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Existence Results for Partial Neutral Differential Equations

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Resultados de Existência para Equações Diferenciais do Tipo Neutro.

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Resumo

Neste artigo estudamos a existência de soluções fracas e regulares para um tipo de equação diferencial do tipo neutro que pode ser modelada como a equação abstrata $\frac{d}{dt}(x(t) + g(t, x(t)) = Ax(t) + f(t, x(t)),$ onde A é o gerador infinitesimal de um semigrupo fortemente contínuo de operadores lineares num espaço de Banach X e f, g são funções apropriadas.

Existence Results for Partial Neutral Differential Equations.

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Abstract

We prove the existence of mild and regular solutions for a class of quasi-linear abstract partial neutral differential equation that can to be modelled in the form $\frac{d}{dt}(x(t)+g(t,x(t)) = Ax(t)+f(t,x(t))$, where A is the infinitesimal generator of an analytic semigroup of bounded linear operators defined on a Banach space X and f, g are appropriate continuous functions.

1 Introduction

The class of equations considered in this paper have the form

$$\frac{d}{dt}(u(t)+g(t,u(t))) = Au(t)+f(t,u(t)), \qquad t>0, \tag{1}$$

$$u(0) = u_0. \tag{2}$$

We consider system (1) as a Cauchy problem on a Banach space X, where A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$, $f,g : [0,T]\times\Omega \to X$ are appropriate continuous functions and Ω is an open subset of X. The equation (1) will be called "Abstract Neutral Differential Equation".

The case $g\equiv 0$ has an extensive literature. The books of Pazy [4], Krein [3], Goldstein [1] and the references contained therein, give a good account of the more important results.

Throughout this paper X will be a Banach space provided with norm $\|\cdot\|$ and the operator $A : D(A)\subset X\to X$ will be the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on X. For the theory of strongly continuous semigroups, refer to [4] and [1]. We mention here only some notations and properties essential to our purpose. In particular, it is well known that there exist $\tilde{M}\geq 1$ and a real number w such that

$$\| T(t) \|\leq \tilde{M}e^{wt}, \qquad t\geq 0. \tag{3}$$

In what follows we assume that $\| T(t) \|$ is uniformly bounded by a positive constant $\tilde{M}$ and that $0 \in \rho(A)$. In that case it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha < 1$, as a closed linear operator with domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha)$ is dense in X and the expression

$$\| x \|_\alpha = \| (-A)^\alpha x \|$$

defines a norm on $D((-A)^\alpha)$. Hereafter we represent by X_α the space $D((-A)^\alpha)$ endowed with the norm $\| \cdot \|_\alpha$. The following properties are well known (see [4]):

Lemma 1 *If the above conditions hold:*

1. *Let* $0 < \alpha \leq 1$. *Then* X_α *is a Banach space.*

2. *If* $0 < \beta \leq \alpha$ *then* $X_\alpha \to X_\beta$ *is continuous and compact if the resolvent operator of* A *is compact.*

3. *For every constant* $a > 0$, *there exists* $C_a > 0$ *such that*

$$\| (-A)^\alpha T(t) \| \leq \frac{C_a}{t^\alpha}, \qquad 0 < t \leq a.$$

4. *For every* $a > 0$ *there exists a positive constant* C_a' *such that*

$$\| (T(t) - I)(-A)^{-\alpha} \| \leq C_a' t^\alpha, \qquad 0 < t \leq a.$$

It is natural to consider the solutions of (1) in a generalized since A is un bounded. Then by analogy with the abstract Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t) + h(t) \\ u(0) &= u_0, \end{aligned} \tag{4}$$

we adopt the following definitions:

Definition 1 *A function* $x \in C([0,T] : X)$ *is a mild solution of the abstract neutral Cauchy problem (1) if:* $x(0) = x_0$; *for each* $0 < t < T$ *the function* $AT(t-s)g(s,x(s))$, $s \in [0,t)$, *is integrable and*

$$\begin{aligned} x(t) &= T(t)(x_0 + g(0,x_0)) - g(t,x(t)) + \int_0^t (-A)T(t-s)g(s,x(s))ds \\ &\quad + \int_0^t T(t-s)f(s,x(s))ds \end{aligned} \tag{5}$$

for every $t \in [0,T]$.

Definition 2 *A function* $x \in C([0,T) : X)$ *is a classical solution of the abstract Cauchy problem (1) if:* $x(0) = x_0$; $x(t) \in D(A)$ *for all* $t \in (0,T)$; $\dot{x}$ *is continuous in* $(0,T)$ *and* $x(\cdot)$ *satisfies equation (1) in* $(0,T)$.

Definition 3 *A function $x \in C([0,T) : X)$ is an N-classical solution of the abstract Cauchy problem (1) if: $x(0) = x_0$; $x(t) \in D(A)$ for all $(0,T)$; $\frac{d}{dt}(x(t) + g(t,x(t)))$ is continuous in $(0,T)$ and $x(\cdot)$ satisfies equation (1) in $(0,T)$.*

The paper is organized as follows. In section 2, we establish the existence of mild solutions. In section 3 we discuss the existence of N-classical and classical solutions to the initial value problem (1). Our results are based on the properties of analytic semigroups and the ideas used in [4] chapter 5. Also in section 3, an application is discussed.

Throughout, we assume that X is an abstract Banach space. The terminology and notations are those generally used in operator theory. In particular, if X, Y are Banach spaces, we indicate by $\mathcal{L}\ (X : Y)$ the Banach space of the bounded linear operator of X into Y and we abbreviate to $\mathcal{L}\ (X)$ whenever $X = Y$. In addition $B_r(x : X)$ will denote the closed ball in space X with center at x and radius r.

For a bounded function $\xi : [0,a] \to X$ and $0 < t < a$ we employ the notation

$$\| \xi(\cdot) \|_{[0,t]} = sup\{\| \xi(s) \| : s \in [0,t]\}.$$

We will write simply ξ_b for $\| \xi(\cdot) \|_{[0,b]}$ when no confusion arises.

Finally for $x_0 \in X$, we will use the notation $x(\cdot, x_0)$ for the mild solution of (1) with $x(0, x_0) = x_0$.

2 Existence of mild solutions

In this section we will study the existence of mild solutions to the abstract Cauchy problem

$$\begin{aligned} \frac{d}{dt}(x(t) + g(t,x(t))) &= Ax(t) + f(t,x(t)), \qquad t > 0, \\ x(0) &= x_0. \end{aligned}$$

The existence of mild and regular solutions will be proved using the fixed-point technique. In particular, for the case of mild solutions, we need the following classical result:

Theorem 1 *Let Γ be a closed, bounded and convex subset of X. If $T = S + U : \Gamma \to \Gamma$ is a continuous function where S is contractive and U is completely continuous, then T has a fixed point in Γ.*

Theorem 2 *Let $x_0 \in X$ and assume that the following conditions hold:*

a) There exist $\beta \in (0,1)$ and $L \geq 0$ such that the function g is X_β-valued and satisfies the Lipschitz condition

$$\| (-A)^\beta g(t,x) - (-A)^\beta g(s,y) \| \leq L(\mid t - s \mid + \| x - y \|) \tag{6}$$

for every $0 \leq s,t < T$ and $x, y \in \Omega$, and

$$\| (-A)^{-\beta} \| L < 1 \tag{7}$$

b) The function f is continuous and takes bounded sets into bounded sets.

c) The semigroup $(T(t))_{t\geq 0}$ is compact.

Then there exists a mild solution $x(\cdot, x_0)$ of the abstract Cauchy problem (1) defined in $[0,r]$ for some $0<r<T$.

Proof. Given that $(-A)^\beta g$ and f are continuous functions and Ω is open, we can assert that there exist $0<r_1<T$ and $0<\rho$, such that $\| (-A)^\beta g(t,x) \| \leq C_1$ and $\| f(t,x) \| \leq C_1$ for a certain constant $C_1>0$ and every $0\leq t<r_1$ and $x\in B_\rho(x_0,X)\subset \Omega$.

Let $0<r<r_1$ and $\epsilon>0$ such that

$$\epsilon + \| (-A)^\beta \| L\rho < \rho \tag{8}$$

$$\| (T(t)-I)(x_0 - g(0,x_0) \| < \frac{\epsilon}{3} \tag{9}$$

$$C_{1-\beta}C_1 \frac{r^\beta}{\beta} < \frac{\epsilon}{3} \tag{10}$$

$$\tilde{M} C_1 r < \frac{\epsilon}{3} \tag{11}$$

$$\| (-A)^{-\beta} \| L + C_{1-\beta}\frac{r^\beta}{\beta} L < 1 \tag{12}$$

where C_β and $C_{1-\beta}$ are the constants in lemma (1). In the set

$$S(r,\rho) := \{u\in C([0,r];X) : u(0)=x_0, \ \| u(t)-x_0 \| \leq \rho, \quad 0\leq t\leq r\},$$

we define the mapping Ψ by the expression:

$$\begin{aligned}\Psi(x)(t) &= T(t)(x_0+g(0,x_0)) - g(t,x(t)) + \textstyle\int_0^t (-A)T(t-s)g(s,x(s))ds \\ &\quad + \textstyle\int_0^t T(t-s)f(s,x(s))ds\end{aligned} \tag{13}$$

for $x\in S(r,\rho)$ and $0\leq t\leq r$. The mapping Ψ, may be decomposed as $\Psi = \Psi_1 + \Psi_2$, where

$$\begin{aligned}\Psi_1(x)(t) &= T(t)(x_0+g(0,x_0)) - g(t,x(t)) + \int_0^t (-A)T(t-s)g(s,x(s))ds \\ \Psi_2(x)(t) &= \int_0^t T(t-s)f(s,x(s))ds.\end{aligned}$$

Next we will prove that Ψ_1 and Ψ_2 are well defined in $C([0,r];X)$, that Ψ_1 satisfies a Lipschitz condition, that Ψ_2 is a compact operator, and that the range of Ψ is contained in $S(r,\rho)$.

It is clear from the continuity of f that Ψ_1 is well defined and that $\Psi_1(u)$ is a continuous function for all $u\in C([0,r]:X)$. On the other hand, since $T(\cdot)$ is analytic, the function $s\to AT(t-s)$ is continuous in the uniform operator topology in $[0,t)$, consequently the function $AT(t-s)g(s,x(s))$ is continuous in $[0,t)$. Moreover from lemma (1) we have:

$$\| (-A)^{1-\beta}T(t-s)(-A)^\beta g(s,x(s)) \| \leq \frac{C_{1-\beta}C_1}{(t-s)^{1-\beta}}$$

which implies that $\| (-A)T(t-s)g(s,x(s)) \|$ is integrable in $[0,t)$. We thus conclude that Ψ_2 is also well defined and with values in $C([0,r]:X)$.

In order to prove that the range of Ψ is included in $S(r,\rho)$ we observe that for $x \in S(r,\rho)$,

$$\begin{aligned}
\| \Psi_1(x)(t) - x_0 \| &\leq \| (T(t)-I)(x_0 - g(0,x_0)) \| + \| g(0,x_0) - g(t,x(t)) \| \\
&\quad + \int_0^t \| (-A)^{1-\beta} T(t-s)(-A)^\beta g(s,x(s)) \| \, ds \\
&\leq \frac{\epsilon}{3} + \| (-A)^{-\beta} \| L \| x_0 - x(t) \| + \int_0^t \frac{C_{1-\beta} C_1}{(t-s)^{1-\beta}} ds \\
&\leq \frac{\epsilon}{3} + \| (-A)^{-\beta} \| L\rho + C_{1-\beta} C_1 \frac{r^\beta}{\beta} \\
&\leq \frac{2\epsilon}{3} + \| (-A)^{-\beta} \| L\rho.
\end{aligned}$$

Hence

$$\| \Psi_1(x)(t) - x_0 \| \leq \frac{2\epsilon}{3} + \| (-A)^\beta \| L\rho. \tag{14}$$

Similarly for the operator Ψ_2 we have

$$\| \Psi_2(x)(t) \| \leq \tilde{M} C_1 r < \frac{\epsilon}{3}. \tag{15}$$

From the inequalities (14) and (15), clearly $\Psi(S(r,\rho)) \subset S(r,\rho)$.

To prove that Ψ_1 satisfies Lipschitz condition we take $x(\cdot)$ and $y(\cdot)$ in $S(r,\rho)$. Then for all $t \in [0,r]$

$$\begin{aligned}
\| \Psi_1(x)(t) - \Psi_1(y)(t) \| &\leq \| g(t,x(t)) - g(t,y(t)) \| \\
&\quad + \int_0^t \frac{C_{1-\beta} L}{(t-s)^{1-\beta}} \| (-A)^\beta g(s,x(s)) - (-A)^\beta g(s,y(s)) \| \, ds \\
&\leq \| (-A)^{-\beta} \| L \| x(t) - y(t) \| + C_{1-\beta} L \int_0^t \frac{\| x(s) - y(s) \|}{(t-s)^{1-\beta}} ds \\
&\leq (\| (-A)^{-\beta} \| L + C_{1-\beta} \frac{r^\beta}{\beta} L) \| x - y \|_{[0,r]} .
\end{aligned}$$

Therefore, from the choice of r, there exists $0 < \theta < 1$ such that

$$\| \Psi_2(x) - \Psi_2(y) \|_{[0,r]} \leq \theta \| x - y \|_{[0,r]} .$$

On the other hand, a classical argument proves that the operator Ψ_2 is compact. We omit this part of the proof.

The preceding remarks prove that the operator Ψ satisfies the condition in Theorem (1). Clearly an fixed point of Ψ is a mild solution of the neutral system (1). This completes the proof.

The next theorem is a classical result, the proof is standard and we will consider here only the principal ideas.

Theorem 3 *Let $x_0 \in X$ and assume that the following conditions hold:*

a) There exist $\beta \in (0,1)$ and $L \geq 0$ such that the function g is X_β-valued and satisfies the Lipschitz condition

$$\| (-A)^\beta g(t,x) - (-A)^\beta g(s,y) \| \leq L(\mid t-s \mid + \| x-y \|) \tag{16}$$

for every $0 \leq s,t < T$ and $x, y \in \Omega$ and

$$\| (-A)^{-\beta} \| L < 1. \tag{17}$$

b) The function f is continuous and there exists $N > 0$ such that

$$\| f(t,x) - f(s,y) \| \leq N(\mid t-s \mid + \| x-y \|) \tag{18}$$

for every $0 \leq s,t < T$ and $x, y \in \Omega$.

Then there exists a unique mild solution $x(\cdot, x_0)$ of the abstract Cauchy problem (1) defined in $[0,r]$ for some $0 < r \leq T$.

Proof. With the notations used in the proof of Theorem (2), for $u,\ v \in C([0,T] : X)$ and $t \in [0,T]$,

$$\begin{aligned}
\| \Psi(u)(t) - \Psi(v)(t) \| \quad & \leq \quad \| g(t,u(t)) - g(t,v(t)) \| \\
& \quad + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| (-A)^\beta g(s,u(s)) - (-A)^\beta g(s,v(s)) \| \, ds \\
& \quad + \tilde{M} \int_0^t \| f(s,u(s)) - f(s,v(s)) \| \, ds \\
& \leq \quad \| (-A)^{-\beta} \| \| (-A)^\beta g(t,u(t)) - (-A)^\beta g(t,v(t)) \| \\
& \quad + (C_{1-\beta} \frac{t^\beta}{\beta} L + \tilde{M} N t) \| u - v \|_{[0,t]} \\
& \leq \quad (\| (-A)^{-\beta} \| L + C_{1-\beta} \frac{t^\beta}{\beta} L + \tilde{M} N t) \| u - v \|_t \, .
\end{aligned}$$

Then for $0 < r < T$

$$\| \Psi(u) - \Psi(v) \|_{[0,r]} \leq (\| (-A)^{-\beta} \| L \| + C_{1-\beta} \frac{r^\beta}{\beta} + \tilde{M} N r) \| u - v \|_{[0,r]} \, . \tag{19}$$

The last inequality implies that for $0 < r < T$ sufficiently small Ψ is a contraction mapping, since $\| (-A)^{-\beta} \| L < 1$, which in turn implies that there exists a unique mild solution of (1) in $C([0,r] : X)$.

Remark 1 *Clearly the assertions in Theorem (3) remain true if the assumptions on f and g are substituted by locally Lipschitz conditions, in particular if $(-A)^\beta F$ and G are continuously differentiable in $[0,T] \times \Omega$.*

3 Regular Solutions

In this section we establish the existence of regular solution to the abstract Cauchy problem (4). The existence of N-classical and classical solutions, requires some additional regularity conditions on the functions g, f.

Assumption(f, g) : There exist $0 < \alpha < \beta < 1$ and an open set $\Omega_\alpha \subset X_\alpha$ such that the functions $f, (-A)^\beta g$ are continuous in $[0,T] \times \Omega_\alpha$ and

- There exist $L > 0$ and $0 < \gamma < 1$ such that for every $(t, x_1), (s, x_2) \in [0,T] \times \Omega_\alpha$

$$\begin{aligned} \| (-A)^\beta g(t,x_1) - (-A)^\beta g(s,x_2) \| &\leq L\{| t-s |^\gamma + \| x_1 - x_2 \|_\alpha\}, \\ \| f(t,x_1) - f(s,x_2) \| &\leq L\{| t-s |^\gamma + \| x_1 - x_2 \|_\alpha\}, \\ L \| (-A)^{-\beta+\alpha} \| &\leq 1. \end{aligned}$$

Theorem 4 *Assume that the Neutral system (1) satisfies the Assumption(f,g), that g is a $D(A)$-valued continuous function and that $1-\beta < Min\{\beta-\alpha, \gamma\}$. Then for every $x_0 \in \Omega_\alpha$ there exists a unique N-classical solution $x(\cdot) \in C([0,r] : X)$ for some $0 < r < T$.*

Proof. Let $0 < r_1 < T$ and $\delta > 0$ such that

$$V = \{(t,x) \in [0,r_1] \times X_\alpha : \| (-A)^\alpha x - (-A)^\alpha x_0 \| < \delta\} \subset [0,T] \times \Omega_\alpha. \tag{20}$$

Assuming that the functions f, $(-A)^\beta g$ are bounded in V by $C_1 > 0$, we choice $0 < r < r_1$ such that:

$$\| (T(\cdot) - I)(-A)^\alpha x_0 \|_{[0,r]} \leq \frac{(1-\mu)\delta}{6}, \tag{21}$$

$$\| (T(\cdot) - I)(-A)^\alpha g(0,x_0) \|_{[0,r]} \leq \frac{(1-\mu)\delta}{6}, \tag{22}$$

$$\frac{r^{\beta-\alpha}}{\beta-\alpha} C_{1-\beta+\alpha} L\delta + \frac{r^{1-\alpha}}{1-\alpha}(C_\alpha N\delta + C_\alpha C_1) \leq \frac{(1-\mu)\delta}{6}, \tag{23}$$

$$\| (-A)^{\alpha-\beta} \| Lr^\gamma + C_{1-\beta+\alpha} C_1 \frac{r^{\beta-\alpha}}{\beta-\alpha} + C_\alpha C_1 \frac{r^{1-\alpha}}{1-\alpha} < \frac{(1-\mu)\delta}{6}, \tag{24}$$

$$LC_{1-\beta+\alpha} \frac{r^{\beta-\alpha}}{\beta-\alpha} + C_\alpha L \frac{r^{1-\alpha}}{1-\alpha} < 1-\mu, \tag{25}$$

where $\mu = \| (-A)^{\alpha-\beta} \| L$. On the space $C([0,r] : X)$, we define the operator

$$\begin{aligned} \Psi(y)(t) &= T(t)(-A)^\alpha(x_0 + g(0,x_0)) - (-A)^\alpha g(t, (--A)^{-\alpha} y(t)) \\ &+ \int_0^t (-A)^{1-\beta+\alpha} T(t-s)(-A)^\beta g(s, (-A)^{-\alpha} y(s)) ds \\ &+ \int_0^t (-A)^\alpha T(t-s) f(s, (-A)^{-\alpha} y(s)) ds. \end{aligned} \tag{26}$$

The same arguments used in the proof of Theorem 2, show that Ψ is well defined on $Y = C([0,r] : X)$. We will prove that the operator Ψ is a contraction in

$$S = \{y \in Y : y(0) = (-A)^\alpha x_0, \ \| y(t) - (-A)^\alpha x_0 \| \leq \delta, \ t \in [0,r]\}.$$

Firstly we prove that the range of Ψ is contained in S. Let $y \in S$. Then for $t \in [0,r]$,

$$\| \Psi(y)(t) - (-A)^\alpha x_0 \|$$

$$\begin{aligned}
&\leq \| (T(t)-I)(-A)^\alpha x_0 \| + \| (T(t)-I)(-A)^\alpha g(0,x_0) \| \\
&\quad + \| (-A)^\alpha g(0,x_0) - (-A)^\alpha g(t,(-A)^{-\alpha} y(t)) \| \\
&\quad + \int_0^t \frac{C_{1-\beta+\alpha}}{(t-s)^{1-\beta+\alpha}} \| (-A)^\beta g(s,(-A)^{-\alpha} y(s)) \| \, ds \\
&\quad + \int_0^t \frac{C_\alpha}{(t-s)^\alpha} \| f(s,(-A)^{-\alpha} y(s)) \| \, ds \\
&\leq \frac{2(1-\mu)\delta}{6} + \| (-A)^{\alpha-\beta} \| L\{r^\gamma + \| y(t) - (-A)^\alpha x_0 \|\} \\
&\quad + \int_0^t \frac{C_{1-\beta+\alpha} C_1}{(t-s)^{1-\beta+\alpha}} ds + \int_0^t \frac{C_\alpha C_1}{(t-s)^\alpha} ds \\
&\leq \frac{2(1-\mu)\delta}{6} + \| (-A)^{\alpha-\beta} \| L\{r^\gamma + \delta\} + C_{1-\beta+\alpha} C_1 \frac{r^{\beta-\alpha}}{\beta-\alpha} + C_\alpha C_1 \frac{r^{1-\alpha}}{1-\alpha}.
\end{aligned}$$

From the choice of r we can conclude that

$$\| \Psi(y) - (-A)^\alpha x_0 \|_r \leq \delta$$

so that, $\Psi(y) \in S$.

On the other hand for $x(\cdot), y(\cdot) \in S$ and $t \in [0,r]$,

$$\| \Psi(y)(t) - \Psi(x)(t) \|$$

$$\begin{aligned}
&\leq \| (-A)^\alpha g(t,(-A)^{-\alpha} y(t)) - (-A)^\alpha g(t,(-A)^{-\alpha} x(t)) \| \\
&\quad + \int_0^t \frac{C_{1-\beta+\alpha}}{(t-s)^{1-\beta+\alpha}} \| (-A)^\beta g(s,(-A)^{-\alpha} y(s)) - (-A)^\beta g(s,(-A)^{-\alpha} x(s)) \| \, ds \\
&\quad + \int_0^t \frac{C_\alpha}{(t-s)^\alpha} \| f(s,(-A)^{-\alpha} y(s)) - f(s,(-A)^{-\alpha} x(s)) \| \, ds \\
&\leq \| (-A)^{\alpha-\beta} \| L \| y(t) - x(t) \| + \int_0^t \frac{C_{1-\beta+\alpha}}{(t-s)^{1-\beta+\alpha}} L \| y - x \|_r \, ds \\
&\quad + \int_0^t \frac{C_\alpha}{(t-s)^\alpha} L \| y - x \|_r \, ds.
\end{aligned}$$

Then

$$\| \Psi(x) - \Psi(y) \|_r \leq (L \| (-A)^{-\beta+\alpha} \| + L C_{1-\beta+\alpha} \frac{r^{\beta-\alpha}}{\beta-\alpha} + C_\alpha L \frac{r^{1-\alpha}}{1-\alpha}) \| x - y \|_r \, .$$

The last estimate and the choice of b imply that Ψ is a contraction mapping in S. Let $y(\cdot)$ be the only fixed point of the operator Ψ in S. We affirm that $y(\cdot)$ is Hölder continuous. In fact, let $\vartheta < Min\{1-\alpha, \beta-\alpha\}$ with $\vartheta+\beta > 1$, and $\tilde{C} > 0$ be the constant guaranteed in lemma (1), such that for all $0 < s < t < T$ and $0 < h < 1$

$$\| (T(h)-I)(-A)^{\alpha}T(t-s) \| \leq \frac{\tilde{C}h^{\vartheta}}{(t-s)^{\vartheta+\alpha}}, \qquad 0 < s < t$$

and

$$\| (T(h)-I)(-A)^{1-\beta+\alpha}T(t-s) \| \leq \frac{\tilde{C}h^{\vartheta}}{(t-s)^{1-\beta+\alpha+\vartheta}}, \qquad 0 < s < t.$$

For $x \in [0, r)$ and $h > 0$ sufficiently small

$$\| y(t+h) - y(t) \|$$

$$\begin{aligned}
\leq\ & \| (T(h)-I)(-A)^{\alpha}T(t)(x_0 - g(0,x_0)) \| \\
& + \| (-A)^{\alpha-\beta} \| L\{h^{\gamma} + \| y(t+h)-y(t) \| \} \\
& + \int_0^t \| (T(h)-I)(-A)^{1-\beta+\alpha}T(t-s)(-A)^{\beta}g(s,(-A)^{-\alpha}y(s)) \| ds \\
& + \int_t^{t+h} \| (-A)^{1-\beta+\alpha}T(t+h-s)(-A)^{\beta}g(s,(-A)^{-\alpha}y(s)) \| ds \\
& + \int_0^t \| (T(h)-I)(-A)^{\alpha}T(t-s)f(s,(-A)^{-\alpha}y(s)) \| ds \\
& + \int_t^{t+h} \| (-A)^{\alpha}T(t+h-s)f(s,(-A)^{-\alpha}y(s)) \| ds \\
\leq\ & \frac{\tilde{C}}{t^{(\alpha+\vartheta)}} \| x_0 - g(0,x_0) \| h^{\vartheta} + L \| (-A)^{\alpha-\beta} \| \{h^{\gamma} + \| y(t+h)-y(t) \| \} \\
& + \int_0^t \frac{\tilde{C}C_1 h^{\vartheta}}{(t-s)^{1-\beta+\alpha+\vartheta}} ds + \int_t^{t+h} \frac{C_1 C_{1-\beta+\alpha}}{(t-s)^{1-\beta+\alpha}} ds \\
& + \int_0^t \frac{\tilde{C}C_1 h^{\vartheta}}{(t-s)^{\alpha+\vartheta}} ds + \int_t^{t+h} \frac{C_{\alpha}C_1}{(t-s)^{\alpha}} ds \\
\leq\ & \frac{C(x_0)h^{\vartheta}}{t^{\vartheta+\alpha}} + C_2 h^{\gamma} + L \| (-A)^{\alpha-\beta} \| \| y(t+h)-y(t) \| + C_3 h^{\vartheta} \\
& + C_4 h^{\beta-\alpha} + C_5 h^{\vartheta} + C_6 h^{1-\alpha},
\end{aligned}$$

where the constants C_i are independent of t. If $\rho = Min\{\vartheta, \gamma\}$, the last inequality can be rewritten in the form

$$\| y(t+h) - y(t) \| \leq \frac{C(\alpha,\beta,\theta,t)}{1-\mu} h^{\rho} \tag{27}$$

since $\mu = L \| (-A)^{\alpha-\beta} \| < 1$. Therefore the function $y(\cdot)$ is ρ-Hölder continuous in $(0, r)$ and $\rho + \beta > 1$.

Now it is easy to prove that $s \to (-A)^{\beta}g(s,(-A)^{-\alpha}y(s))$ and $s \to f(s,(-A)^{-\alpha}y(s))$ are ρ-Hölder continuous in $(0, r)$. From this remark, Theorem (4.3.2) in [4] and Lemma

(2) below, we infer that the function,

$$\begin{aligned} x(t) &= T(t)(x_0 + g(0, x_0)) - g(t, (-A)^{-\alpha} y(t)) \\ &+ \int_0^t (-A)^{1-\beta} T(t-s)(-A)^{\beta} g(s, (-A)^{-\alpha} y(s)) ds \\ &+ \int_0^t T(t-s) f(s, (-A)^{-\alpha} y(s)) ds \end{aligned}$$

is X_α-valued, that the integral terms are functions in $C^1([0, r] : X)$ and that $x(t) \in D(A)$ for all $t \in (0, r)$. Operating on $x(\cdot)$ with $(-A)^{\alpha}$, we conclude that $(-A)^{-\alpha} y = x$ and hence that $x(t) + g(t, x(t))$ is a C^1 function in $(0, b)$. This completes the proof.

Lemma 2 *Let $0 < \beta < 1$ and $g \in C([0, T] : X_{1-\beta})$. Assume that $g : [0, T] \to X$ is θ-Hölder continuous in $(0, T)$ with $\beta + \theta > 1$. If $y : [0, T] \to X$ is defined by*

$$y(t) = \int_0^t (-A)^{1-\beta} T(t-s) g(s) ds$$

then $y(t) \in D(A)$ for every $t \in [0, T)$ and $\dot{y} \in C((0, T) : X)$.

Proof. The proof is entirely analogous to the proof of Theorem (2.4.1) in [1], we include only the principal ideas. For $t \in [0, T)$ we rewrite $y(t)$ in the form

$$\int_0^t (-A)^{1-\beta} T(t-s)(g(s) - g(t)) ds + \int_0^t (-A)^{1-\beta} T(t-s) g(t) ds = v(t) + w(t). \qquad (28)$$

Clearly, $Aw(t) = T(t)(-A)^{1-\beta} g(t) - (-A)^{1-\beta} g(t) \in C([0, T] : X)$. Now for $\epsilon > 0$ sufficiently small we define the function

$$v_\epsilon(t) := \begin{cases} \int_0^{t-\epsilon} (-A)^{1-\beta} T(t-s)(g(s) - g(t)) ds, & \text{for } t \in [\epsilon, T), \\ 0 & \text{for } t \in [0, \epsilon]. \end{cases} \qquad (29)$$

It is clear that $v_\epsilon(t) \in D(A)$. Moreover for $0 < \delta_1 < \delta_2$

$$\begin{aligned} \| Av_{\delta_2}(t) - Av_{\delta_1}(t) \| &\leq \int_{t-\delta_2}^{t-\delta_1} \| (-A)^{2-\beta} T(t-s)(g(s) - g(t)) \| ds \\ &\leq \int_{t-\delta_2}^{t-\delta_1} \frac{C_{2-\beta}}{(t-s)^{2-\beta}} |s - t|^{\theta} ds \\ &\leq C_{2-\beta} (\delta_2^{\beta+\theta-1} - \delta_1^{\beta+\theta-1}). \end{aligned} \qquad (30)$$

The last inequality proves that $A(v_\delta)(t)$ is convergent, $\beta + \theta > 1$, and therefore

$$A(v_\epsilon)(t) = \int_0^t A^{2-\beta} T(t-s)(g(s) - g(t)) ds, \qquad (31)$$

since A is a closed operator.

The continuity of $A(y)$ is proved as in [1] and we omit more details. Finally the assertion $\partial_t y \in C([0, T) : X)$ follows in the usual manner.

In the rest of this paper for a function $j : [0,b] \times X \to X$ and $h \in \mathbb{R}$ we denote by $\partial_h j$ to the function

$$\partial_h j(t) = \frac{j(t+h) - j(t)}{h}.$$

Moreover, if j is differentiable we will employ the following decomposition:

$$j(t+s,y) - j(t,y) = D_1 j(t,y)s + W_1(j,t,s,y) \tag{32}$$

and

$$j(t,y+y_1) - j(t,y) = D_2 j(t,y) \cdot y_1 + W_2(j,t,y,y_1) \tag{33}$$

where

$$\frac{W_1(j,t,s,y)}{|\, t-s \,|} \to 0 \qquad as \qquad s \to t,$$
$$\frac{W_1(j,t,y,y_1)}{\|\, y - y_1 \,\|} \to 0 \qquad as \qquad y_1 \to y.$$

In order to prove the next theorem, we need the following technical lemma:

Lemma 3 *Assume that the assumption of Theorem (3) holds. If $x_0 \in D(A)$ and g is $D(A)$-valued continuous then $x(\cdot) = x(\cdot, x_0)$ is Lipschitz on closed intervals.*

Proof. We consider only the principal ideas. Let $x(\cdot, x_0)$ be a mild solution of (1) and assume that $x(\cdot)$ is defined in $[0,b]$. Using the continuity of Ag and f we can to assert that $Ag(s,x(s))$ and $f(s,x(s))$ are bounded by $C_1 > 0$ in $[0,b]$. From the assumption $x_0 \in D(A)$ for $t \in [0,b)$ and $h > 0$ we have

$$\|\, x(t+h) - x(t) \,\|$$

$$\begin{aligned}
&\leq C_2 h + \|\, g(t+h, x(t+h)) - g(t,x(t)) \,\| \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \|\, (-A)^\beta g(s+h, x(s+h)) - (-A)^\beta g(s,x(s)) \,\| \, ds \\
&+ \int_0^h \|\, T(t+h-s)(-A) g(s,x(s)) \,\| \, ds \\
&+ \tilde{M} \int_0^t \|\, f(s+h, x(s+h)) - f(s,x(s)) \,\| \, ds + \tilde{M} \int_0^h \|\, f(s,x(s)) \,\| \, ds \\
&\leq C_1 h + L \,\|\, (-A)^{-\beta} \,\|\, \{h + \|\, x(t+h) - x(t) \,\|\} \\
&+ \int_0^t \frac{C_{1-\beta} L}{(t-s)^{1-\beta}} \{\|\, x(s+h) - x(s) \,\| + h\} ds + \tilde{M} C_1 h \\
&+ N\tilde{M} \int_0^t \{h + \|\, x(s+h) - x(s) \,\|\} ds + \tilde{M} C_1 h
\end{aligned}$$

so that, there exists $C_4 > 0$, independent of $t \in [0,b]$, such that

$$\begin{aligned}
\|\, x(t+h) - x(t) \,\| \;\; &\leq \;\; C_4 h + \|\, (-A)^{-\beta} \,\|\, L \,\|\, x(t+h) - x(t) \,\| \\
&+ \int_0^t \{\frac{C_{1-\beta} L}{(t-s)^{1-\beta}} + N\tilde{M}\} \,\|\, x(s+h) - x(s) \,\| \, ds.
\end{aligned}$$

Then

$$\| x(t+h) - x(t) \| \leq \frac{C_4 h}{1-\mu} + \frac{1}{1-\mu}\int_0^t \{\frac{C_{1-\beta}L}{(t-s)^{1-\beta}} + \tilde{M}\} \| x(s+h) - x(s) \| ds.$$

since $\mu = \| (-A)^{-\beta} \| L < 1$. Finally the assertion follows from the Gronwall Bellman inequality.

Next we establish a different condition for the existence of classical solutions. In this case we assume that the functions f and $(-A)^{\beta} g$ are continuously differentiable.

Theorem 5 *Assume that $(-A)^{1-\beta} g(\cdot)$ and $f(\cdot)$ are continuously differentiable functions in $[0,T] \times \Omega$. If $x_0 \in D(A)$, $g(\cdot)$ is $D(A)$-valued continuous and $\| D_2 g(0, x_0) \|_{\mathcal{L}(X)} < 1$ then there exists a unique classical solution of (1) defined in $[0, r]$ for some $0 < r < T$.*

Proof: Let $x(\cdot) = x(\cdot, x_0)$ be the mild solution of the abstract Cauchy problem (1) and $z(\cdot)$ the solution of the integral equation

$$\begin{aligned} z(t) &= T(t)(Ax_0 + f(0, x_0)) + h(t) + D_2 g(t, x(t)) z(t) \\ &+ \int_0^t (-A)^{1-\beta} T(t-s) D_2 (-A)^{\beta} g(s, x(s)) z(s) ds \\ &+ \int_0^t T(t-s) D_2 f(s, x(s)) z(s) ds \end{aligned} \tag{34}$$

with

$$z(0) = Ax_0 + f(0, x_0) \tag{35}$$

where

$$\begin{aligned} h(t) &= D_1 g(t, x(t)) + \int_0^t (-A)^{1-\beta} T(t-s) D_1 (-A)^{\beta} g(s, x(s)) ds \\ &+ \int_0^t T(t-s) D_1 f(s, x(s)) ds. \end{aligned}$$

The existence and uniqueness of a local solution for the integral equation (34) is a consequence of the contraction mapping principle and the condition $\| D_2 g(0, x_0) \|_{\mathcal{L}(X)} < 1$; we omit more details. In what follows we assume that $x(\cdot)$ and $z(\cdot)$ are defined in $[0, 2b]$ where $0 < 2b < T$ and $\| D_2 g(t, x_t) \|_{2b} < \eta < 1$. Using the notations introduced previously, for t in $[0, b]$ and $h > 0$ sufficiently small we have

$$\xi(t,h) = \| \frac{x(t+h) - x(t)}{h} - z(t) \|$$

$$\begin{aligned} \leq & \| T(t)(\frac{T(h) - I}{h} x_0) - A(x_0) \| + \| \frac{1}{h}\int_0^h T(t+h-s) f(s, x(s)) ds - T(t) f(0, x_0) \| \\ &+ \| T(t)(\frac{T(h) - I}{h}) g(0, x_0) + \frac{1}{h}\int_0^h (-A)^{1-\beta} T(t+h-s)(-A)^{\beta} g(s, x(s)) ds \| \\ &+ \| D_2 g(t, x(t)) \xi(t,h) \| + \| D_1 g(t, x(t+h) - D_1 g(t, x(t)) \| \end{aligned}$$

$$
\begin{aligned}
&+ \| \frac{W_2(g,t,x(t),x(t+h)}{h} \| + \| \frac{W_1(g,t,t+h,x(t+h)}{h} \| \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_1(-A)^\beta g(s,x(s+h)) - D_1(-A)^\beta g(s,x(s)) \| ds \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_2(-A)^\beta g(s,x(s))\xi(s,h) \| ds \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| \frac{W_1((-A)^\beta g,s,s+h,x(s+h))}{h} \| ds \\
&+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| \frac{W_2((-A)^\beta g,s,x(s),x(s+h))}{h} \| ds \\
&+ \int_0^t \tilde{M} \| D_1 f(s,x(s+h)) - D_1 f(s,x(s)) \| ds \\
&+ \int_0^t \tilde{M} \| D_2 f(s,x(s))\xi(t,h) \| ds \\
&+ \tilde{M} \int_0^t \| \frac{W_1(f,s,s+h,x(s+h))}{h} \| ds + \tilde{M} \int_0^t \| \frac{W_2(f,s,x(s),x(s+h))}{h} \| ds.
\end{aligned}
$$

On the other hand, from lemma (3), we know that $x(\cdot)$ is Lipschitz. Therefore,

$$
\frac{W_2((-A)g,s,x(s),x(s+h))}{\| x(s+h)-x(s) \|} \cdot \frac{\| x(s+h)-x(s) \|}{h} \to 0 \qquad as\ h \to 0
$$

and

$$
\frac{W_2(f,s,x(s),x(s+h))}{\| x(s+h)-x(s) \|} \cdot \frac{\| x(s+h)-x(s) \|}{h} \to 0 \qquad as\ h \to 0.
$$

This enables us to rewrite the last inequality in the form,

$$
\begin{aligned}
\| \xi(t,h) \| \quad \leq \quad & \rho(t,h) + \| D_2 g(t,x(t)) \| \| \xi(t,h) \| \\
& + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_2(-A)^\beta g(s,x(s)) \| \| \xi(s,h) \| ds \\
& + \tilde{M} \int_0^t \| D_2 f(s,x(s) \| \| \xi(s,h) \| ds
\end{aligned}
$$

where

$$
\rho(t,h) \to 0 \qquad as\ h \to 0.
$$

From this inequality and the choice of b, it follows that,

$$
\begin{aligned}
\| \xi(t,h) \| \quad \leq \quad & \frac{1}{1-\eta}\rho(t,h) + \frac{1}{1-\eta}\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| \partial_2(-A)^\beta g(s,x(s)) \| \| \xi(s,h) \| ds \\
& + \frac{1}{1-\eta}\tilde{M} \int_0^t \| \partial_2 f(s,x(s) \| \| \xi(s,h) \| ds.
\end{aligned}
$$

Finally, the Gronwall Bellman inequality shows that $\xi(t,h) \to 0$ as $h \to 0$. This proves that $x(\cdot, x_0) = z(\cdot)$ and completes this proof.

Example 1

We conclude this section with an application of the results of this work. In the space $X := L^2([0,\pi])$ we consider the operator $Ax := x''$ with domain

$$D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}.$$

The operator A is the infinitesimal generator of a strongly continuous semigroup $(T(\cdot))_{t\geq 0}$; moreover the semigroup is compact, analytic and self-adjoint. The spectrum of A is discrete, the eigenvalues are $-n^2, n \in \mathbb{N}$ with eigenvectors $z_n(\xi) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(n\xi)$ and the following properties hold:

1. $\{z_n : n \in \mathbb{N}\}$ is an orthonormal base of X.
2. For $x \in X$, $A(x) = -\sum_{n=1}^{\infty} n^2(x, z_n)z_n$.
3. For $x \in X$, $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}(x, z_n)z_n$.
4. For $0 < \rho \leq 1$ and $x \in X$, $(-A)^{\rho}x = \sum_{n=1}^{\infty} n^{2\rho}(x, z_n)z_n$ and

$$D((-A)^{\rho}) = \{x \in X : \sum_{n=1}^{\infty} n^{4\rho}(x, z_n)^2 < \infty\}.$$

5. For $x \in X$, $(-A)^{\frac{1}{2}}x = \sum_{n=1}^{\infty} n(x, z_n)z_n$.
6. For $f \in X$, $(-A)^{-\frac{1}{2}}x = \sum_{n=1}^{\infty} \frac{1}{n}(x, z_n)z_n$ and $\| (-A)^{-\frac{1}{2}} \| = 1$.

Let $a : I = [0,\pi] \times [0,\pi] \to \mathbb{R}$ be a sufficiently regular function with $a(0,\cdot) \equiv a(\pi,\cdot) \equiv 0$. Now we define the operator $\Lambda : [0,T] \times X \to X$ by the expression

$$\Lambda(\varphi)(t) = \int_0^{\pi} a(t,s)\varphi(s)ds. \tag{36}$$

Clearly Λ is a bounded linear operator in X. A simple calculation proves that for

$$\int_0^{\pi}\int_0^{\pi} (\frac{\partial^2 a(t,s)}{\partial t^2})^2 ds dt < \infty,$$

sufficiently small, the range of Λ is contained in $D(-A)$ and $\| A\Lambda \| < 1$.

Let $f : [0,T] \times X \to X$ be continuously differentiable. The existence of a mild and N-classical solutions for the abstract Cauchy problem

$$\begin{aligned} \frac{d}{dt}(u(t) + \Lambda(u)(t)) &= Au(t) + f(t, u(t)), \qquad t > 0, \\ u(0) &= u_0, \end{aligned} \tag{37}$$

follows from Theorem (2) or Theorem (3) and Theorem (5) respectively.

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