following lemma this expression for the multiplicity only runs with Newton non-degenerate ideals.

Corollary 3.10. *Let $I \subseteq \mathcal{A}$ be an ideal of finite codimension. Then, the relation $e(I) = n!w(I)$ holds if and only if $\Gamma_+(I) = \mathrm{C}(\overline{I})$.*

Proof. The *if* part is an immediate consequence of (2). If $e(I) = n!w(I)$, then K_I is a reduction of $\overline{I}$, since $e(K_I) = n!\mathrm{V}_n(\Gamma_-(K_I)) = n!w(I)$. Then, the integral closure of K_I is equal to $\overline{I}$, which implies that $\overline{I}$ is a monomial ideal. □

Now, we can summarize the above results as follows.

Corollary 3.11. *Let $I \subseteq \mathcal{A}$ be an ideal of finite codimension. Then, the following conditions are equivalent:*

(i) *I is Newton non-degenerate;*
(ii) *$e(I) = n!v(I)$;*
(iii) *$e(I) = n!w(I)$;*
(iv) *$\overline{I}$ is generated by monomials;*
(v) *I admits a Newton non-degenerate reduction;*
(vi) *$I_0 \subseteq \overline{I}$;*
(vii) *$\overline{I} = \{f \in \mathcal{A} : \Gamma_+(f) \subseteq \Gamma_+(I)\}$.*

Next, we give a characterization of non-degenerate systems with respect to a given Newton polyhedron that is related with the Newton non-degeneracy property.

For a fixed Newton polyhedron Γ_+ and a system of generators $\{g_1, \dots, g_n\}$ of an ideal I with $\mathrm{d}(g_i) = d_i$, we set $d = d_1 \cdots d_n$.

Theorem 3.12. *Under the same hypothesis as in Theorem 2.5, the following conditions are equivalent:*

(i) *the system $\{g_1, \dots, g_n\}$ is non-degenerate on Γ_+;*
(ii) *the ideal $J = \langle g_1^{d/d_1}, \dots, g_n^{d/d_n}\rangle$ is Newton non-degenerate and $\Gamma_+(J) = \Gamma_+(\mathcal{A}_d)$.*

Proof. Suppose that (i) holds. From part (ii) of Theorem 2.5, we have that all inequalities in (1) become equalities. In particular, we have $v(J) = v(\mathcal{A}_d)$ and this implies that $\Gamma_+(J) = \Gamma_+(\mathcal{A}_d)$. Therefore, J is a Newton non-degenerate ideal, by Corollary 3.11.

Now, suppose condition (ii). From the condition $\Gamma_+(J) = \Gamma_+(\mathcal{A}_d)$, the Newton polyhedron of J is face-wise parallel to Γ_+. Then, if Δ is a face of Γ_+ and we denote the elements g^{d/d_i} by h_i, for $i = 1, \dots, n$, we have that the ideal in $\mathcal{A}_\Delta$ generated by $\mathrm{in}_\Delta\, h_1, \dots, \mathrm{in}_\Delta\, h_n$ has finite codimension. Moreover,

$$\langle \mathrm{in}_\Delta h_1, \dots, \mathrm{in}_\Delta h_n\rangle = \langle (\mathrm{in}_\Delta g_1)^{d/d_1}, \dots, (\mathrm{in}_\Delta g_n)^{d/d_n}\rangle \subseteq \langle \mathrm{in}_\Delta g_1, \dots, \mathrm{in}_\Delta g_n\rangle.$$

Then $\langle \mathrm{in}_\Delta\, g_1, \dots, \mathrm{in}_\Delta\, g_n\rangle$ has also finite codimension in $\mathcal{A}_\Delta$ and $\{g_1, \dots, g_n\}$ verifies the definition of non-degeneracy with respect to the polyhedron Γ_+. □

Example 3.13. Consider the ideal in $\mathcal{O}_2$ given by $I = \langle x^\alpha + x^r y^s, x^s y^r + y^\beta\rangle$. Suppose that $r > s$ and that the parameters r, s are such that the polyhedron $\Gamma_+(I)$ has three compact faces. The vertices of $\Gamma_+(I)$ are $(\alpha, 0)$, (r, s), (s, r) and $(0, \beta)$. By Theorem 3.12 we see that the generating system given above is always degenerate with respect to any Newton polyhedron. Anyway, we can use the filtration induced by $\Gamma_+(I)$ and part (i) of Theorem 2.5 to give the inequality

$$\dim_{\mathbb{C}} \mathcal{A}/I > r^2 + s^2 + s(\alpha + \beta - 2s).$$

To end this section we show how to compute the multiplicity of a product of Newton non-degenerate ideals and the local version of Bernstein Theorem.

3.1. **Multiplicity of products of ideals.** In [23, p.132], Teissier states that if I and J are monomial ideals of $\mathcal{A}$, then the number $v(I^rJ^s) = \mathrm{V}_n\left(\mathbb{R}^n_+ \setminus \Gamma_+(I^rJ^s)\right)$ admits the following expression:

$$v(I^rJ^s) = \sum_{i=0}^{n} \binom{n}{i} v_i\, r^i s^{n-i},$$

for certain positive rational numbers v_i. These numbers are called *mixed covolumes of* $\Gamma_+(I)$ *and* $\Gamma_+(J)$ and are also denoted by $v_i(I,J)$. It can be proved that $v_0(I,J) = v(J)$ and $v_n(I,J) = v(I)$.

Since the integral closure of a Newton non-degenerate ideal is a monomial ideal, if I and J are Newton non-degenerate ideals, the function $v(I^rJ^s)$ can also be expressed as an homogeneous polynomial in r and s of degree n. In fact, we have $v(I^rJ^s) = \sum_{i=0}^{n}\binom{n}{i} v_i\, r^i s^{n-i}$, because

$$\Gamma_+(I^rJ^s) = \Gamma_+\left(\overline{I^rJ^s}\right) = \Gamma_+\left(\overline{\overline{I}^r\,\overline{J}^s}\right) = \Gamma_+\left(\overline{I}^r\,\overline{J}^s\right).$$

Then, the numbers $v_i(I,J)$ are also defined when I and J are Newton non-degenerate.

Moreover, as can be seen in [23, p.132], given two ideals I and J of finite codimension in $\mathcal{A}$, the multiplicity $e(I^rJ^s)$ is expressed as

$$e(I^rJ^s) = \sum_{i=0}^{n} \binom{n}{i} e_i\, r^i s^{n-i},$$

for certain positive integers e_i. These are also denoted by $e_i(I,J)$, for all $i = 0,\dots,n$ and are called *mixed multiplicities of* I *and* J. It is also proved in [22, p. 307] that $e_0(I,J) = e(J)$ and $e_n(I,J) = e(I)$.

Corollary 3.14. *Suppose that I and J are Newton non-degenerate ideals of $\mathcal{A}$, then IJ is also Newton non-degenerate.*

Proof. Since I and J are Newton non-degenerate ideals, we have that $\overline{I}$ and $\overline{J}$ are generated by monomials. The integral closure of IJ can be always expressed as $\overline{IJ} = \overline{\overline{I}\,\overline{J}}$. Then, $\overline{IJ}$ is also generated by monomials and IJ is Newton non-degenerate, by Corollary 3.11. □

Corollary 3.15. *Suppose that I and J are Newton non-degenerate ideals of $\mathcal{A}$, then $e_i(I,J) = n!v_i(I,J)$, for all $i = 1,\dots,n$.*

Proof. By Corollary 3.14 we know that I^rJ^s is Newton non-degenerate for all r, s. Then $e(I^rJ^s) = n!v(I^rJ^s)$, for all r, s. □

This result can be generalized to the case of more than two ideals. Let I_1, ..., I_m be ideals of $\mathcal{A}$. We define the numbers $v_{i_1,\dots,i_m}(I_1,\dots,I_m)$ and $e_{i_1,\dots,i_m}(I_1,\dots,I_m)$ by

$$\begin{aligned} v(I_1^{r_1}\cdots I_m^{r_m}) &= \sum_{i_1+\cdots+i_m=n} \binom{n}{i_1\ \cdots\ i_m} v_{i_1,\dots,i_m}(I_1,\dots,I_m) r_1^{i_1}\cdots r_m^{i_m}, \quad \text{and} \\ e(I_1^{r_1}\cdots I_m^{r_m}) &= \sum_{i_1+\cdots+i_m=n} \binom{n}{i_1\ \cdots\ i_m} e_{i_1,\dots,i_m}(I_1,\dots,I_m) r_1^{i_1}\cdots r_m^{i_m}, \end{aligned}$$

where $\binom{n}{i_1\ \cdots\ i_m} = n!/(i_1!\cdots i_m!)$.

Corollary 3.16. *If $I_1,\dots,I_m$ are Newton non-degenerate, then we have*

$$e_{i_1,\dots,i_m}(I_1,\dots,I_m) = n!v_{i_1,\dots,i_m}(I_1,\dots,I_m).$$

When $m = n$, the number $v_{1,\dots,1}(I_1,\dots,I_n)$ is nothing but the mixed covolume of the polyhedra $\Gamma_+^{(j)} = \Gamma_+(I_j)$, for $j = 1,\dots,n$, denoted by $V(\Gamma_+^{(1)},\dots,\Gamma_+^{(n)})$, which we treat in the next subsection.

3.2. **Local Bernstein's theorem.** By the theory due to D.N. Bernstein [2] and A.G. Khovanskii [13, 14] it is possible to compute the Euler characteristics of the zero set of a complete intersection system of polynomials using the Newton polyhedra of such polynomials. We state here the local version of Bernstein's theorem, which is very useful in our context. Even though we might say it is stated in §5.1 of [14] implicitly, we believe it is worth to state here the formula explicitly, because of its importance.

Let $\Gamma_+^{(i)}$, $i = 1,\dots,n$, be n Newton polyhedra. We denote by $V_n(\Gamma_+)$ the n-dimensional volume of $\Gamma_- = \mathbb{R}_+^n - \Gamma_+$.

The *mixed covolume* of $\{\Gamma_+^{(1)},\dots,\Gamma_+^{(n)}\}$, denoted by $V\left(\Gamma_+^{(1)},\dots,\Gamma_+^{(n)}\right)$, is the number

$$\begin{aligned} V\left(\Gamma_+^{(1)},\dots,\Gamma_+^{(n)}\right) = \frac{1}{n!}\Big((-1)^{n-1}\sum_i V_n(\Gamma_+^{(i)}) + (-1)^{n-2}\sum_{i<j} V_n(\Gamma_+^{(i)}+\Gamma_+^{(j)}) + \cdots \\ \cdots + V_n(\Gamma_+^{(1)}+\cdots+\Gamma_+^{(n)})\Big). \end{aligned}$$

The definition of mixed covolume given here is a non-compact polyhedral version of the mixed volume given in [2] (up to the multiplication by the constant $\frac{1}{n!}$). See Chapter 7, §4,5 in [6] for the definition and more information about the mixed volume. Comparing Bernstein's theorem with the definition of the mixed volume in [6] (Definition 4.11 on page 322), we conclude the mixed covolume we have defined is equal to $v_{1,\dots,1}(I_1,\dots,I_n)$ that we treated at the end of §3.1.

Let $f_1,\dots,f_n$ be analytic functions in variables $x_1,\dots,x_n$ so that the Newton polyhedron of f_i is in $\Gamma_+^{(i)}$ for $i = 1,\dots,n$. For an integral vector $\xi = (\xi_1,\dots,\xi_n)$, $\xi_i \geq 0$, $\Delta_i(\xi)$ denotes the face of $\Gamma_+^{(i)}$ supported by the vector ξ. We set $f_i^\xi = f_{i\Delta_i(\xi)}$.

Definition 3.17. The system $f_1,\dots,f_n$ is said to be *non-degenerate* on $\{\Gamma_+^{(1)},\dots,\Gamma_+^{(n)}\}$, if for any $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}_+^n$, and for any solution $z \in (\mathbb{C}\setminus\{0\})^n$ of the system of equations $f_1^\xi = \cdots = f_n^\xi = 0$, the differentials $df_1^\xi(z), \ldots, df_n^\xi(z)$ are linearly independent in the tangent space to the point z.

Theorem 3.18 (Local Bernstein's theorem). *If the system $f_1, \ldots, f_n$ is non-degenerate on $\{\Gamma_+^{(1)}, \ldots, \Gamma_+^{(n)}\}$, then*

$$\dim_{\mathbb{C}} \mathcal{A}/\langle f_1, \ldots, f_n\rangle = n!V\left(\Gamma_+^{(1)}, \ldots, \Gamma_+^{(n)}\right).$$

Proof. We first remark that the left hand side is equal to the local intersection number of the divisors defined by $f_i = 0$, $i = 1, \ldots, n$, at 0. By standard argument of finite determinacy (see Theorem 2.1 in [26]), we may assume that $f_1, \ldots, f_n$ are polynomials without changing their Newton principal parts. Now we consider $g_i := f_i + x_1^N + \cdots + x_n^N$, $i = 1, \ldots, n$, for sufficiently large N. We denote the Newton polyhedron (as polynomials) of g_i by Δ_i. Let D_i denote the divisor in the complex projective space $P^n(\mathbb{C})$ defined by $g_i = 0$. We remark that the intersection number of $D_1, \ldots, D_n$ is N^n. Let X be the projective toric variety corresponding to the compact polyhedron $\Delta_1 + \cdots + \Delta_n$ and $\pi : X \to P^n(\mathbb{C})$ the corresponding toric modification. Let D_i' denote the strict transform of D_i by π. Since the system $\{g_1, \ldots, g_n\}$ defines a non-degenerate system at 0, $D_1' \cap \cdots \cap D_n'$ is nonsingular near $\pi^{-1}(0)$. This implies that $D_1' \cap \cdots \cap D_n'$ does not intersect with $\pi^{-1}(0)$. So we have

$$(D_1, \ldots, D_n)_0 + (D_1', \ldots, D_n') = (D_1, \ldots, D_n) = N^n.$$

Let Δ denote the convex hull of the set

$$\{(0, \ldots, 0), (N, 0, \ldots, 0), (0, N, 0, \ldots, 0), \ldots, (0, \ldots, 0, N)\}.$$

Then we have

$$N^n = n!V(\Delta, \ldots, \Delta) = \sum_I (-1)^{n-|I|} \mathrm{V}_n\Big(\sum_{i\in I} \Delta\Big),$$

where I is a subset of $\{1, \ldots, n\}$ and $|I|$ denotes the number of elements of I.

By Bernstein theorem [2], we have

$$(D_1', \ldots, D_n') = n!V(\Delta_1, \ldots, \Delta_n) = \sum_I (-1)^{n-|I|} \mathrm{V}_n\Big(\sum_{i\in I} \Delta_i\Big).$$

Thus,

$$\begin{aligned}
(D_1, \ldots, D_n)_0 &= n!\,(V(\Delta, \ldots, \Delta) - V(\Delta_1, \ldots, \Delta_n)) \\
&= \sum_I (-1)^{n-|I|} \Big(\mathrm{V}_n\Big(\sum_{i\in I} \Delta\Big) - \mathrm{V}_n\Big(\sum_{i\in I} \Delta_i\Big)\Big) \\
&= \sum_I (-1)^{n-|I|} \mathrm{V}_n\Big(\sum_{i\in I} \Gamma_+^{(i)}\Big) \\
&= n!V\left(\Gamma_+^{(1)}, \ldots, \Gamma_+^{(n)}\right).
\end{aligned}$$

□

Example 3.19. Let $f_1 = x^8 + x^2y^2 + y^5$, and $f_2 = x^{11} + x^4y^2 + y^9$. Then we have

$$\dim_{\mathbb{C}} \mathbb{C}[[x, y]]/\langle f_1, f_2\rangle = V_2(\Gamma_+^{(1)} + \Gamma_+^{(2)}) - V_2(\Gamma_+^{(1)}) - V_2(\Gamma_+^{(2)}) = 78 - 13 - 19 = 36.$$

4. Graded Algebras induced by Newton filtrations

In [15], Kouchnirenko uses the Koszul complex in order to compute the codimension of an ideal of the form $I = \langle x_1\partial f/\partial x_1, \dots, x_n\partial f/\partial x_n\rangle \subset \mathcal{F}_n$. The problem we treat here is how to apply the method of Kouchnirenko to other complexes. It is natural to expect similar computations work for some other complexes, and we will show this is actually true for complex coming from a free resolution of a perfect module. As a particular case we give the proof of part (ii) of Theorem 2.5 and compute the codimension of some determinantal ideals which are non-degenerate on a given Newton polyhedron.

In the sequel we shall consider the ring $\mathcal{F}_n$ and denote this ring by $\mathcal{A}$. We set the problem in a more general context. That is, we consider finite $\mathcal{A}$-modules of projective dimension $\leq n$. We denote the projective dimension of a module M by $\operatorname{pd} M$.

4.1. **Preliminaries.** Let R be a Noetherian ring, I an ideal of R, and M a finite R-module. We define the *depth* $\operatorname{depth}(I, M)$ (or *grade* of I with respect to M) by

$$\operatorname{depth}(I, M) = \operatorname{depth}_R(I, M) = \text{the length of a maximal } M\text{-sequence in } I.$$

We often use the fact that $\operatorname{depth}(I, M)$ is determined by the radical ideal $\sqrt{I}$ and M. We recall two theorems in commutative algebra which are important in this section. Let $\mathcal{F}$ denote a complex of finite free R-module:

$$\mathcal{F}\ :\ 0 \longrightarrow F_\ell \xrightarrow{\varphi_\ell} F_{\ell-1} \xrightarrow{\varphi_{\ell-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0.$$

Theorem 4.1 (Buchsbaum-Eisenbud acyclicity criterion). *The complex $\mathcal{F}$ is acyclic, if and only if* $\operatorname{depth}(I_{r_i}(\varphi_i), R) \geq i$ *for* $i = 1, \dots, \ell$.

Reference for a proof. See [5, (16.15)], [4, Theorem 1.4.12]. □

Theorem 4.2 (Transfer of perfection). *Let M be a perfect R-module with* $\operatorname{pd} M = \ell$. *Let S be a Noetherian R-algebra such that* $\operatorname{depth}(\operatorname{Ann}(M \otimes S), S) \geq \ell$ *and $M \otimes S \neq 0$. Then $M \otimes S$ is perfect with* $\operatorname{pd} M = \ell$. *Furthermore $\mathcal{F} \otimes S$ is a free resolution of $M \otimes S$ for every free resolution $\mathcal{F}$ of M of length ℓ.*

Reference for a proof. See [5, (3.5)]. □

4.2. **Graded algebras.** Let Γ_+ be a Newton polyhedron and $\{\mathcal{A}_q\}_{q\geq 0}$ the filtration of $\mathcal{A}$ induced by this polyhedron. Consider the graded ring $A = \operatorname{gr}\mathcal{A} = \bigoplus_q A_q$, where $A_q = \mathcal{A}_q/\mathcal{A}_{q+1}$, for all $q \geq 0$. As a consequence of Theorem 4.5.6 of [4], we have that the dimension of A is equal to n. We also consider the following graded rings associated to A:

(i) the *Rees ring* $\quad R(\mathcal{A}) = \bigoplus_{q\geq 0} \mathcal{A}_q T^q \subset \mathcal{A}[T]$, $\quad T$ an indeterminate

(ii) the *extended Rees ring* $\quad \widehat{R}(\mathcal{A}) = R(\mathcal{A}) \oplus \bigoplus_{q<0} \mathcal{A}T^q \subset \mathcal{A}[T, T^{-1}]$.

We understand these rings as graded rings graded by the degree of the indeterminate T. Observe that one has the representation $A = \widehat{R}(\mathcal{A})/T^{-1}\widehat{R}(\mathcal{A})$. The natural map $\operatorname{Proj}(R(\mathcal{A})) \to \operatorname{Spec}(\mathcal{A})$ is the "formal completion" of the projective toric modification defined by Γ_+ and $\operatorname{Proj}(A)$ is its exceptional set.

Proposition 4.3. *The rings $\widehat{R}(\mathcal{A})$ and A are Cohen-Macaulay.*

Proof. Since A is a quotient of $\widehat{R}(\mathcal{A})$ by a nonzero divisor T^{-1}, the ring $\widehat{R}(\mathcal{A})$ is Cohen-Macaulay if and only if A is so.

By Proposition 4.10 of [12], it is enough to show that the localization A_{A_+} is Cohen-Macaulay, where $A_+ = \bigoplus_{q>0} A_q$. Therefore, by [7] Lemma 18.1, it is enough to show that there is an A-sequence in A_+ of length n. Using [15], we can show that this is true.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a Newton non-degenerate function germ such that the Newton polyhedron $\Gamma_+(f)$ is exactly the one which we consider for the initial Newton filtration.

Let M be the degree of f with respect to this filtration. Then, the Koszul complex generated by $x_1\partial f/\partial x_1, \dots, x_n\partial f/\partial x_n$ is strict and $x_1\partial f/\partial x_1 T^M, \dots, x_n\partial f/\partial x_n T^M$ form a regular sequence in $\widehat{R}(\mathcal{A})$. This also represents a regular sequence in A. □

Next, we give some preliminary results from [15] that we shall use in this section.

For a given g in $\mathcal{A}$, we call *the initial form of* g, the image in A of the principal part of g.

For each face Δ of Γ_+, we denote by A_Δ the graded ring induced by the Newton filtration to the ring $\mathcal{A}_\Delta$. The *initial form of* g *over* Δ is defined as the image in A_Δ of the principal part of g over Δ.

We also denote by $\operatorname{in} g$ (resp. $\operatorname{in}_\Delta g$) the initial form of g (resp. the initial form of g over Δ), since there is no risk of confusion. Remember that if $d = \mathrm{d}(g)$, then $\operatorname{in} g = g + \mathcal{A}_{d+1} \in A_d$, therefore $\operatorname{in}_\Delta g = g_{C(\Delta)} + (\mathcal{A}_\Delta)_{d+1} \in (A_\Delta)_d$.

Let $\overline{I}_q$ be the family of all q-dimensional faces of Γ not contained in the union of the coordinate planes, for any $q \geq 0$. Now, we define the rings

$$C_q = \bigoplus_{\Delta \in \overline{I}_q} A_\Delta, \qquad \text{for any } q \geq 0.$$

These rings are very important, since they take part on an exact sequence that provides a method for the calculation of the Hilbert series of A.

Proposition 4.4 ([15, 2.4]). *For each face $\Delta \subseteq \Gamma$, there exists a grade preserving epimorphism $\pi_\Delta : A \to A_\Delta$. Moreover, for each pair $\Delta \subset \Delta_1$ there exists a grade preserving epimorphism $\pi_{\Delta,\Delta_1} : A_{\Delta_1} \to A_\Delta$ such that $\pi_{\Delta,\Delta_1} \circ \pi_{\Delta_1} = \pi_\Delta$.*

The construction of the morphisms of the above proposition is quite simple. The image of an element $f + \mathcal{A}_{q+1} \in A_q$ through π_Δ is given by the image in A_Δ of $f_{C(\Delta)}$. The morphisms π_{Δ,Δ_1}, where $\Delta \subset \Delta_1$, are defined in an analogous way.

Theorem 4.5 ([15, 2.6]). *There exists an exact sequence of graded A-modules with grade preserving morphisms of the form*

$$0 \longrightarrow A \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0. \tag{3}$$

Theorem 4.6 ([15, 2.9]). *Let M be the value of the filtration over Γ, then*

(i) *given a q-dimensional face of Γ, the Hilbert series of A_Δ is written as*

$$H_{A_\Delta}(t) = \frac{p(t)}{(1 - t^M)^{q+1}},$$

where $p(t)$ is a polynomial with rational coefficients.

(ii) *Moreover, if* Δ *is* $(n-1)$*-dimensional, we have that*

$$\lim_{t\to 1}(1-t^M)^n H_{A_\Delta}(t) = n!\,\mathrm{V}_n(P(\Delta)),$$

where $P(\Delta)$ *is the pyramid of base* Δ *and vertex at* 0.

Corollary 4.7. *Under the same hypotheses as in the previous theorem, we have the following equivalent equalities:*

(i) $\lim_{t\to 1}(1-t)^n H_A(t) = n!\,\mathrm{V}_n(\phi^{-1}([0,1]))$,

(ii) $\lim_{t\to 1}(1-t^M)^n H_A(t) = n!\,\mathrm{V}_n(\Gamma_-)$.

We recall that a morphism α between filtered rings $\mathcal{A}$ and $\mathcal{B}$, with filtrations $\{\mathcal{A}_q\}_{q\in\mathbb{Z}}$ and $\{\mathcal{B}_q\}_{q\in\mathbb{Z}}$ respectively, is said to be *strict* if $\alpha(\mathcal{A}_q)=\alpha(\mathcal{A})\cap\mathcal{B}_q$, for each $q\in\mathbb{Z}$.

Lemma 4.8 ([15, 4.3]). *Let* $\mathcal{K}\to\mathcal{A}\to\mathcal{B}$ *be a complex of filtered* $\mathbb{C}$*-modules which preserve filtrations. If the induced morphism* $\operatorname{gr}\mathcal{K}\to\operatorname{gr}\mathcal{A}\to\operatorname{gr}\mathcal{B}$ *is exact, then the morphism* $\mathcal{A}\to\mathcal{B}$ *is strict.*

Lemma 4.9 ([15, 4.4]). *Let* $\alpha:\mathcal{A}\to\mathcal{B}$ *be a strict morphism between filtered* $\mathbb{C}$*-modules. Then, there is a* $\mathbb{C}$*-module isomorphism* $\operatorname{gr}(\mathcal{B}/\alpha(\mathcal{A}))\cong\operatorname{gr}\mathcal{B}/\operatorname{gr}(\alpha)(\operatorname{gr}\mathcal{A})$.

Now we introduce some notation. If $R=\bigoplus_{q\geq 0}R_q$ is a graded ring, we denote by $R(-a)$ the graded ring given by $R(-a)_q=R_{q-a}$, for all $q\geq 0$. Consider the graded complex of free R-modules

$$\mathcal{F}:\quad 0\longrightarrow\bigoplus_{i=1}^{e_\ell}R(-a_i^{(\ell)})\longrightarrow\cdots\longrightarrow\bigoplus_{i=1}^{e_1}R(-a_i^{(1)})\longrightarrow\bigoplus_{i=1}^{e_0}R(-a_i^{(0)})\longrightarrow 0$$

so that all differentials are of degree 0. Then, we define the polynomial

$$q_{\mathcal{F}}(t)=\sum_{i=0}^{\ell}(-1)^i(t^{a_1^{(i)}}+\cdots+t^{a_{e_k}^{(i)}}).$$

Remember that for two graded complexes $\mathcal{F}$ and $\mathcal{G}$, $q_{\mathcal{F}\otimes\mathcal{G}}(t)=q_{\mathcal{F}}(t)q_{\mathcal{G}}(t)$.

4.3. **Comparing exactness of complexes.** We first consider the following complex of free $\mathcal{A}$-modules

$$0\longrightarrow\mathcal{A}^{e_\ell}\xrightarrow{\varphi_\ell}\cdots\xrightarrow{\varphi_2}\mathcal{A}^{e_1}\xrightarrow{\varphi_1}\mathcal{A}^{e_0}. \tag{4}$$

Let Γ_+ be a Newton polyhedron and the filtration induced by Γ_+. As usual, denote by $\mathrm{d}(g)$ the level of a series g with respect to this filtration. Let $(g_{ij}^{(s)})$ be a representation matrix of the map $\varphi_s:\mathcal{A}^{e_s}\to\mathcal{A}^{e_{s-1}}$, $s=0,1,\dots,\ell$. We also consider non-negative integers $a_i^{(s)}$ for $s=0,1,\dots,\ell$, $i=1,\dots,e_s$ such that $\mathrm{d}(g_{ij}^{(s)})\geq a_j^{(s)}-a_i^{(s-1)}$, for any s,i,j, and understand $\mathcal{A}^{e_s}$ as filtered modules by

$$(h_1,\dots,h_{e_s})\in(\mathcal{A}^{e_s})_q\quad\text{if and only if}\quad h_j\in\mathcal{A}_{q-a_j^{(s)}}\ \text{for } j=1,\dots,e_s.$$

Then, complex (4) becomes a filtered complex.

We next consider the graded free complex

$$0\longrightarrow\widehat{F}_\ell\xrightarrow{\widehat{\varphi}_\ell}\widehat{F}_{\ell-1}\xrightarrow{\widehat{\varphi}_{\ell-1}}\cdots\xrightarrow{\widehat{\varphi}_2}\widehat{F}_1\xrightarrow{\widehat{\varphi}_1}\widehat{F}_0. \tag{5}$$

Here we set the grading of each free module $\widehat{F}_s$ by

$$\widehat{F}_s = \bigoplus_{i=1}^{e_s} \widehat{R}(\mathcal{A})(-a_i^{(s)}), \quad s = 0, 1, \dots, \ell,$$

and we assume that the maps $\widehat{\varphi}_s : \widehat{F}_s \longrightarrow \widehat{F}_{s-1}$ are defined by the matrix

$$\left(g_{ij}^{(s)} \, T^{a_j^{(s)} - a_i^{(s-1)}} \right)_{i=1,\dots,e_s;\ j=1,\dots,e_{s-1}}$$

and are of degree 0.

Thirdly, we set $F_s = \bigoplus_{i=1}^{e_s} A(-a_i^{(s)})$, and consider the complex

$$\text{(6)} \qquad 0 \longrightarrow F_\ell \xrightarrow{\overline{\varphi}_\ell} F_{\ell-1} \xrightarrow{\overline{\varphi}_{\ell-1}} \cdots \xrightarrow{\overline{\varphi}_2} F_1 \xrightarrow{\overline{\varphi}_1} F_0,$$

where $\overline{\varphi}_s : F_s \to F_{s-1}$ is the map induced by the map $\widehat{\varphi}_s : \widehat{F}_s \to \widehat{F}_{s-1}$.

Let $\Delta \in \overline{I}_q$, $q = 0, 1, \dots, n-1$. We set

$$F_{s\Delta} = \bigoplus_{i=1}^{e_s} A_\Delta(-a_i^{(s)}), \quad s = 0, 1, \dots, \ell,$$

and consider the complex induced from the previous one

$$(6_\Delta) \qquad 0 \longrightarrow F_{\ell\Delta} \xrightarrow{\overline{\varphi}_\ell|_\Delta} F_{\ell-1\Delta} \xrightarrow{\overline{\varphi}_{\ell-1}|_\Delta} \cdots \xrightarrow{\overline{\varphi}_2|_\Delta} F_{1\Delta} \xrightarrow{\overline{\varphi}_1|_\Delta} F_{0\Delta}.$$

Lemma 4.10. *If the complex (4) is exact and each φ_s is strict then the complexes (5) and (6) are exact.*

Proof. This is proved in a straightforward way, and we omit the details. □

We set the number $r_k = \sum_{i=k}^{\ell} (-1)^{i-k} e_i$, for all $k = 0, \dots, \ell$.

Proposition 4.11. *The following conditions are equivalent:*

(i) *the complex (4) is exact and each φ_s is strict;*
(ii) *the complex (6) is exact;*
(iii) *the complex (6_Δ) is exact in dimensions $\geq n-q$, for any $\Delta \in \overline{I}_q$, $q = 0, 1, \dots, n-1$;*
(iv) *the complex (5) is exact.*

Proof. The previous lemma implies (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iv). We first show (ii) $\Longrightarrow$ (*i*). If we suppose that the complex (6) is exact, by Lemma 4.8 we have that (4) is a complex and each φ_s is strict. We remark that, by the exactness of (6), for any $a_q \in (\mathcal{A}^{e_i})_q$ with $\varphi_i(a_q) = 0$, there exists $b_q \in (\mathcal{A}^{e_{i+1}})_{q+1}$ such that $a_q - \varphi_{i+1}(b_q) \in (\mathcal{A}^{e_i})_{q+1}$. Let $a \in \mathcal{A}^{e_i}$ with $\varphi_i(a) = 0$. We choose q_0 so that $a \in (\mathcal{A}^{e_i})_{q_0} \setminus (\mathcal{A}^{e_i})_{q_0+1}$. Setting $a_{q+1} = a_q - \varphi_{i+1}(b_q)$, for $q \geq q_0$, we have $\varphi_{i+1}(\sum_{q \geq q_0} b_q) = a$, and we are done by completeness of $\mathcal{A}$.

For the proof of (iii)$\Longrightarrow$(ii), we consider the two (horizontal and vertical) spectral sequences of the double complex which is the tensor product of the complex (6) and the complex in Theorem 4.5. The (p, q)-terms in ${}^0E$ terms are isomorphic to $F_{-q} \otimes C_{n-p}$. Since the horizontal sequences in the double complex are exact, we obtain ${}_{\mathrm{hor}}^{\ 1}E^{p,q} = 0$, and we thus obtain all the cohomologies of the total complex vanish. Next we observe that the condition (iii) implies ${}_{\text{vert}}^{1}E^{p,q} = 0$ if $p+q \leq 0$ and $p \geq 1$. So we obtain

$$ {}_{\text{vert}}^{1}E^{0,q} = {}_{\text{vert}}^{2}E^{0,q} = \cdots = {}_{\text{vert}}^{\infty}E^{0,q} = 0, \quad \text{if } q < 0, $$

and therefore the complex (6) is exact.

We show here that (ii)$\Longrightarrow$(iii). Suppose that (6) is exact and let $\Delta \in \overline{I}_q$. We first remark that

$$ \begin{aligned} \dim A_\Delta / I_{r_i}(\overline{\varphi}_i|_\Delta) &\leq \dim A / I_{r_i}(\overline{\varphi}_i) \\ &\leq n - \operatorname{ht} I_{r_i}(\overline{\varphi}_i) \\ &\leq n - \operatorname{depth}(I_{r_i}(\overline{\varphi}_i), A), && \text{(by [4] Proposition 1.2.14)} \\ &\leq n - i && \text{(by acyclicity of the complex (6)).} \end{aligned} $$

Since A_Δ is Cohen-Macaulay [11], we obtain

$$ \operatorname{depth}(I_{r_i}(\overline{\varphi}_i|_\Delta), A_\Delta) = \dim A_\Delta - \dim A_\Delta / I_{r_i}(\overline{\varphi}_i|_\Delta) \geq q + 1 - n + i. $$

Then, item (iii) follows from Buchsbaum-Eisenbud acyclicity criterion.

We assume (iv). Since $\widehat{R}(\mathcal{A})$ is Cohen-Macaulay, we have that

$$ \begin{aligned} \dim A / I_{r_i}(\overline{\varphi}_i) + 1 &= \dim \widehat{R}(\mathcal{A}) / I_{r_i}(\widehat{\varphi}_i) \\ &= n + 1 - \operatorname{depth}(I_{r_i}(\widehat{\varphi}_i), \widehat{R}(\mathcal{A})) \\ &\leq n + 1 - i && \text{(by acyclicity of (5)).} \end{aligned} $$

Since A is Cohen-Macaulay, we obtain

$$ \operatorname{depth}(I_{r_i}(\overline{\varphi}_i), A) = \dim A - \dim A / I_{r_i}(\overline{\varphi}_i) \geq n - n + i = i. $$

Then, by Buchsbaum-Eisenbud acyclicity criterion, we have (iv)$\Longrightarrow$(ii). □

4.4. **Non-degeneracy of the matrix (g_{ij}).** In this section we assume that (g_{ij}) is the matrix defining the morphism $\varphi_1 : \mathcal{A}^{e_1} \longrightarrow \mathcal{A}^{e_0}$ of complex (4). We also denote the map φ_1 by φ. Suppose that $\operatorname{rank} \varphi = e_0$ and $\operatorname{depth}(I_{e_0}(\varphi)) = \ell$. Assuming $g_{ij} = \sum_k a_k x^k$ and $\Delta \in I_q$, we set

$$ \operatorname{in}_\Delta(g_{ij}) = \sum \left\{ a_k x^k : k \in C(\Delta),\ \mathrm{d}(x^k) = a_j^{(1)} - a_i^{(0)} \right\}. $$

Definition 4.12. We say that the matrix (g_{ij}) is *non-degenerate on* Γ_+ if $I_{e_0}(\operatorname{in}_\Delta(g_{ij}))$ is an ideal of depth $\ell - n + q + 1$ in A_Δ, for each face $\Delta \in \overline{I}_q$.

We remember that $\operatorname{depth} I_{e_0}(\operatorname{in}_\Delta(g_{ij}))$ is just the codimension (or height) of $I_{e_0}(\operatorname{in}_\Delta(g_{ij}))$ in A_Δ because A_Δ is Cohen-Macaulay.

Proposition 4.13. *Suppose that the complex (6_Δ) is exact in dimensions $\geq n - q$, for all $\Delta \in \overline{I}_q$, $q = 0, 1, \ldots, n-1$. Then, the matrix (g_{ij}) is non-degenerate on Γ_+.*

Proof. Let $\Delta \in \overline{I}_q$. We choose $\Delta_1 \in \overline{I}_{n-1}$ such that Δ is a face of Δ_1. Since (6) is exact and $\sum_{i=0}^{\ell}(-1)^i e_i = 0$, we have $\sqrt{I_{r_\ell}(\overline{\varphi}_\ell|_{\Delta_1})} = \sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})}$, by [7] Corollary 20.12.

Therefore

$$\begin{aligned}
\operatorname{depth}(I_{r_1}(\overline{\varphi}_1|_\Delta), A_\Delta) &= \operatorname{depth}(I_{r_1}(\overline{\varphi}_1|_{\Delta_1})A_\Delta, A_\Delta) && \text{(by [7] Corollary 20.5)}\\
&= \operatorname{depth}(\sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})}A_\Delta, A_\Delta)\\
&= \operatorname{depth}(\sqrt{I_{r_\ell}(\overline{\varphi}_\ell|_{\Delta_1})}A_\Delta, A_\Delta) && \text{(by the above equality)}\\
&= \operatorname{depth}(I_{r_\ell}(\overline{\varphi}_1|_{\Delta_1})A_\Delta, A_\Delta)\\
&= \operatorname{depth}(I_{r_\ell}(\overline{\varphi}_1|_\Delta), A_\Delta) && \text{(by [7] Corollary 20.5)}\\
&\geq \ell - n + q + 1,
\end{aligned}$$

where the last inequality is a consequence of the hypothesis and the Buchsbaum-Eisenbud acyclicity criterion applied to complex (6_Δ). □

4.5. **The case for complexes coming from a free resolution of perfect modules.** Set $R = \mathbb{C}[X_1, ..., X_m]$ and we understand R is a graded ring with grading by $\deg X_i = d_i$. Let $\mathfrak{M}$ be a graded perfect R-module with projective dimension $\ell \geq 1$. We consider a graded free resolution of $\mathfrak{M}$:

$$(7) \qquad 0 \longrightarrow \widetilde{F}_\ell \xrightarrow{\widetilde{\varphi}_\ell} \cdots \xrightarrow{\widetilde{\varphi}_2} \widetilde{F}_1 \xrightarrow{\widetilde{\varphi}_1} \widetilde{F}_0 \longrightarrow \mathfrak{M} \longrightarrow 0,$$

where $\widetilde{F}_s = \bigoplus_{i=1}^{e_s} R(-\tilde{a}_i^{(s)})$ and the map $\widetilde{\varphi}_s : \widetilde{F}_s \longrightarrow \widetilde{F}_{s-1}$ are of degree 0.

The perfect R-module $\mathfrak{M}$ has nonzero annihilator, since

$$\operatorname{depth}(\operatorname{Ann}(\mathfrak{M}), R) = \ell.$$

By Theorem 19.8 of [16], we have that $\sum_{i=0}^{\ell}(-1)^i e_i = 0$.

We consider homogeneous elements $f_1, ..., f_m \in \widehat{R}(\mathcal{A})$, and understand $\widehat{R}(\mathcal{A})$ an R-module via the substitution $X_i \mapsto f_i$. Then we can consider the tensor product $\mathfrak{M} \otimes_R \widehat{R}(\mathcal{A})$.

Proposition 4.14. *Suppose that the complex* (5) *is obtained from* (7) *by tensoring* $\widehat{R}(\mathcal{A})$ *over* R, *that is,* $(5) = (7) \otimes_R \widehat{R}(\mathcal{A})$. *We suppose that* $\operatorname{ht}(I_{r_1}(\varphi_1)) = \ell$. *Then, the conditions in Proposition 4.11 are equivalent to the non-degeneracy of* (g_{ij}) *on* Γ_+.

Proof. It is enough to prove that condition (iii) of Proposition 4.11 is obtained from the hypothesis of non-degeneracy. By Lemma 4.10 we have that $\operatorname{pd}(\operatorname{coker}\widehat{\varphi}) \leq \ell$. Since $\mathcal{A} = \widehat{R}(\mathcal{A})/\,(T^{-1} - 1)\,\widehat{R}(\mathcal{A})$, we have

$$\operatorname{depth}(I_{e_0}(\widehat{\varphi}), \widehat{R}(\mathcal{A})) \geq \operatorname{depth}(I_{e_0}(\varphi), \mathcal{A}).$$

Since $\operatorname{ht}(I_{r_1}(\varphi)) = \ell$, $\operatorname{depth}(I_{e_0}(\varphi), \mathcal{A}) = \ell$. Then, we have that $\operatorname{coker}\widehat{\varphi}$ is a perfect $\widehat{R}(\mathcal{A})$-module.

Let $\Delta \in \overline{I}_{n-1}$, consider A_Δ as an $\widehat{R}(\mathcal{A})$-module through the natural maps $\widehat{R}(\mathcal{A}) \to A \to A_\Delta$. Since $\operatorname{coker}\widehat{\varphi}$ is a perfect $\widehat{R}(\mathcal{A})$-module and $\operatorname{depth}(I_{e_0}(\overline{\varphi}|_\Delta), A_\Delta) \geq \ell$, Theorem 4.2 implies that the complex (6_Δ) is exact.

Let $\Delta \in \overline{I}_q$, $q = 0, 1, \ldots, n-2$. By Buchsbaum-Eisenbud acyclicity criterion, the complex (6_Δ) is acyclic in dimensions $\geq n - q$ if and only if the following conditions hold.

$$(E_\Delta) \qquad \operatorname{depth}(I_{r_i}(\overline{\varphi}_i|_\Delta), A_\Delta) \geq i - n + q + 1, \quad \text{for} \quad i = n - q, \ldots, \ell.$$

We choose $\Delta_1 \in \overline{I}_{n-1}$ such that Δ is a face of Δ_1. Because (6_{Δ_1}) is exact, we have $\sqrt{I_{r_i}(\overline{\varphi}_i|_{\Delta_1})} \supseteq \sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})}$ for $i = 1, \ldots, \ell$. Then, we obtain the following inequalities:

$$\begin{aligned}
\operatorname{ht}(I_{r_i}(\overline{\varphi}_i|_\Delta)) &= \operatorname{depth}(I_{r_i}(\overline{\varphi}_i|_\Delta), A_\Delta) && \text{(since } A_\Delta \text{ is Cohen-Macaulay)}\\
&= \operatorname{depth}(\sqrt{I_{r_i}(\overline{\varphi}_i|_\Delta)}, A_\Delta) &&\\
&\geq \operatorname{depth}(\sqrt{I_{r_i}(\overline{\varphi}_i|_{\Delta_1})}A_\Delta, A_\Delta) && \text{(since } \sqrt{I_{r_i}(\overline{\varphi}_i|_\Delta)} \supseteq \sqrt{I_{r_i}(\overline{\varphi}_i|_{\Delta_1})}A_\Delta\text{)}\\
&\geq \operatorname{depth}(\sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})}A_\Delta, A_\Delta) && \text{(since } \sqrt{I_{r_i}(\overline{\varphi}_i|_{\Delta_1})} \supseteq \sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})}\text{)}\\
&= \operatorname{ht}(\sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})}A_\Delta) && \text{(since } A_\Delta \text{ is Cohen-Macaulay)}\\
&= \operatorname{ht}(I_{r_1}(\overline{\varphi}_\Delta)) = \ell - n + q + 1.
\end{aligned}$$

In the last equality we have applied the hypothesis. Then, we see that the conditions in (E_Δ) are satisfied and we obtain condition (iii) of Proposition 4.11. □

We summarize the result we obtained as follows.

Theorem 4.15. *Suppose that there exist a perfect R-module $\mathfrak{M}$, and a resolution (7) of $\mathfrak{M}$ so that (5) = (7) $\otimes_R \widehat{R}(\mathcal{A})$. Then the following conditions are equivalent:*

(i) *the complex (4) is exact and each φ_s is strict;*
(ii) *the complex (6) is exact;*
(iii) *the complex (6_Δ) is acyclic in dimension $\geq n - q$ for any $\Delta \in \overline{I}_q$, $q = 0, 1, \ldots, n-1$;*
(iv) *the complex (5) is exact;*
(v) *the matrix (g_{ij}) is non-degenerate.*

Theorem 4.16. *If we suppose that $\operatorname{coker}\varphi$ has finite codimension, then it is automatically perfect with projective dimension n. Suppose that there exist a perfect R-module $\mathfrak{M}$, and a resolution (7) of $\mathfrak{M}$ so that (5) = (7) $\otimes_R \widehat{R}(\mathcal{A})$. Suppose that the matrix (g_{ij}) is non-degenerate on Γ_+ and that $\operatorname{coker}\varphi$ has finite codimension, then*

$$\dim_{\mathbb{C}} \operatorname{coker}\varphi = (-1)^n \frac{q^{(n)}(1)}{M^n} \mathrm{V}_n(\Gamma_-) \quad \textit{where} \quad q(t) = q_{(6)}(t) = \sum_{s=0}^{n}\sum_{i=1}^{e_s}(-1)^s t^{a_i^{(s)}},$$

M denotes the common value of the filtration induced by Γ_+ along its boundary and $q^{(n)}(1)$ is the n-th derivative of $q_{(6)}$ at 1.

Proof. By Proposition 4.14 the complex (6) is exact, and the Hilbert series of $\operatorname{coker}\overline{\varphi}$ is given by $H_{\operatorname{coker}\overline{\varphi}}(t) = H_A(t)q(t)$, where $H_A(t)$ is the Hilbert series of A and $q(t)$ is the polynomial described above. Then, we conclude

$$\begin{aligned}
\dim_{\mathbb{C}} \operatorname{coker} \varphi &= \dim_{\mathbb{C}} \operatorname{coker} \overline{\varphi} && \text{(by Lemmas 4.8 and 4.9)} \\
&= \lim_{t\to 1} H_A(t) q(t) \\
&= n!\, \mathrm{V}_n(\Gamma_-) \lim_{t\to 1} \frac{q(t)}{(1-t^M)^n} && \text{(by Corollary 4.7)} \\
&= (-1)^n \frac{q^{(n)}(1)}{M^n} \mathrm{V}_n(\Gamma_-).
\end{aligned}$$

□

Proof of Theorem 2.5. We apply here the above theorem to the Koszul complex associated to the system $\{g_1, \dots, g_n\}$. Since they generate an ideal of finite codimension, this complex is acyclic. It is given by the tensor product of

$$0 \longrightarrow A(-d_i) \xrightarrow{g_i} A \longrightarrow 0.$$

Therefore, the polynomial $q(t)$ of Theorem 4.16 associated to this complex is given by $q(t) = \prod_{i=1}^n (1 - t^{d_i})$. Moreover,

$$\lim_{t\to 1} \frac{q(t)}{(1-t)^n} = \lim_{t\to 1} \prod_{i=1}^{n} (1 + t + \cdots + t^{d_i - 1}) = d_1 \cdots d_n,$$

and we obtain that

$$\dim_{\mathbb{C}} \mathcal{A}/\langle g_1, \dots, g_n\rangle = \frac{d_1 \cdots d_n}{M^n} n! \mathrm{V}_n(\Gamma_-).$$

If the above equality holds, then the inequalities in (1) are actually equalities and this implies that the ideal $\langle g_1^{d/d_1}, \dots g_n^{d/d_n}\rangle$ is Newton non-degenerate by Corollary 3.11. Then, as a consequence of Theorem 3.12, the system $\{g_1, \dots, g_n\}$ is non-degenerate on Γ_+. □

5. Complexes

To deduce formulas for $\dim_{\mathbb{C}} \mathcal{A}/\langle g_1, \dots, g_{e_1}\rangle$, we consider free resolutions of perfect modules which are explicitly described. Then we can determine the grade shifting (i.e., $(-a_i^{(s)})$) at each term so that each differential is of degree 0. As we have seen, it can be done easily for the Koszul complex. Other complexes that we also consider here are the Eagon-Northcott complex, which gives the resolution for determinantal ideals of maximal order and the Buchsbaum-Rim complex, which provides a resolution for coker φ where $\varphi : \mathcal{A}^p \to \mathcal{A}^s$ is a morphism of free modules. We also remark that the Lascoux complex [19], a resolution for determinantal ideals which involve minors of lower order, can also be used to obtain formulas for the codimension of these ideals.

5.1. **Eagon-Northcott complex.** Let R be a Noetherian ring and $U = (g_{ij})$ a p by s matrix, $p \geq s$, with $g_{ij} \in R$, $I_s(U)$ denotes the ideal generated by the maximal minors of U. We set

$$F_k = \wedge^s (R^s)^* \otimes S_{k-1}(R^s)^* \otimes \wedge^{s+k-1} R^p, \quad k = 1, \dots, p - s + 1,$$

where $S_k(R^s)$ denotes the k-th symmetric algebra of R^s, and $F_0 = R$. Then, the Eagon-Northcott complex is given by

$$EN : 0 \longrightarrow F_{p-s+1} \longrightarrow F_{p-s} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0.$$

More details on the construction of this complex are given in [8]. This is a free resolution of $R/I_s(U)$ if $\operatorname{depth}(I_s(U), R) = p - s + 1$. Let $w_1, \ldots, w_p$, $d_1, \ldots, d_s$ be non-negative integers. If R is a graded ring and g_{ij} are homogeneous elements of degree $d_j - w_i$, this is a graded complex and

$$q_{EN}(t) = 1 + \sum_{k=1}^{p-s+1} (-1)^k \sum_{I_k, J_k} t^{d+d_{J_k} - w_{I_k}},$$

where $d = d_1 + \cdots + d_s$, $d_{J_k} = d_{j_1} + \cdots + d_{j_{k-1}}$ for $J_k = (j_1 \leq \cdots \leq j_{k-1})$, and $w_{I_k} = w_{i_1} + \cdots + w_{i_{s+k-1}}$, for $I_k = (i_1 < \cdots < i_{s+k-1})$.

Thus we obtain that

$$q_{EN}^{(p-s+1)}(1) = \sum_{k=1}^{p-s+1} (-1)^k \sum_{I_k, J_k} (d + d_{J_k} - w_{I_k})_{p-s+1},$$

where $(n)_k = n(n-1)\cdots(n-k+1)$.

Example 5.1. Consider the matrix with entries in $\mathcal{A} = \mathbb{C}[[x, y]]$ given by

$$U = \begin{pmatrix} x^\alpha + x^r y^s + 2y^\beta & -x^r y^s + y^\beta \\ -x^\alpha & x^\alpha + x^r y^s - y^\beta \\ 2x^\alpha - x^r y^s + y^\beta & x^\alpha + x^r y^s + 2y^\beta \end{pmatrix}$$

Let us consider the polyhedron Γ_+ of the ideal generated by $\{x^\alpha, x^r y^s, y^\beta\}$. The Newton filtration associated to Γ_+ is given in Example 2.9, then we assume that $\alpha s < \beta(\alpha - r)$.

We observe that this matrix is non-degenerate on Γ_+ and that $I_2(U)$ has finite codimension, then the associated Eagon-Northcott complex is acyclic. In this case, the Eagon-Northcott complex has the form

$$0 \longrightarrow \mathcal{A}^2 \xrightarrow{\varphi_2} \mathcal{A}^3 \xrightarrow{\varphi_1} \mathcal{A} \longrightarrow \mathcal{A}/I_2(U) \longrightarrow 0.$$

Here the morphism $\varphi_2 : \mathcal{A}^2 \to \mathcal{A}^3$ is defined by the matrix U and $\varphi_1 : \mathcal{A}^3 \to \mathcal{A}$ is given by $(J_{2,3}\ J_{3,1}\ J_{1,2})$, where $J_{i,j}$ is the minor of the matrix U constructed taking rows i and j. Let us denote the graded ring associated to the mentioned filtration on $\mathcal{A}$ by A. Then, the associated graded complex is given by

$$0 \longrightarrow F_2 \xrightarrow{\overline{\varphi}_2} F_1 \xrightarrow{\overline{\varphi}_1} F_0 \longrightarrow F_0/I_2(U') \longrightarrow 0,$$

where $F_2 = A(-\alpha\beta rs) \oplus A(-\alpha\beta rs)$, $F_1 = A(-2\alpha\beta rs) \oplus A(-2\alpha\beta rs) \oplus A(-2\alpha\beta rs)$, $F_0 = A$ and U' is the matrix whose entries are the initial forms of the entries of U. Then, by Theorem 4.16 we obtain that $\dim_{\mathbb{C}} \mathcal{A}/I_2(U) = \frac{2q^{(2)}(1)}{M^2} V_2(\Gamma_-) = 3(\alpha s + \beta r)$.

5.2. **Buchsbaum-Rim complex.** Let R be a Noetherian ring, and $U = (g_{ij})$ a p by s matrix, $p \geq s$, with $g_{ij} \in R$. The matrix U defines a map $\varphi : R^p \to R^s$. We set

$$F_k = \wedge^s (R^s)^* \otimes S_{k-2}(R^s)^* \otimes \wedge^{s+k-1} R^p, \quad k = 2, \ldots, p-s+1,$$

$F_1 = R^p$, $F_0 = R^s$, and consider the Buchsbaum-Rim complex

$$BR : 0 \longrightarrow F_{p-s+1} \longrightarrow F_{p-s} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0.$$

More details on the construction of this complex are given in [7, pp. 594–595].

This is a free resolution of $\operatorname{coker}\varphi$ if $\operatorname{depth}(I_s(\varphi), R) = p - s + 1$. Let $w_1, \ldots, w_p$, $d_1, \ldots, d_s$ be non-negative integers. If R is a graded ring and g_{ij} are homogeneous elements of degree $w_j - d_i$, this is a graded complex and

$$q_{BR}(t) = \sum_{i=1}^{s} t^{d_i} - \sum_{j=1}^{p} t^{w_j} + \sum_{k=2}^{p-s+1} (-1)^k \sum_{I_k, J_k} t^{d + d_{J_k} - w_{I_k}},$$

where $d = d_1 + \cdots + d_s$, $d_{J_k} = d_{j_1} + \cdots + d_{j_{k-2}}$ for $J_k = (j_1 \leq \cdots \leq j_{k-2})$, and $w_{I_k} = w_{i_1} + \cdots + w_{i_{s+k-1}}$ for $I_k = (i_1 < \cdots < i_{s+k-1})$. Thus we obtain that

$$q_{BR}^{(p-s+1)}(1) = \sum_{i=1}^{s} (d_i)_{p-s+1} - \sum_{j=1}^{s} (w_j)_{p-s+1} + \sum_{k=2}^{p-s+1} (-1)^k \sum_{I_k, J_k} (d + d_{J_k} - w_{I_k})_{p-s+1}.$$

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