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Ovaloids of R^3 and their Umbilics: a Differential Equation Approach

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OVALÓIDES DE $\mathbb{R}^3$ E SEUS UMBÍLICOS:

UM ESTUDO VIA EQUAÇÕES DIFERENCIAIS

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RESUMO

Propomos e resolvemos - em casos bastante especiais - a seguinte extensão da Conjectura de Carathéodory sobre Umbílicos: Seja $N : \mathbb{M} \to \mathbb{S}^2$ a aplicação de Gauss de um ovalóide $\mathbb{M} \subset \mathbb{R}^3$ de classe C^3. Seja $\mathcal{A}$ o conjunto dos umbílicos $q \in \mathbb{M}$ tais que $N(q) = q$. Se A é uma componente conexa de $\mathcal{A}$, então, cada componente conexa de $\mathbb{M} \setminus A$ contém pelo menos um umbílico de $\mathbb{M}$.

OVALOIDS OF $\mathbb{R}^3$ AND THEIR UMBILICS: A DIFFERENTIAL EQUATION APPROACH.

R. GARCIA AND C. GUTIERREZ

Dedicated to Jack Hale

ABSTRACT. It is proposed and solved -in very special cases- the following extension of Carathéodory Conjecture on umbilics: Let $N : \mathbb{M} \to \mathbb{S}^2$ be the Gauss map of a C^3 ovaloid $\mathbb{M} \subset \mathbb{R}^3$. Let $\mathcal{A}$ be the set of umbilics $q \in \mathbb{M}$ such that $N(q) = q$. If A is a connected component of $\mathcal{A}$, then, each connected component of $\mathbb{M} \setminus A$ contains at least an umbilic point of $\mathbb{M}$.

§1 INTRODUCTION

The study of umbilic points and lines of curvature of surfaces embedded in $\mathbb{R}^3$ goes back to the works of Monge [Mon], Dupin [Dup], Cayley [Cay], Darboux [Dar] and Gullstrand [Gul] among others. In this context appears the challenging Carathéodory Conjecture which states that every smooth convex embedding of a 2-sphere in $\mathbb{R}^3$, i.e. an ovaloid, must have at least two umbilics. A local conjecture stronger than Carathéodory's, known as the Bol-Loewner conjecture, states that there are no umbilics of index greater than one. This conjecture has been asserted to be true for analytic surfaces by several authors, among whom are H. Hamburger [Ha1], [Ha2], G. Bol [Bol], T. Klotz [Klo], C. J. Titus [Tit] and Scherbel [Sch], implying therefore Carathéodory Conjecture for compact analytic surfaces. A more recent work on the subject can be found in the articles of C. Gutierrez, F. Mercuri, F. Sanchez-Bringas [GMS], [GS1],

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[GS2], [GS3], J. Sotomayor and L. F. Mello [SM] and B. Smyth, F. Xavier [SX1]. Also V. Topogonov [Top] considered the problem of existence of umbilic points on complete convex surfaces diffeomorphic to the plane.

The subject of lines of curvature and umbilic points has been renewed, from the point of view of Differential Equations and Geometry [GS1]-[GS3], [SG1]-[SG3], [B-F], [B-T] and Global Analysis [SX1]. The requirements of Computer Aided Design has also aroused pertinent work on umbilics and lines of curvature [MPW].

In this paper it is proposed and solved -in very special cases- the following extension of Carathéodory Conjecture on umbilics:

Conjecture. *Let $N : \mathbb{M} \to \mathbb{S}^2$ be the Gauss map of a C^3 ovaloid $\mathbb{M} \subset \mathbb{R}^3$. Let $\mathcal{A}$ be the set of umbilics $q \in \mathbb{M}$ such that $N(q) = q$. If A is a connected component of $\mathcal{A}$, then, each connected component of $\mathbb{M} \setminus A$ contains at least an umbilic of $\mathbb{M}$.*

Notice that in this conjecture, if $q \in \mathcal{A}$, then $q = N(q)$ and so $\mathbb{S}^2$ and $\mathbb{M}$ meet tangentially at q.

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§2 Ovaloids and the Bonnet function

Orient the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ so that the positive unitary normal vector at $p \in \mathbb{S}^2$ is p itself. Let $\mathbb{M} \subset \mathbb{R}^3$ be a *C^r-ovaloid*. By this we mean that $\mathbb{M}$ is an oriented C^r-embedded surface such that its Gauss map $N : \mathbb{M} \to \mathbb{S}^2$ is an orientation preserving diffeomorphism. This definition implies that $\mathbb{M}$ is convex, compact and that its Gaussian curvature is positive everywhere. We define the *support function of* $\mathbb{M}$ as the map $\sigma : \mathbb{S}^2 \to \mathbb{R}$ given by

$$\sigma(p) = p \cdot N^{-1}(p),$$

where the dot stands for the usual inner product, some times also denoted by $<,>$.

Given $\delta \in \{-,+\}$, let $\Pi^\delta : \mathbb{R}^2 \to \mathbb{S}^2 \setminus \{(0,0,\delta(1))\}$ be the diffeomorphism given by

$$\Pi^\delta(x,y) = (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{\delta(x^2+y^2-1)}{1+x^2+y^2}).$$

That is, Π^δ is the inverse map of the corresponding stereographic projection. The map

$$\Phi^\delta(x,y) = (X^\delta(x,y), Y^\delta(x,y), Z^\delta(x,y)) = N^{-1} \circ \Pi^\delta(x,y),$$

defined in $\mathbb{R}^2$, provides a global C^{r-1} parametrization of $\mathbb{M} \setminus \{N^{-1}(0,0,\delta(1))\}$ called *Bonnet chart associated to* $(\mathbb{M}, \Pi^\delta)$. Given the support function σ of $\mathbb{M}$, associated to $(\mathbb{M}, \Pi^\delta)$ we define the *Bonnet function*

$$\beta^\delta(x,y) = (1+x^2+y^2)\sigma(\Pi^\delta(x,y)).$$

The proof of the following lemma can be found in [GS1]

Lemma 2.1.

Let $\mathbb{M} \subset \mathbb{R}^3$ be a C^r ovaloid, $r \geq 3$. Then the support function σ of $\mathbb{M}$ is of class C^r and the differential equation of the principal lines of curvature of $\mathbb{M}$ in its Bonnet chart, associated to $(\mathbb{M}, \Pi^\delta)$, is given by $\omega^\delta = 0$ where

$$\omega^\delta = \beta^\delta_{xy} dx^2 + (\beta^\delta_{yy} - \beta^\delta_{xx}) dx dy - \beta^\delta_{xy} dy^2 \tag{1}$$

The following will be needed later

Lemma 2.2. *Let $\delta \in \{-,+\}$ and let*

$$\beta^\delta(x,y) = (1+x^2+y^2)\sigma(\Pi^\delta(x,y))$$

be the functions determined by σ as above. If $\mathcal{I}$: $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is the inversion

$$\mathcal{I}(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}),$$

then, for all $(x, y) \in \mathbb{R}^2 \setminus \{0\}$,

$$(x^2 + y^2)\beta^- \circ \mathcal{I}(x, y) = \beta^+(x, y) \tag{2}$$

Proof. The result follows immediately from the identity

$$\sigma(\Pi^+(x, y)) = \sigma(\Pi^- \circ \mathcal{I}(x, y)). \qquad \square$$

Theorem 2.3. *([GS1, Theorem 2.2]) Let* $\mathbb{M} \subset \mathbb{R}^3$ *be a* C^r *ovaloid,* $r \geq 3$. *Then the inverse* $N^{-1} : \mathbb{S}^2 \to \mathbb{M}$, *of the Gauss map* N, *can be written as follows:*

$$N^{-1}(u, v, w) = f(u, v, w) \cdot (u, v, w) + \mathcal{A}(u, v, w) \cdot \nabla f(u, v, w)$$

where $f : \mathbb{S}^2 \to \mathbb{R}$ *denotes the support function of* $\mathbb{M}$, ∇f *its gradient vector field and*

$$\mathcal{A}(u, v, w) = \begin{pmatrix} v^2 + w^2 & -uv & -uw \\ -uv & u^2 + w^2 & -vw \\ -uw & -vw & u^2 + v^2 \end{pmatrix}$$

Conversely, given a C^r *function* $f : \mathbb{S}^2 \to \mathbb{R}$, $r \geq 3$, *there exists a constant* $c > 0$ *such that* $f + c$ *is the support function of an ovaloid of class* C^r.

Remark 2.4. *Let* $f : \mathbb{S}^2 \to \mathbb{R}$ *be a* C^r *function,* $r \geq 3$, *and let* $F = F(u, v, w)$ *be an arbitrary extension of* f *to a neighborhood of* $\mathbb{S}^2$. *As* $A(u, v, w) \cdot (u, v, w) \equiv 0$, *it follows that*

$$\mathcal{A}(u, v, w) \cdot \nabla f(u, v, w) = \mathcal{A}(u, v, w) \cdot \nabla F(u, v, w)$$

where ∇f *and* ∇F *are the gradient vector fields of* f *and of* F, *respectively.*

Proposition 2.5. *Let* $\beta^+ : \mathbb{R}^2 \to \mathbb{R}$ *be a* C^3 *function. Then*

$$\beta^-(x, y) = (x^2 + y^2)\beta^+ \circ \mathcal{I}(x, y)$$

extends in class C^3 *to* $(0, 0)$ *and satisfies:* $\beta^-(0, 0) = \beta^-_x(0, 0) = \beta^-_y(0, 0) = \beta^-_{xy}(0, 0) = \beta^-_{xx}(0, 0) - \beta^-_{yy}(0, 0) = 0$ *iff there exists a* C^3-*ovaloid* $\mathbb{M} \subset \mathbb{R}^3$ *tangent to* $\mathbb{S}^2$ *at its* *umbilic point* $(0,0,1)$ *and such that the differential equation of the principal lines of curvature of* $\mathbb{M}$ *in its Bonnet chart, associated to* $(\mathbb{M}, \Pi^+)$, *is given by* $\omega^+ = 0$, *where* ω^+ *is as in (1) above.*

Proof. We shall only prove the necessity of the condition.

Let $f : \mathbb{S}^2 \setminus \{(0,0,1)\} \to \mathbb{R}$ be given by

$$f(u,v,w) = K(1-w)\beta^+(\frac{u}{1-w}, \frac{v}{1-w}) + 1$$

where $K > 0$ is a constant. Observe that by defining $f(0,0,1) = 1$, the assumptions imply that f is C^3 and extends C^3 to $\{(0,0,1)\}$. If $K > 0$ is small enough and if $N^{-1} : \mathbb{S}^2 \to \mathbb{R}^3$ is defined as in Theorem 2.3, then $\mathbb{M} = N^{-1}(\mathbb{S}^2)$ is an ovaloid. Also, by Lemmas 2.1 and 2.2, $(0,0,1)$ is an umbilic point of $\mathbb{M}$. Moreover, it follows from Theorem 2.3 and Remark 2.4 that $(0,0,1) \in \mathbb{M}$ and that $N^{-1}(0,0,1) = (0,0,1)$. The last statement of the proposition follows from Lemma 2.1. □

In view of proposition 2.5, Theorems 2.6 and 2.7 below deal with functions $\beta^+ : \mathbb{R}^2 \to \mathbb{R}$. In this context (besides $(0,0,1)$, umbilics are the solutions of the system of equations:

$$\beta^+_{xy} = 0 \qquad\qquad \beta^+_{yy} - \beta^+_{xx} = 0$$

Under conditions of theorem below, it may be seen that ∂R is made up of umbilics.

Theorem 2.6. *Let* $H, G : \mathbb{R} \to \mathbb{R}$ *be* C^3 *functions such that the support of* H *(resp.* G*) is contained in* $[a,b] \subset \mathbb{R}$ *(resp* $[c,d] \subset \mathbb{R}$*). If* $\beta^+ = \beta^+(x,y) = H(x)G(y)$ *is positive in* $R = (a,b) \times (c,d)$, $G(c) = G_y(c) = G_{yy}(c) = 0$, $G(d) = G_y(d) = G_{yy}(d) = 0$, $H(a) = H_x(a) = H_{xx}(a) = 0$ *and* $H(b) = H_x(b) = H_{xx}(b) = 0$ *Then there exist at least an umbilic point in* R.

Proof. In the conditions above the set of umbilic points is given by the solutions of the system of equations:

$A(x,y) = H(x)G''(y) - G(y)H''(x) = 0,$

$B(x,y) = H'(x)G'(y) = 0.$

By Rolle's theorem there exist points $x_0 \in (a,b)$ and $y_0 \in (c,d)$, local maximum of H and G respectively, such that $H'(x_0) = G'(y_0) = 0$. Hence, $\{(x_0,t),\ c \leq t \leq d\} \cup \{(s,y_0),\ a \leq t \leq b\}$ is contained in $B^{-1}(0)$.

Also, by Rolle's theorem, there exist points in the intervals (a,x_0), (x_0,b) (resp. in the intervals $(c,y_0),(y_0,d)$) where the second derivative of H (resp. of G) vanishes.

Let

$x_1 = sup\{x \in (a,x_0)|H''(x) = 0\}$, $\qquad x_2 = inf\{x \in (x_0,b)|H''(x) = 0\}$,

$y_1 = sup\{y \in (c,y_0)|G''(y) = 0\}$, $\qquad y_2 = inf\{y \in (y_0,d)|G''(y) = 0\}$.

We can suppose that $x_1 < x_0 < x_2$ and $y_1 < y_0 < y_2$. Otherwise, for example, if $x_1 = x_0$ the points (x_0,y_1) and (x_0,y_2) are umbilics.

Now consider the subrectangle $R_1 = [x_1,x_2] \times [y_1,y_2]$.

By the construction of R_1 we have $A(x_i,y_j) = 0,\ \ i,j = 1,2.$

Also $A(x,y_1) > 0,\ \ A(x,y_2) > 0$ for all $x \in (x_1,x_2)$ and $A(x_1,y) < 0,\ \ A(x_2,y) < 0$ for all $y \in (y_1,y_2)$.

In fact, $G(y_1) > 0$ and x_0 is a maximum of H, so $A(x,y_1) = -G(y_1)H''(x) > 0$.

Now consider the path $c(t)$, with $0 \leq t \leq 2$, given by

$$(x_1,y_0)+t(x_0-x_1,0),\ 0 \leq t \leq 1,$$

$$(x_0,y_0)+(t-1)(0,y_1-y_0),\ 1 \leq t \leq 2,$$

We have that $A \circ c(0) < 0$ and $A \circ c(2) > 0$. By the Intermediate Value Theorem, there exist $t_1 \in (0,2)$ such that $A \circ c(t_1) = 0$. The point $c(t_1)$ is an umbilic. $\square$

Remark 2.7. *If $c(1) = (x_0,y_0)$ is not an umbilic point we have at least two umbilic points in the rectangle R_1.*

We observe that (x_0,y_0) is a critical point of β and so near this point $\beta(x,y) = a(x-x_0)^2 + b(x-x_0)(y-y_0) + c(y-y_0)^2 + h.o.t.$, and generically $(a-c)b \neq 0$ and therefore it is not an umbilic point.

Remark 2.8. *Theorem 2.6 is also true for the disk supposing that the support function β, in polar coordinates, is the product of separable variables functions.*

Theorem 2.9. *Let $H, G : \mathbb{R} \to [0, \infty)$ be C^3 functions. Suppose $\beta^+(x, y) = H(x)G(y) \geq 0$ satisfies the conditions of Proposition 2.5 (i.e., β^+ is the Bonnet function of a C^3 ovaloid). Then there exist at least one umbilic point $p_0 \in \mathbb{R}^2$.*

Proof. We can apply the Rolle's theorem in this situation to reduce the analysis to a rectangle R_1 as in the proof of Theorem 2.6. □

§2.1 Principal Curvature Lines near Umbilic Points.

We recall, in proposition below, the main properties of Darbouxian umbilics (see either [SG1], [SG3] or [B-F]).

Let 0 be an umbilic point of a C^4 surface parametrized by $(x, y, h(x, y))$ where,

$$h(x,y) = \frac{k}{2}(x^2+y^2) + \frac{a}{6}x^3 + \frac{b}{2}xy^2 + \frac{c}{6}y^3 + O(4)$$

Then differential equation of curvature lines is given by:

$$-[by + P_1(x,y)]dy^2 + [(b-a)x + cy + P_2(x,y)]dxdy + [by + P_3(x,y)]dx^2 = 0$$

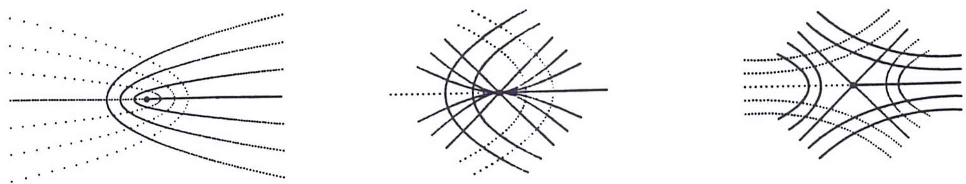
Proposition 2.10. *Suppose that $b(b-a) \neq 0$, define $\Delta_p = 4b(a-2b)^3 - c^2(a-2b)^2$ and consider the conditions below.*

i) D_1, $\Delta_p > 0$

ii) D_2, $\Delta_p < 0$ *and* $\frac{a}{b} > 1$

iii) D_3, $\frac{a}{b} < 1$

Then the principal curvature lines have in a neighborhood of 0, in the cases D_1, D_2 and D_3 respectively, one hyperbolic sector, one hyperbolic and one parabolic sector and three hyperbolic sectors. The topological index of each principal foliation is respectively, *1/2, 1/2 and −1/2. The umbilics are called Darbouxians of type D_1, D_2 and D_3, respectively.*



Type D_1 Type D_2 Type D_3

Figure 1

Example 2.11. *Consider $\beta^+(x,y) = x^2 exp(-x^2) exp(-y^2)$. The conditions of Proposition 2.5 are satisfied and (besides (0,0,1)) we have four umbilic points, all of which are of Darbouxian type D_1, see [SG1].*

In fact, we have that

$$A(x,y) = \beta^+_{xx} - \beta^+_{yy} = [4x^2y^2 - 4x^4 + 8x^2 - 2] exp(-x^2) exp(-y^2)$$

$$B(x,y) = \beta^+_{xy} = [4xy(x^2-1)] exp(-x^2) exp(-y^2)$$

Therefore the four umbilic points are given by $(p_i, 0)$ where,

$$p_1 = \frac{1}{2}\sqrt{4-2\sqrt{2}},\ \ p_2 = -\frac{1}{2}\sqrt{4-2\sqrt{2}},\ \ p_3 = \frac{1}{2}\sqrt{4+2\sqrt{2}},\ \ p_4 = -\frac{1}{2}\sqrt{4+2\sqrt{2}}.$$

Near each $(p_i, 0)$, the linear part of the differential equation of curvature lines is equivalent to $4x dx dy - y(dx^2 - dy^2) = 0$. Therefore by Proposition 2.10, all umbilic points $(p_i, 0)$, $i = 1, \cdots, 4$, are of type D_1.

§3 OVALOIDS AND DARBOUX FRAME

In this section it will be considered an ovaloid $\mathbb{M}$ of class C^r, $r \geq 4$, which is tangent, quadratically, to the unitary sphere $\mathbb{S}^2$ along a regular curve γ. Semilocal hypothesis are imposed on $\mathbb{M}$ in order to obtain the existence of umbilics.

Denote by $\mathcal{F}_i(\mathbb{M})$, $i = 1, 2$, the principal foliations of a regular and oriented surface $\mathbb{M}$ of class C^4 and by $\mathcal{U}_{\mathbb{M}}$ the set of umbilic points of $\mathbb{M}$. The leaves of $\mathcal{F}_i(\mathbb{M})$ are the curvature lines outside the umbilic set $\mathcal{U}_{\mathbb{M}}$.

Lemma 3.1. *Let $\mathbb{M}$ be a C^4 surface tangent to $\mathbb{S}^2$ along a closed regular curve γ. Then the oriented Darboux frame $\{N, T, N \wedge T\}$, where N is the unitary normal to the surface $\mathbb{M}$ and $T = \gamma'$ is the unitary tangent to the curve γ parametrized by arc length, satisfies the following differential equation:*

$$\begin{aligned} T' &= k_g N \wedge T + N \\ (N \wedge T)' &= -k_g T \\ N' &= -T \end{aligned} \tag{3}$$

where k_g is the geodesic curvature of $\gamma \subset \mathbb{M}$.

Proof. By assumptions above, the curve γ is a curvature line of $\mathbb{M}$. As γ is also a curve of $\mathbb{S}^2$ it follows that the normal curvature of γ is equal to 1. Therefore, the Darboux frame is as stated; see also [Spi, pp. 278]. $\square$

Lemma 3.2. *Let $\mathbb{M}$ be a C^4 surface tangent, with quadratic contact, to $\mathbb{S}^2$ along a closed regular curve γ. Then the expression*

$$\alpha(u,v) = \gamma(u) + v(N \wedge T)(u) + [\frac{v^2}{2} + a(u)\frac{v^3}{6} + A(u,v)v^4]N(u) \tag{4}$$

where, $A(u, 0) = 0$, defines a chart (u, v) of class C^1 of $\mathbb{M}$, in a neighborhood of γ. Moreover $\gamma \subset \mathcal{U}_{\mathbb{M}}$.

Proof. Consider the map $\alpha : [0, L] \times (-\epsilon, \epsilon) \subset \mathbb{R}^2 \to \mathbb{R}^3$ defined by:

$$\alpha(u, v) = \gamma(u) + v(N \wedge T)(u) + W(u, v)N(u),$$

where $W(u, 0) = 0$.

Direct calculations show that $\alpha_u(u, 0) = \gamma'(u) = T(u)$ and $\alpha_v(u, 0) = (N \wedge T)(u)$. So, by the Inverse Function Theorem, it follows that (u, v) is a local chart. As the contact between $\mathbb{M}$ and $\mathbb{S}^2$ is quadratic along γ, it follows that the normal curvature $k_n(N \wedge T, \gamma(u)) = 1$. To obtain the expansion of $W(u, v) =< \alpha(u, v), N(u) >$ as stated, we apply Hadamard's Lemma.

By lemma 3.1 γ is a curvature line with normal curvature $k_n(T, \gamma(u)) = 1$. Therefore any point of γ is an umbilic point. See also [SG1] for a similar chart near a principal cycle. $\square$

Remark 3.3. *In the lemma 3.2 the higher order terms of the function $W = W(u, v)$ can be evaluated in terms of the Gaussian curvature $\mathcal{K}$ and of the Mean curvature $\mathcal{H}$ of the surface $\mathbb{M}$. For example, we have that $\mathcal{H}_v \mid_{\{v=0\}} = a/2$ and $\mathcal{K}_v \mid_{\{v=0\}} = a$.*

Theorem 3.4. *Let $\mathbb{M}$ be a C^r, $r \geq 4$, ovaloid quadratically tangent to $\mathbb{S}^2$ along a closed regular curve γ. Consider the chart (u, v) given by lemma 3.2. Suppose that γ bounds an open disc $D_\gamma \subset \mathbb{M}$ and that, for every u, $[\mathcal{K}_v(u, 0)]^2 + [\mathcal{K}_{uv}(u, 0)]^2 > 0$. Then $\gamma \subset \mathcal{U}_{\mathbb{M}}$ and $\#(\mathcal{U}_{\mathbb{M}} \cap D_\gamma) \geq 1$.*

Proof. In the chart (u, v) we have that the coefficients of the first fundamental form $E =< \alpha_u, \alpha_u >$, $F =< \alpha_u, \alpha_v >$ and $G =< \alpha_v, \alpha_v >$ and the coefficients $e =< \alpha_{uu}, \alpha_u \wedge \alpha_v >$, $f =< \alpha_{uv}, \alpha_u \wedge \alpha_v >$ and $g =< \alpha_{vv}, \alpha_u \wedge \alpha_v >$, which are proportional to the coefficients of the second fundamental form, are given by:

$$E(u,v) = 1 - 2k_g v + (k_g^2 - 1)v^2 + (3k_g - a)\frac{v^3}{3} + O(4)$$

$$F(u,v) = a'\frac{v^4}{6} + a'a\frac{v^5}{12} + O(6)$$

$$G(u,v) = 1 + v^2 + av^3 + O(4)$$

$$e(u,v) = 1 - 3k_g v + (6k_g^2 - ak_g - 2)\frac{v^2}{2} + O(3)$$

$$f(u,v) = a'\frac{v^2}{2} - a'k_g\frac{v^3}{3} + O(4)$$

$$g(u,v) = 1 + (a - k_g)v - (2ak_g + 1)\frac{v^2}{2} + O(3)$$

Then, as $\mathcal{K} = (eg - f^2)/(EG - F^2)^2$ and $\mathcal{H} = (Eg + Ge - 2fF/[2(EG - F^2)^{\frac{3}{2}}]$ it follows that $\mathcal{K}_v(u,0) = 2\mathcal{H}_v(u,0) = a(u)$.

The differential equation of curvature lines in the chart (u,v) is given by, see [Str, Spi], $L(u,v)dv^2 + M(u,v)dudv + N(u,v)du^2 = 0$, where,

$$L(u,v) = (Fg - Gf)(u,v) = v^2[-a'(u) + \frac{1}{3}a'k_g v + v^2 L_1(u,v)]$$

$$M(u,v) = (Eg - Ge)(u,v) = v[a(u) - (3 + 5k_g)\frac{v}{2} + v^2 M_1(u,v)]$$

$$N(u,v) = (Ef - Fe)(u,v) = v^2[a'(u) - \frac{4}{3}a'k_g v + v^2 N_1(u,v)]$$

Therefore the curve γ is contained in the umbilic set $\mathcal{U}_{\mathbb{M}}$.

So, in a neighborhood of $\{v = 0\}$, the differential equation of curvature lines is given by:

$$(5) \qquad v[-a'(u) + \cdots]dv^2 + [a(u) - \frac{1}{2}(3 + 5k_g)v + \cdots]dudv + v[a' + \cdots]du^2 = 0$$

In the equation above we observe the division by v of the functions L, M and N and that $\{v = 0\}$ is an invariant set of solutions of (5).

If $a(u) \neq 0$ then γ is a regular solution of (5). Therefore by Poincaré Index Theorem, applied to the principal foliations $\mathcal{F}_i(\mathbb{M})$, we must have at least an umbilic point in D_γ. Recall that the Poincaré Index Theorem, see [T-B] or [GMR], for the disk D_γ gives that

$$\sum_k Ind(\mathcal{F}_i(\mathbb{M}), p_k) + \frac{(\#(e) - \#(i))}{2} = \chi(D_\gamma) = 1,$$

where $\#(i)$ and $\#(e)$ means, respectively, the number of internal and external contact points with the boundary of D_γ and $Ind(\mathcal{F}_i(\mathbb{M}), p_k)$ denotes the topological index of $\mathcal{F}_i(\mathbb{M})$ at an isolated umbilic point $p_k \in D_\gamma$, see [Spi, chapter IV, Addendum 2].

In the case where the equation $a(u) = 0$ has only transversal zeroes, say $2n$, these points are singularities of type D_3 (Darbouxian umbilic). In fact, near a singular point $(u_0, 0)$, of (5) the linear part of this equation is given by:

$$-\lambda v dv^2 + [\lambda(u - u_0) - \frac{1}{2}(3 + 5k_g(u_0))v]dudv + \lambda v du^2 = 0$$

where $\lambda = a'(u_0) \neq 0$. So, by the classification of Darbouxian umbilic points, see Proposition 2.10, [SG1], [SG3] or [B-F] it follows that $(u_0, 0)$ is a singular point of index $-\frac{1}{2}$ (D_3 Darbouxian type).

Taking into account that in boundary of D_γ we have $2n$ singular points of the differential equation (5), all having index $-\frac{1}{2}$ and therefore n internal contact points and zero external contact point with the boundary of a disk sufficiently close to D_γ, it follows from the Poincaré Index Formula that

$$\sum_k^m Ind(\mathcal{F}_i(\mathbb{M}), p_k) = 1 + \frac{n}{2} = \frac{n+2}{2},$$

where $p_k \subset D_\gamma$ $k = 1, \cdot, m$ are isolated umbilic points. □

Corollary 3.5. *Suppose, in the conditions above, that $\mathcal{U}_{\mathbb{M}}$ contains only Darbouxian umbilic points in $int(D_\gamma)$. Then the following holds:*

i) If $a(u) = (\mathcal{K}_v)(u,0) \neq 0$ *then* $\#(\mathcal{U}_M \cap D_\gamma) \geq 2$.

ii) If $a(u) = (\mathcal{K}_v)(u,0)$ *has* $2n$ *transversal zeroes, then* $\#(\mathcal{U}_{\mathbb{M}} \cap D_\gamma) \geq n+2$.

Proof. We observe that a Darbouxian umbilic point has index $-\frac{1}{2}$ (D_3 type) or $\frac{1}{2}$ (D_1 and D_2 types). Therefore the result follows applying the Poincaré Index Theorem. □

Summarizing, it has been proved, in Section 2, that the conjecture presented in this article is true when the ovaloid is parametrized by a Bonnet chart such that the associated Bonnet function $\beta : \mathbb{R}^2 \to \mathbb{R}$ is the product $\beta(x,y) = \varphi(x) \cdot \psi(y)$ of two functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ which have compact support. It would be very interesting to give an answer to this conjecture when the associated Bonnet function is compactly supported and has the form $\beta(x,y) = \varphi_1(x) \cdot \psi_1(y) + \varphi_2(x) \cdot \psi_2(y) + \cdots + \varphi_n(x) \cdot \psi_n(y)$.

In section 3, the considered conjecture is proved in the case that a connect component of $\mathcal{A}$ is diffeomorphic to a disk, having regular boundary, with an additional generic condition which somehow determines the behavior of the curvature lines near this boundary. Whitin this context, the conjecture may be considered by studying discs with either less regular boundary or under weaker generic conditions.

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